On the limits of oscillation of a function and its Cesàro means

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There is a group of Tauberian theorems of which the simplest is one due to K. Ananda Rau [Theorem 2 of the paper numbered **1** in the list of references at the end of the note]. More complicated theorems of the same group are discussed in a paper by S. Minakshisundaram and myself to be published by the London Mathematical Society [**4**].

There is another group of Tauberian theorems of which the most general is one of Karamata's [**2**, Satz 1], its special cases being the well-known oscillation theorems of Fekete and Winn [*Proc. London Math. Soc.* (2), 33 (1932), 488-513; *Journal London Math. Soc.*, 8 (1933), 27-32].

The precise nature of the relation between the two groups of Tauberian theorems does not seem to have been studied till now. This note shows that Karamata's results admit of generalizations whose basic idea is also that of the first group of Tauberian theorems. This idea has been recently embodied by L. S. Bosanquet [**1**] in certain difference formulae of which Lemmas 1, 2 are simplified versions sufficient for the purposes of the note.

1. In a well-known theorem [**2**, Satz 1], Karamata has refined and elaborated Fejér's relation between the intervals of oscillation of a sequence and its first order Cesàro mean. The theorems which appear below suggest a formal generalization of Karamata's results, different from the one due to S. Minakshisundaram [**3**, Theorem 2] mentioned in § 3.

Theorem A. *For any function* $s(t)$ *defined in* $(0, \infty)$ *and of bounded variation in every finite interval, let*

$$\sigma_p(x) = \frac{s_p(x)}{x^p} = \frac{p}{x^p}\int_0^x (x-t)^{p-1} s(t)\,dt \quad (p = 1, 2, 3, \ldots.), \qquad (1)$$

[1] Numerals in thick type appearing within square brackets indicate references given at the end of the note.

$$\sigma_0(x) = s_0(x) = s(x), \tag{1_0}$$

$$\bar{\underline{\sigma}}_p = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\ \sigma_p(x),\ l_p = \bar{\sigma}_p - \underline{\sigma}_p \quad (p = 0, 1, 2, \ldots).$$

If
$$\limsup_{t\to\infty} \operatorname*{Max}_{t\leq t'\leq\lambda t} \{s(t') - s(t)\} \leqq W^+(\lambda) > 0, \tag{2}$$

and $0 < \theta < 1 < \lambda$, *then*

$$-\left(\frac{\lambda-1}{p}\right)^p \underline{\sigma}_0 \leqq \frac{A_p(\lambda)\,\underline{\sigma}_p + B_p(\lambda)\,\bar{\sigma}_p}{\Gamma(p+1)} + \left(\frac{\lambda-1}{p}\right)^{p-1} \int_1^\lambda W^+(t)\,dt,\ p \geqq 1, \tag{3}$$

where $A_p(\lambda)$, $B_p(\lambda)$ *are polynomials of degree* p *in* λ *such that*

$$A_p(\lambda) < 0,\ B_p(\lambda) > 0,\quad \frac{A_p(\lambda) + B_p(\lambda)}{\Gamma(p+1)} = -\left(\frac{\lambda-1}{p}\right)^p;$$

$$\left(\frac{1-\theta}{p}\right)^p \bar{\sigma}_0 \leqq \frac{C_p(\theta)\,\underline{\sigma}_p + D_p(\theta)\,\sigma_p}{\Gamma(p+1)} + \left(\frac{1-\theta}{p}\right)^{p-1} \int_\theta^1 W^+\left(\frac{1}{t}\right) dt,\ p \geqq 1, \tag{4}$$

where $C_p(\theta)$, $D_p(\theta)$ *are polynomials of degree* p *in* θ *such that*

$$C_p(\theta) < 0,\ D_p(\theta) > 0,\quad \frac{C_p(\theta) + D_p(\theta)}{\Gamma(p+1)} = \left(\frac{1-\theta}{p}\right)^p;$$

further, when $\theta = 1/\lambda$,

$$\left(\frac{\lambda-1}{p}\right)^p l_0 \leqq \frac{B_p(\lambda) - A_p(\lambda)}{\Gamma(p+1)} l_p + \left(\frac{\lambda-1}{p}\right)^{p-1} \int_1^\lambda \left\{W^+(t) + W^+\left(\frac{\lambda}{t}\right)\right\} dt,\ p \geqq 1. \tag{5}$$

THEOREM B. *If in theorem A, we have, in addition to* (2),

$$\liminf_{t\to\infty} \operatorname*{Min}_{t\leq t'\leq\lambda t} \{s(t') - s(t)\} \geqq -W^-(\lambda) < 0. \tag{6}$$

then

$$\left\{\left(\frac{\lambda-1}{p}\right)^p + \left(\frac{1-\theta}{p}\right)^p\right\} \bar{\sigma}_0 \leqq \frac{E_p(\lambda,\theta)\,\underline{\sigma}_p + F_p(\lambda,\theta)\,\bar{\sigma}_p}{\Gamma(p+1)}$$
$$+ \left(\frac{\lambda-1}{p}\right)^{p-1} \int_1^\lambda W^-(t)\,dt + \left(\frac{1-\theta}{p}\right)^{p-1} \int_\theta^1 W^+\left(\frac{1}{t}\right) dt,\ p \geqq 1, \tag{7}$$

$$-\left\{\left(\frac{\lambda-1}{p}\right)^p + \left(\frac{1-\theta}{p}\right)^p\right\} \underline{\sigma}_0 \leqq -\frac{E_p(\lambda,\theta)\,\bar{\sigma}_p + F_p(\lambda,\theta)\,\underline{\sigma}_p}{\Gamma(p+1)}$$
$$+ \left(\frac{\lambda-1}{p}\right)^{p-1} \int_1^\lambda W^+(t)\,dt + \left(\frac{1-\theta}{p}\right)^{p-1} \int_\theta^1 W^-\left(\frac{1}{t}\right) dt,\ p \geqq 1, \tag{8}$$

where $E_p(\lambda, \theta)$, $F_p(\lambda, \theta)$ *are polynomials of degree* p *in* λ, θ *such that*

$$E_p(\lambda,\theta) = -B_p(\lambda) + C_p(\theta) + \tfrac{1}{2}\{1 + (-1)^{p+1}\},$$
$$F_p(\lambda,\theta) = -A_p(\lambda) + D_p(\theta) - \tfrac{1}{2}\{1 + (-1)^{p+1}\};$$

further

$$\left\{\left(\frac{\lambda-1}{p}\right)^p+\left(\frac{1-\theta}{p}\right)^p\right\} l_0 \leqq \frac{F_p(\lambda,\theta)-E_p(\lambda,\theta)}{\Gamma(p+1)} l_p$$

$$+\,2\left(\frac{\lambda-1}{p}\right)^{p-1}\int_1^\lambda W(t)\,dt+2\left(\frac{1-\theta}{p}\right)^{p-1}\int_\theta^1 W\left(\frac{1}{t}\right)dt,\ p\geqq 1, \qquad (9)$$

where $$2\,W(\lambda)=W^+(\lambda)+W^-(\lambda).$$

1. 1. These theorems reduce to Karamata's theorem when $p=1$. Though unwieldy they bring to light a natural connection between Karamata's theorem and a group of Tauberian theorems discussed in two papers recently presented to the London Mathematical Society [1, 4].[1] This connection turns on an idea expressed by L. S. Bosanquet [1] in certain difference formulae which (to suit our immediate needs) can be simplified as follows.

LEMMA 1. *For any function $F(x)$ of x and $\phi>0$, let us write*

$$\Delta_\phi^p F(x)=\sum_{\nu=0}^{p}(-1)^\nu\binom{p}{\nu}F(x+\overline{p-\nu}\,\phi),\quad p\geqq 1.$$

Then $$\Delta_\phi^p s_p(t_0)=\Gamma(p+1)\int_{t_0}^{t_0+\phi}dt_1\int_{t_1}^{t_1+\phi}dt_2\int_{t_2}^{t_2+\phi}\cdots\int_{t_{p-1}}^{t_{p-1}+\phi}s(t)\,dt.$$

LEMMA 2. *For any function $F(x)$ of x and $\omega>0$, let*

$$\Delta_{-\omega}^p F(x)=\sum_{\nu=0}^{p}(-1)^\nu\binom{p}{\nu}F(x-\nu\omega),\quad p\geqq 1.$$

Then $$\Delta_{-\omega}^p s_p(t_0)=\Gamma(p+1)\int_{t_0-\omega}^{t_0}dt_1\int_{t_1-\omega}^{t_1}dt_2\int_{t_2-\omega}^{t_2}\cdots\int_{t_{p-1}-\omega}^{t_{p-1}}s(t)\,dt.$$

2. PROOF OF THEOREM A. To prove (3) we put $\phi=(\lambda-1)\,x/p$ in Lemma 1 and write it in the form

$$-\,\phi^p s(x)=-\frac{\Delta_\phi^p s_p(x)}{\Gamma(p+1)}+\int_x^{x+\phi}dt_1\int_{t_1}^{t_1+\phi}\cdots\int_{t_{p-1}}^{t_{p-1}+\phi}\{s(t)-s(x)\}\,dt$$

$$=\quad I+J\quad(\text{say}). \qquad (10)$$

[1] One of the Tauberian theorems in question [4, Theorem 2] is a direct generalization of the result: *given* (2), $\sigma_p(x)=O(1)$ *as* $x\to\infty$ *implies* $\sigma_0(x)=O(1)$. Theorem A, on the other hand, is an attempt to develop this conclusion without going beyond the hypothesis (2).

Now, in J, $x \leqq t_1 \leqq t \leqq t_1 + (p-1)\phi$ and hence

$$J \leqq \int_x^{x+\phi} \operatorname*{Max}_{x \leqq t \leqq t_1 + (p-1)\phi} \{s(t) - s(x)\}\, \phi^{p-1}\, dt,$$

$$= \int_{1+(p-1)\frac{\phi}{x}}^{\lambda} \operatorname*{Max}_{x \leqq t \leqq t' x} \{s(t) - s(x)\}\, \phi^{p-1}\, x\, dt'$$

$$\leqq \phi^{p-1}\, x \int_{1+(p-1)\frac{\phi}{x}}^{\lambda} W^{+}(t')\, dt' + o(\phi^{p-1} x) \quad (x \to \infty). \quad (11)$$

Next

$$I = -\frac{x_p}{\Gamma(p+1)} \sum_{\nu=0}^{p} (-1)^{\nu} \binom{p}{\nu} \frac{s_p(x + \overline{p-\nu}\,\phi)}{(x+\overline{p-\nu}\,\phi)^p} \left(1 + \overline{p-\nu}\,\frac{\phi}{x}\right)^p.$$

Writing $c_\nu = \binom{p}{\nu}\left(1 + \overline{p-\nu}\,\frac{\phi}{x}\right)^p$ = a positive quantity independent of x, we get

$$I \leqq -\frac{x^p}{\Gamma(p+1)} \{c_0 + c_2 + \ldots . (\quad)\}\, \underline{\sigma}_p + \frac{x^p}{\Gamma(p+1)} \{c_1 + c_3 + \ldots . (\quad)\}\, \bar{\sigma}_p$$
$$+ o(x^p) \quad (x \to \infty),$$

or

$$I \leqq x^p \frac{A_p(\lambda)\, \underline{\sigma}_p + B_p(\lambda)\, \bar{\sigma}_p}{\Gamma(p+1)} + o(x^p) \quad (x \to \infty) \quad (12)$$

where

$$A_p(\lambda) = -\{c_0 + c_2 + \ldots . (\quad)\},\ B_p(\lambda) = \{c_1 + c_3 + \ldots . (\quad)\}$$

so that

$$A_p(\lambda) + B_p(\lambda) = -\sum_{\nu=0}^{p} (-1)^{\nu} \binom{p}{\nu} \left(\frac{x + \overline{p-\nu}\,\phi}{x}\right)^p = -\frac{\Delta_\phi^p x^p}{x^p}$$
$$= -\Gamma(p+1)\left(\frac{\phi}{x}\right)^p.$$

Using (11) and (12) in (10) we obtain

$$-\left(\frac{\phi}{x}\right)^p s(x) < \frac{A_p(\lambda)\, \underline{\sigma}_p + B_p(\lambda)\, \bar{\sigma}_p}{\Gamma(p+1)} + \left(\frac{\phi}{x}\right)^{p-1} \int_1^{\lambda} W^{+}(t)\, dt$$
$$+ o(1) \quad (x \to \infty),$$

whence (3) results immediately.

To prove (4), choose $\omega = (1-\theta)\, x/p$ in Lemma 2 and write the lemma

$$\omega^p s(x) = \frac{\Delta_{-\omega}^p s_p(x)}{\Gamma(p+1)} + \int_{x-\omega}^{x} dt_1 \int_{t_1-\omega}^{t_1} \ldots \int_{t_{p-1}-\omega}^{t_{p-1}} \{s(x) - s(t)\}\, dt$$
$$= I' + J' \text{ (say).} \quad (13)$$

In J', $t_1 - (p-1)\,\omega \leqq t \leqq t_1 \leqq x$ and so

$$J' \leqq \int_{x-\omega}^{x} \max_{t_1-(p-1)\omega \leqq t \leqq x} \{s(x) - s(t)\}\, \omega^{p-1}\, dt$$

$$= \int_{\theta}^{1-(p-1)\frac{\omega}{x}} \max_{t'x \leqq t \leqq x} \{s(x) - s(t)\}\, \omega^{p-1}\, x\, dt'$$

$$\leqq \omega^{p-1}\, x \int_{\theta}^{1-(p-1)\frac{\omega}{x}} W^{+}\left(\frac{1}{t'}\right) dt' + o\,(\omega^{p-1}\, x) \quad (x \to \infty). \tag{14}$$

Next

$$I' \leqq \frac{x^p}{\Gamma(p+1)} \sum_{\nu=0}^{p} (-1)^{\nu} \binom{p}{\nu} \frac{s_p(x-\nu\omega)}{(x-\nu\omega)^p} \left(1 - \nu\frac{\omega}{x}\right)^p$$

whence setting $d_\nu = \binom{p}{\nu}\left(1 - \nu\dfrac{\omega}{x}\right)^p$ = a positive quantity independent of x, we find

$$I' \leqq \frac{x^p}{\Gamma(p+1)}\{d_0 + d_2 + \ldots.(\;)\}\,\bar{\sigma}_p - \frac{x^p}{\Gamma(p+1)}\{d_1 + d_3 + \ldots.(\;)\}\,\underline{\sigma}_p$$
$$+\, o\,(x^p) \quad (x \to \infty). \tag{15}$$

Using (14) and (15) in (13) we can get (4) exactly as we got (3) formerly. (5) follows by addition from (3) and (4) since

$$\lambda^p\, C_p\,(1/\lambda) = -\,B_p\,(\lambda),\; \lambda^p\, D_p\,(1/\lambda) = -\,A_p\,(\lambda).$$

PROOF OF THEOREM B. (7) is obtained by a repetition of the preceding arguments, after we have put $\phi = (\lambda - 1)\, x/p$, $\omega = (1-\theta)\, x/p$ in the following combination of Lemmas 1, 2.

$$(\phi^p + \omega^p)\, s(x) = \frac{\Delta_\phi^p\, s_p(x) + \Delta_{-\omega}^p\, s_p(x)}{\Gamma(p+1)} - \int_x^{x+\phi} dt_1 \int_{t_1}^{t_1+\phi} \ldots \int_{t_{p-1}}^{t_{p-1}+\phi} \{s(t) - s(x)\}\, dt$$
$$+ \int_{x-\omega}^{x} dt_1 \int_{t_1-\omega}^{t_1} \ldots \int_{t_{p-1}-\omega}^{t_{p-1}} \{s(x) - s(t)\}\, dt\;[1] \tag{16}$$

(8) is deduced from (7) by considering $-\,s(x)$ instead of $s(x)$. Finally (9) is obtained by adding (7) and (8).

3. For the sake of completeness I conclude with the statement

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[1] When p is odd, the term $-\,s_p(x)$ in $\Delta_\phi^p s_p(x)$ cancels out $s_p(x)$ in $\Delta_{-\omega}^p s_p(x)$. Hence, in the passage from (15) to its limiting inequality form, we have to add $(\underline{\sigma}_p - \bar{\sigma}_p)/\Gamma(p+1)$ to the right-hand member when p is odd. This accounts for the expressions for $E_p(\lambda, \theta)$, $F_p(\lambda, \theta)$ in terms of $A_p(\lambda)$, $B_p(\lambda)$, $C_p(\theta)$, $D_p(\theta)$.

of two theorems which are easy deductions from Karamata's special case $p = 1$ of Theorems A, B.

THEOREM C. *If $\sigma_k(x)$ is defined as in* (1) *for any positive k (not necessarily integral) and if* (2) *is postulated, then, for $0 < \theta < 1 < \lambda$,*

$$-(\lambda - 1)\underline{\sigma}_k \leqq A_1(\lambda)\,\underline{\sigma}_{k+1} + B_1(\lambda)\,\overline{\sigma}_{k+1} + \int_1^\lambda W^+(t)\,dt. \tag{17}$$

$$(1 - \theta)\,\overline{\sigma}_k \leqq C_1(\theta)\,\underline{\sigma}_{k+1} + D_1(\theta)\,\overline{\sigma}_{k+1} + \int_\theta^1 W^+(1/t)\,dt. \tag{18}$$

THEOREM D. *Postulating* (2), (6) *and $0 < \theta < 1 < \lambda$, we have*

$$(\lambda - \theta)\,\overline{\sigma}_k \leqq E_1(\lambda, \theta)\,\underline{\sigma}_{k+1} + F_1(\lambda, \theta)\,\overline{\sigma}_{k+1} + \int_1^\lambda W^-(t)\,dt + \int_\theta^1 W^+\left(\frac{1}{t}\right)dt, \tag{19}$$

$$-(\lambda - \theta)\,\underline{\sigma}_k \leqq -E_1(\lambda, \theta)\,\overline{\sigma}_{k+1} - F_1(\lambda, \theta)\,\underline{\sigma}_{k+1} + \int_1^\lambda W^+(t)\,dt + \int_\theta^1 W^-\left(\frac{1}{t}\right)dt. \tag{20}$$

The first of these theorems appears elsewhere [3, Theorem 2] with some misprints. Either theorem leads to a relation between l_k and $l_{(k)}$ where $k = k -$ the greatest integer not exceeding k.

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