A STRUCTURE THEOREM FOR SI-MODULES

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An associative ring R is called a *left SI-ring* if every singular left R-module is injective. In Goodearl [**4**] it is shown that these rings have a finite ring decomposition into a ring K with $K/\mathrm{Soc}\, K$ left semisimple, and simple rings which are Morita equivalent to left SI-domains.

For an R-module M denote by $\sigma[M]$ the full subcategory of R-Mod subgenerated by M. Extending the definition of SI-rings, we call an R-module M an *SI-module* if every singular module in $\sigma[M]$ is M-injective. This also generalizes a similar notion in Yousif [**11**]. We obtain that every finitely generated, self-projective SI-module M has a decomposition

$$M = K \oplus V_1 \oplus \cdots \oplus V_n,$$

with fully invariant submodules K, V_i, such that $K/\mathrm{Soc}\, K$ is a semisimple R-module, and, for $i = 1, \ldots, n$, $\mathrm{End}_R(V_i)$ is a simple ring, and the category $\sigma[V_i]$ is equivalent to T_i-Mod for an SI-domain T_i.

1. Preliminary results. Let R be an associative ring with unit and R-Mod the category of unital left R-modules. For $M \in R$-Mod we denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are submodules of M-generated modules. M is called *self-projective* if it is M-projective. $\mathrm{Soc}\, M$ (resp. $\mathrm{Rad}\, M$) denotes the socle (resp. the radical) of the module M. An R-submodule of M is said to be *fully invariant* (or *characteristic*) if it is invariant under any R-endomorphism of M.

Morphisms are written on the opposite side to the scalars. For basic notions see [**10**].

The following elementary observations will be useful.

1.1. Proposition. *Consider a self-projective R-module M with $S = \mathrm{End}_R(M)$.*

(1) *If* $\mathrm{Rad}\, M = 0$, *then S has zero (Jacobson) radical.*

(2) *Assume M is finitely generated. Then M has no non-trivial fully invariant submodules if and only if S is a simple ring.*

Proof. (1) This follows from the fact that, for any simple homomorphic image E of M, $\mathrm{Hom}_R(M, E)$ is a simple left $\mathrm{End}_R(M)$-module.

(2) For every ideal $I \subset S$, $MI \subset M$ is fully invariant. Since M is self-projective, $I = \mathrm{Hom}_R(M, MI)$ by [**10**, 18.4] and hence $MI \neq M$ for $I \neq S$.

For every fully invariant submodule $U \subset M$, $\mathrm{Hom}_R(M, U)$ is an ideal in S.

If $K \subset M$ is an essential submodule, we write $K \trianglelefteq M$.

Let M and N be R-modules. N is called *singular in* $\sigma[M]$ or *M-singular* if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [**9**]).

By definition, every M-singular module belongs to $\sigma[M]$. For $M = R$ the notion R-singular is identical to the usual definition of singular for R-modules.

The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (e.g. [**10**, 17.3 and 17.4]). Hence every module $N \in \sigma[M]$ contains a *largest M-singular submodule* which we denote by $Z_M(N)$. The following properties of M-singular modules are shown in [**9**, 1.1] and [**8**, 2.4].

1.2. PROPOSITION. *Let M be an R-module.*

(1) *A simple R-module E is M-singular or M-projective.*

(2) *If* $\operatorname{Soc} M = 0$, *then every simple module in $\sigma[M]$ is M-singular.*

(3) *If M is self-projective and $Z_M(M) = 0$, then the M-singular modules form a hereditary torsion class in $\sigma[M]$.*

We extend the definition of a left SI-ring (see [**4**]) to modules.

DEFINITION. An R-module M is called an *SI-module* if every M-singular module is M-injective.

In Yousif [**11**], M is called an SI-module if every singular module in R-Mod is M-injective. Since M-singular modules are singular in R-Mod, this is a stronger condition than the one given above.

Though for $M = R$ the two notions coincide, in general SI-modules in our sense need not be SI-modules in the sense of Yousif (compare the example after [**9**, 2.2]).

Let T be a left SI-ring which is not left semisimple (for examples see [**4**], [**1**]), and R the ring of lower triangular (2,2)-matrices over T. The map

$$R = \begin{pmatrix} T & 0 \\ T & T \end{pmatrix} \to T, \qquad \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mapsto a,$$

is a surjective ring homomorphism whose kernel is essential as left ideal in R. Hence every left T-module is singular as an R-module and all T-singular modules are T-injective, i.e. T is a SI-module over R. Since T is not left semisimple, not every R-singular module is T-injective. Hence T is not an SI-module over R in the sense of Yousif.

Every left module over a left SI-ring is an SI-module in the sense of Yousif and hence is an example of an SI-module in our sense.

In Smith [**7**], R is called a *left RIC-ring* if every cyclic singular left R-module is injective. It is observed in [**5**, Corollary 5] that RIC-rings are SI-rings. By [**9**, 3.8 and 3.10] and [**2**, Lemma 2], we have more general statements in our next proposition which also include Proposition 3.1 and 3.6 in [**4**]. For this we recall two definitions.

An R-module M is called *hereditary in $\sigma[M]$* if every submodule of M is projective in $\sigma[M]$ (see [**10**, 39.1]). M is a *GCO-module* (*generalized co-semisimple*) if every M-singular simple R-module is M-injective (see [**9**, 2.2]).

1.3. PROPOSITION. *For a finitely generated, self-projective R-module M the following conditions are equivalent*:

(a) *M is an SI-module*;

(b) *every cyclic M-singular module is M-injective*;

(c) *M/K is semisimple for every $K \trianglelefteq M$ and $Z_M(M) = 0$*;

(d) *M is hereditary in $\sigma[M]$ and M-singular modules are semisimple*;

(e) *M is a GCO-module, $M/\operatorname{Soc} M$ is noetherian and* $\operatorname{Soc}(M/K) \neq 0$ *for every $K \trianglelefteq M$.*

We will need the following lemma.

1.4. LEMMA. *Let M be a self-projective SI-module with finite uniform dimension and* $\operatorname{Rad} M = 0$. *Then M contains no proper fully invariant submodule which is essential as an R-submodule.*

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Proof. Assume $V \subset M$ is a fully invariant submodule and $V \trianglelefteq_R M$. Since $\text{Rad}\, M = 0$, $S := \text{End}_R(M)$ has zero radical by Proposition 1.1. $\text{Rad}\, M = 0$ also implies that $\text{Hom}_R(M, U) \neq 0$ for non-zero $U \subset M$ (see [**8**], p. 1475, (iv)). With this knowledge we derive from Theorem 3.7 in [**8**] that there exists a monomorphism $f: M \to V$, and since M has finite uniform dimension, the image of every monomorphism in S is essential in M. Hence the image of f^2 is essential in M. Therefore M/Mf^2 is a semisimple module and the following exact sequence splits:

$$0 \to Mf/Mf^2 \to M/Mf^2 \to M/Mf \to 0.$$

Applying the functor $\text{Hom}_R(M, -)$ and the isomorphisms $Sg \simeq \text{Hom}(M, Mg)$ for any $g \in S$ (see [**10**], 18.4), we obtain that Sf/Sf^2 is a direct summand in the S-module S/Sf^2. Hence there exists a submodule $Sf^2 \subset U \subset S$ with $Sf + U = S$ and $Sf \cap U = Sf^2$. This yields $\text{id} = rf + u$ for some $r \in S$ and $u \in U$ and hence $f = frf + fu$. Since $fu \in Sf \cap U = Sf^2$ we finally have $f = frf + sf^2$ for some $s \in S$. Since f is monic this means $\text{id} = fr + sf$ and $M = Mfr + Msf \subset V$.

2. Structure theorem. Let us first describe uniform SI-modules with zero socle.

2.1. PROPOSITION. *For a finitely generated, self-projective R-module M, the following are equivalent*:

(a) *M is a uniform SI-module with* $\text{Soc}\, M = 0$;

(b) *M is a self-generator and* $\text{End}_R(M)$ *is a left SI-domain which is not a division ring.*

Under this condition, M has no fully invariant submodules and $\text{End}_R(M)$ *is a simple ring.*

Proof. (a)$\Rightarrow$(b). If M is an SI-module with zero socle, all simple modules in $\sigma[M]$ are M-singular (by Proposition 1.2), hence M-injective and M-generated. Therefore M is a projective generator in $\sigma[M]$. This implies that $\sigma[M]$ is equivalent to S-Mod (see [**10**, 18.5 and 46.2]) and S is a left SI-ring.

Since $Z_M(M) = 0$, every $f \in \text{End}_R(M)$ is a monomorphism.

(b)$\Rightarrow$(a). The functor $\text{Hom}_R(M, -)$ is an equivalence.

The last part follows from Lemma 1.4 and Proposition 1.1.

Now we investigate the decomposition of SI-modules with zero socles.

2.2. THEOREM. *For a finitely generated, self-projective R-module M and* $S = \text{End}_R(M)$, *the following are equivalent*:

(a) *M is an SI-module and* $\text{Soc}\, M = 0$;

(b) *M is a generator in* $\sigma[M]$ *and S is a left SI-ring with zero left socle*;

(c) $M = M_1 \oplus \ldots \oplus M_n$, *with* M_i *minimal fully invariant submodules, and* $\sigma[M_i] = \sigma[L_i]$ *for some finitely generated, self-projective and uniform SI-module* L_i *with zero socle*;

(d) $M = M_1 \oplus \ldots \oplus M_n$, *with* M_i *fully invariant submodules,* $\text{End}_R(M_i)$ *simple rings and* $\sigma[M_i]$ *equivalent to* T_i-Mod, *for left SI-domains* T_i *which are not division rings.*

Proof. (a)$\Leftrightarrow$(b). As observed in the proof of Proposition 2.1, (a) implies that M is a projective generator in $\sigma[M]$. Hence $\sigma[M]$ is equivalent to S-Mod (see [**10**, 18.5 and 46.2]) and M is an SI-module if and only if S is a left SI-ring.

(a)$\Rightarrow$(c). As noted above, M is a generator in $\sigma[M]$, and by Proposition 1.3, M is noetherian and hereditary in $\sigma[M]$. Every essential submodule of M is an intersection of maximal submodules, and $\operatorname{Soc} M = 0$ implies $\operatorname{Rad} M = 0$ and S has zero radical by Proposition 1.1.

By Theorem 3.7 in [**8**], the endomorphism ring of the M-injective hull $\hat{M}$ is semisimple artinian, i.e. $\operatorname{End}_R(\hat{M}) = T_1 \oplus \ldots \oplus T_n$ with simple artinian rings T_i. Denoting by e_i the unit in T_i, we have $e_1 + \ldots + e_n = \mathrm{id}_{\hat{M}}$ and, since the e_i are in the center of $\operatorname{End}_R(\hat{M})$,

$$\hat{M} = \hat{M}e_1 \oplus \ldots \oplus \hat{M}e_n$$

is a decomposition into fully invariant submodules. The intersection $M_i := M \cap \hat{M}e_i$ is a fully invariant submodule of M and $M_1 \oplus \ldots \oplus M_n \trianglelefteq_R M$. As we have seen in Lemma 1.4, this means

$$M_1 \oplus \ldots \oplus M_n =_R M.$$

Since $\operatorname{Hom}_R(M_i, M_j) = 0$ for $i \neq j$, we observe that $\hat{M}e_i$ is the injective hull of M_i in $\sigma[M_i]$. Moreover M_i is a self-projective self-generator with $Z_M(M) = 0$ and, again applying Theorem 3.7 in [**8**], we know that $\operatorname{End}_R(\hat{M}e_i) \simeq T_i$ is the classical left quotient ring of $\operatorname{End}_R(M_i)$. Hence $\operatorname{End}_R(M_i)$ has no non-trivial central idempotents and M_i has no non-trivial decomposition into fully invariant submodules.

To study properties of the summands M_i we may assume that M itself has no non-trivial decomposition into fully invariant submodules. We want to show that M has no proper fully invariant submodules.

Let $X \subset M$ be fully invariant. First consider a non-zero R-submodule $Y \subset M$ with $X \cap Y = 0$. We show that X and Y do not have isomorphic uniform submodules: assume, for a uniform submodule $U \subset X$, there exists a monomorphism $g: U \to Y$. Since $\operatorname{Hom}_R(M, U)$ is a non-zero left ideal in S and $\operatorname{Rad} S = 0$, we can find $f: M \to U$ with $f^2 \neq 0$ and hence $(U)f \neq 0$. Then $(U)fg \subset Y$ and, by the invariance of X, also $(U)fg \subset X$ implying $(U)f = 0$, a contradiction.

Now let $\{Y_\lambda\}_\Lambda$ denote the family of all submodules of M, with no uniform submodules isomorphic to submodules of X, and put $Y = \sum_\Lambda Y_\lambda$.

Assume Y contains a uniform submodule U isomorphic to a submodule of X. Since M is hereditary in $\sigma[M]$, we may suppose $U \subset \bigoplus_\Lambda Y_\lambda$, and we conclude that U has an isomorphic copy in one of the Y_λ's (compare [**10**, 39.7]), a contradiction. Obviously, $X \cap Y = 0$ and, by the above observation, $X \oplus Y \trianglelefteq_R M$.

Hereditariness of M also implies that, for any $f \in S$, $(Y)f$ has no uniform submodules isomorphic to submodules in X. Hence $(Y)f \subset Y$, i.e. Y and $X \oplus Y$ are fully invariant in M. By Lemma 1.4, we have $X \oplus Y = M$. This means by assumption $X = M$.

Now choose a uniform submodule $U \subset M$ and a non-zero $f \in \operatorname{Hom}_R(M, U)$. Then $L := (M)f$ is uniform and M-projective. The trace $\operatorname{Tr}(L, M)$ of L in M is fully invariant and hence $\operatorname{Tr}(L, M) = M$, implying $\sigma[M] = \sigma[L]$.

(c)$\Rightarrow$(d). Each of the L_i is a progenerator in $\sigma[M_i] = \sigma[L_i]$ (see proof of (a)$\Leftrightarrow$(b)). Hence $\sigma[M_i]$ is equivalent to T_i-Mod where $T_i := \operatorname{End}_R(L_i)$ is a left SI-domain by Proposition 2.1.

According to Proposition 1.1, $\operatorname{End}_R(M_i)$ is a simple ring.

(d)$\Rightarrow$(b). By the given equivalences, every M_i is an SI-module and $\sigma[M_i]$ contains an M_i-projective generator L_i with zero socle. Then also M_i has zero socle and is a progenerator in $\sigma[M_i]$ (see proof of (a)$\Leftrightarrow$(b)), and $\text{End}_R(M_i)$ is a left SI-ring. As a product of these rings, $\text{End}_R(M)$ is also a left SI-ring.

REMARK. For the proof of (b)$\Rightarrow$(c) we could have used part of Goodearl's structure theorem for left SI-rings in [**4**, 3.11]. For $M = R$ our proof provides an alternative to Goodearl's proof of the corresponding part.

Finally we are ready to prove the following extension of Goodearl's characterization of SI-rings in [**4**, 3.11].

2.3. STRUCTURE THEOREM. *For a finitely generated, self-projective R-module M and* $S = \text{End}_R(M)$, *the following are equivalent*:

(a) *M is an SI-module*;

(b) $Z_M(M) = 0$ *and M has a decomposition*

$$M = K \oplus V_1 \oplus \dots \oplus V_n$$

with fully invariant submodules K, V_i, *such that* $K/\text{Soc}\, K$ *is a semisimple R-module, and, for* $i = 1, \dots$, $\text{End}_R(V_i)$ *is a simple ring and the category* $\sigma[V_i]$ *is equivalent to* T_i-Mod, *for an SI-domain* T_i *which is not a division ring.*

Under the given conditions, S is a left SI-ring.

Proof. (a)$\Rightarrow$(b). Assume M is an SI-module. As already observed in Proposition 1.3, M is hereditary and $\bar{M} := M/\text{Soc}\, M$ is noetherian.

As noted in Proposition 1.2, the M-singular modules form a torsion class in $\sigma[M]$. Let K denote the R-submodule $\text{Soc}\, M \subset K \subset M$ such that $K/\text{Soc}\, M$ is the torsion submodule of $\bar{M}$ in this torsion theory. Since $\text{Soc}\, M$ is fully invariant in M and $K/\text{Soc}\, M$ is fully invariant in $M/\text{Soc}\, M$, K is fully invariant in M.

By construction, $\text{Soc}\, K = \text{Soc}\, M$. Also $K/\text{Soc}\, M$ is an SI-module and $\text{Soc}\, M \trianglelefteq K$ since K is projective in $\sigma[M]$ (M hereditary). Hence $K/\text{Soc}\, M$ is semisimple by 1.3 and M-injective by assumption. Therefore

$$\bar{M} = K/\text{Soc}\, M \oplus N/\text{Soc}\, M$$

for some R-submodule $N \subset M$ containing $\text{Soc}\, M$. Since $\text{Soc}\, M$ is a fully invariant submodule, $\bar{M}$ is self-projective. As M/L is semisimple for $L \trianglelefteq M$, and $\text{Soc}\, M$ is the intersection of all $L \trianglelefteq M$, we conclude $\text{Rad}\, \bar{M} = 0$.

Hence $M/K \simeq N/\text{Soc}\, M$ is a self-projective SI-module with zero radical. By definition of K, M/K contains no M-singular submodules. Therefore every simple submodule of M/K is M-projective by Proposition 1.2. Since $\text{Soc}\, M \subset K$, we conclude $\text{Soc}(M/K) = 0$.

Denote by $\{H_\lambda\}_\Lambda$ the family of all submodules of M with $\text{Soc}\, H_\lambda = 0$ and set $V = \sum_\Lambda H_\lambda$. Since all simple submodules of $V \subset M$ are M-projective (by Proposition 1.2) and $\bigoplus_\Lambda H_\lambda$ has zero socle, also $\text{Soc}\, V = 0$ and $K \cap V = 0$. The M-projectivity of simple submodules of M also implies that, for every $f \in S$, $(V)f$ has zero socle and hence $(V)f \subset V$, i.e. V is fully invariant. It is obvious from the definitions and the properties derived that $\text{Soc}\, M \oplus V \trianglelefteq K \oplus V \trianglelefteq_R M$ and that $K \oplus V$ is a fully invariant submodule of M.

Passing to the factor module, we have that $(K \oplus V)/K$ is a fully invariant submodule of M/K which is essential as an R-submodule. Recalling the properties of M/K shown above, by Lemma 1.4, this implies $K \oplus V = M$.

The composition of V is now obtained from Theorem 2.2.

(b)$\Rightarrow$(a). Obviously, for every essential submodule $U \subset M$, M/U is semisimple and hence M is an SI-module by Proposition 1.3.

It remains to show that S is a left SI-ring. Since M is hereditary in $\sigma[M]$, S is left semi-hereditary by [**10**, 39.14] and hence left non-singular.

By (b), $\mathrm{End}_R(M) = \mathrm{End}_R(K) \times \mathrm{End}_R(V_1) \times \cdots \times \mathrm{End}_R(V_n)$. In the proof of Theorem 2.2 we have shown that all $\mathrm{End}_R(V_i)$ are left SI-rings. Therefore it is enough to show that $S_1 = \mathrm{End}_R(K)$ is also a left SI-rng.

From the exact sequence $0 \to \mathrm{Soc}\, M \to M \to \bar{M} \to 0$, we derive the exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Soc}\, M) \to S \to \mathrm{Hom}(M, \bar{M}) \to 0.$$

Since $\bar{M} \simeq K/\mathrm{Soc}\, K$ is semisimple, $\mathrm{Hom}(M, \bar{M})$ is a semisimple left S-module. From $\mathrm{Hom}(M, \mathrm{Soc}\, M) \subset \mathrm{Soc}\, S$ we conclude that $S/\mathrm{Soc}\, S$ is left semisimple and S is a left Sl-ring by Proposition 1.3.

Remark. For $M = R$, our Structure Theorem yields Goodearl's Structure Theorem for SI-rings (see [**4**, 3.11]), which was also proved in Theorem 2.7 of Baccella [**1**] in a different way.

Obviously any SI-module is a GCO-module (compare 1.3). By our Structure Theorem we obtain that self-projective GCO-modules with descending chain condition on essential submodules are SI-modules. Referring to [**9**, 3.11] we have the following corollary.

2.4. Corollary. *For a finitely generated, self-projective R-module M, the following are equivalent*:

(a) *M is an SI-module with dcc on essential submodules*;

(b) *M is a GCO-module with dcc on essential submodules*;

(c) *$M/\mathrm{Soc}\, M$ is semisimple and $Z_M(M) = 0$.*

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