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$\mathcal{C}^p$-parametrization in O-minimal Structures

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Abstract. We give a geometric and elementary proof of the uniform $\mathcal{C}^p$-parametrization theorem of Yomdin and Gromov in arbitrary o-minimal structures.

1 Introduction

Fix any o-minimal expansion of a real closed field R (see [14] or [4] for fundamental definitions and results concerning o-minimal structures). Let p be any positive integer. We will be discussing definable subsets and mappings referring to this o-minimal structure. The aim of this note is to give a geometric and elementary proof of the following uniform $\mathcal{C}^p$-parametrization theorem.

Uniform $\mathcal{C}^p$-Parametrization Theorem *Let X be a definable subset of $R^m \times R^n$. Let $X_t := \{x \in R^n : (t,x) \in X\}$ for any $t \in R^m$ and put $T := \{t \in R^m : X_t \neq \varnothing\}$. Let k and p be positive integers. Assume that all X_t $(t \in T)$ are closed, of pure dimension k, and commonly bounded;* i.e., *there exists $r > 0$ such that $|x| \leqslant r$, for each $t \in T$ and $x \in X_t$.*

Then there exists a finite decomposition $T = T_1 \cup \cdots \cup T_s$ of T into definable $\mathcal{C}^p$-cells in R^m and for each $i \in \{1,\dots,s\}$ a finite family of definable $\mathcal{C}^p$-mappings

$$\varphi_{i\varkappa}\colon T_i \times [0,1]^k \ni (t,\xi) \longmapsto \varphi_{i\varkappa}(t,\xi) \in X (\varkappa \in K_i)$$

such that

(i) *$(\pi \circ \varphi_{i\varkappa})(t,\xi) = t$, where $(t,\xi) \in T_i \times [0,1]^k$ and $\pi\colon R^m \times R^n \to R^m$ is the natural projection;*
(ii) *$X_t = \bigcup_{\varkappa \in K_i} \varphi_{i\varkappa}(\{t\} \times [0,1]^k)$ for each $t \in T_i$;*
(iii) *$\varphi_{i\varkappa}|T_i \times (0,1)^k$ is a $\mathcal{C}^p$-diffeomorphism onto a definable $\mathcal{C}^p$-submanifold of $R^m \times R^n$ open in $X \cap (T_i \times R^n)$;*
(iv) *$\varphi_{i\varkappa}(T_i \times (0,1)^k) \cap \varphi_{i\lambda}(T_i \times (0,1)^k) = \varnothing$, whenever $\varkappa, \lambda \in K_i$, $\varkappa \neq \lambda$;*
(v) *all the partial derivatives $\partial^{|\alpha|}\varphi_{i\varkappa}/\partial\xi^\alpha(t,\xi)$, where $t \in T_i$, $\xi \in [0,1]^k$, $\alpha \in \mathbb{N}^k$, $0 < |\alpha| \leqslant p$, are bounded by a constant independent of t.*

The above theorem in the semialgebraic case originated in the papers of Yomdin [15,16] and Gromov [5] (with some estimates on the number of mappings $\varphi_{i\varkappa}$, which are important from the point of view of applications). Now there are quite a lot of papers connected with it (see [1–3,12,17,18]), where applications in dynamics, analysis, diophantine, and computational geometry are given. Of course this theorem brings to mind (and perhaps can be even considered as a generalization of) the classical

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Hironaka rectilinearization theorem [6, Theorem 7.1] and, from the point of view of the proof (see below), the Puiseux desingularization (*cf.* [10]). A proof of the Uniform $\mathcal{C}^p$-Parametrization Theorem for arbitrary o-minimal structures was given by Pila and Wilkie [12, Corollary 5.2]. Nevertheless, in view of differences between our approach and that of [12], we think that our paper may still be of interest.

Remark 1.1 If the o-minimal structure is that of semialgebraic sets, the number of needed mappings $\varphi_{i\varkappa}$ can be estimated from above by an integer that depends on p, on the degrees of polynomials describing X, the radius r, the dimension n, and the number of parameters m (*cf.* [5, Section 4.5] and Remark 2.6).

For fundamental definitions and results concerning o-minimal geometry, we refer the reader to [14] or [4]. We limit ourselves here to reviewing the notions of a cell and that of a cell decomposition, because they will play particularly important roles in our approach.

A subset C of R^n is called a *cell* (a $\mathcal{C}^p$*-cell*) in R^n if $C = \{a\}$, where $a \in R$ or $C = (a, b)$, where $a, b \in \overline{R}, a < b$, in the case $n = 1$, and, in the case $n > 1$, if either $C = \{(x', f(x')) : x' \in C'\}$, where $x' = (x_1, \dots, x_{n-1})$, C' is a cell (a $\mathcal{C}^p$-cell) in R^{n-1} and $f: C' \to R$ is a definable continuous (a definable $\mathcal{C}^p$-)function or $C = \{(x', x_n) : x' \in C', f_1(x') < x_n < f_2(x')\}$, where C' is a cell (a $\mathcal{C}^p$-cell) in R^{n-1} and each of the functions $f_i: C' \to \overline{R}$ $(i \in \{1, 2\})$ is either a definable continuous (a definable $\mathcal{C}^p$-) function $f_i: C' \to R$ or $f_i \equiv -\infty$, or $f_i \equiv +\infty$ and $f_1(x') < f_2(x')$ for each $x' \in C'$. It is clear that any $\mathcal{C}^p$-cell in R^n is a $\mathcal{C}^p$-submanifold of R^n.

Let X be any definable subset of R^n. By a *cell decomposition* (a $\mathcal{C}^p$*-cell decomposition*) of X, we mean any finite decomposition $\mathcal{C}$ of X into cells in the case $n = 1$ and, in the case $n > 1$, any finite decomposition $\mathcal{C}$ of X into cells ($\mathcal{C}^p$-cells) such that $\{\pi(C) : C \in \mathcal{C}\}$ is a cell decomposition (a $\mathcal{C}^p$-cell decomposition) of $\pi(X)$, where $\pi: R^n = R^{n-1} \times R \ni (x', x_n) \mapsto x' \in R^{n-1}$ is the natural projection.

2 Preparatory Assertions

A key role is played by the following lemma (*cf.* [8, Lemmata 1 and 2]) mimicking an idea of Yomdin and Gromov (*cf.* [5, Section 4.1]).

Lemma 2.1 *Let $\lambda: (a, b) \to R$ be a definable $\mathcal{C}^{p+1}$-function, where $p \in \mathbb{N}$, $p \geqslant 1$, defined on an open interval $(a, b) \subset R$ such that, for each $\nu \in \{2, \dots, p+1\}$, $\lambda^{(\nu)} \geqslant 0$ on (a, b) or $\lambda^{(\nu)} \leqslant 0$ on (a, b). Then for any closed interval $[t-r, t+r] \subset (a, b)$, where $r \in R$ and $r > 0$,*

$$|\lambda^{(p)}(t)| \leqslant 2^{\binom{p+2}{2}-2} \sup_{[t-r,t+r]} |\lambda| \frac{1}{r^p}.$$

Proof First consider the case $p = 1$. Without any loss of generality, we can assume that $\lambda'' \leqslant 0$; *i.e.,* λ is concave. Hence,

$$\frac{\lambda(t) - \lambda(s)}{t - s} \leqslant \frac{\lambda(t) - \lambda(t-r)}{r} \leqslant 2 \sup_{[t-r,t+r]} \frac{|\lambda|}{r},$$

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when $t - r < s < t$. It follows that

$$\lambda'(t) \leqslant 2 \sup_{[t-r,t+r]} \frac{|\lambda|}{r}.$$

Applying this to $\lambda(-t)$, we obtain

$$-\lambda'(t) \leqslant 2 \sup_{[t-r,t+r]} \frac{|\lambda|}{r}; \quad \text{consequently,} \quad |\lambda'(t)| \leqslant 2 \sup_{[t-r,t+r]} \frac{|\lambda|}{r}.$$

Now the lemma follows by induction on p. ∎

Applying Lemma 2.1 to λ' in the place of λ and $\mu - 1$ in the place of p, we have the following corollary.

Corollary 2.2 *Under the assumptions of Lemma 2.1,*

$$|\lambda^{(\mu)}(t)| \leqslant C_p \sup_{(a,b)} |\lambda'| \frac{1}{|t-a|^{\mu-1}},$$

for each $t \in \left(a, \frac{a+b}{2}\right)$ *and* $\mu \in \{2,\dots,p\}$, *where* $C_p := 2^{\binom{p+1}{2}-2}$. *In particular, if* λ' *is bounded;* i.e., $|\lambda'| \leqslant M$, *where* $M \in R$ *and* $M > 0$, *then*

$$|\lambda^{(\mu)}(t)| \leqslant C_p M \frac{1}{|t-a|^{\mu-1}} \quad \textit{for each } t \in \left(a, \frac{a+b}{2}\right), \mu \in \{2,\dots,p\}.$$

Lemma 2.3 *Let* $\lambda\colon (0,1] \to R$ *be a definable* $\mathcal{C}^p$*-function such that*

$$|\lambda^{(\mu)}(t)| \leqslant C \frac{1}{t^{\mu-1}} \quad \textit{for each } t \in (0,1],\ \mu \in \{1,\dots,p\} \tag{2.1}$$

where $C \in R$ *is a positive constant. Fix* $m \in \mathbb{N}$, $m \geqslant p+1$. *Put* $\varphi(\tau) := \lambda(\tau^m)$ *for each* $\tau \in (0,1]$.

Then there exists a positive constant L *depending only on* C *and* m *such that* $|\varphi^{(\mu)}(\tau)| \leqslant L$, *for each* $\tau \in (0,1]$ *and* $\mu \in \{1,\dots,p\}$.

Proof For each $\mu \in \{1,\dots,p\}$,

$$\begin{aligned}\varphi^{(\mu)}(\tau) = a_{1\mu}\tau^{m-\mu}\lambda'(\tau^m) + a_{2\mu}\tau^{2m-\mu}\lambda''(\tau^m)\\ + a_{3\mu}\tau^{3m-\mu}\lambda^{(3)}(\tau^m) + \dots + a_{\mu\mu}\tau^{\mu m-\mu}\lambda^{(\mu)}(\tau^m),\end{aligned}$$

where $a_{i\mu}$ are positive integers defined inductively by the following formulae:

$$a_{1\mu} = \frac{m!}{(m-\mu)!}, \quad a_{i\mu} = m a_{(i-1)(\mu-1)} + (im - \mu + 1) a_{i(\mu-1)}, \quad a_{\mu\mu} = m^\mu.$$

By (2.1), it follows that

$$\begin{aligned}|\varphi^{(\mu)}(\tau)| &\leqslant a_{1\mu}\tau^{m-\mu}C + a_{2\mu}\tau^{2m-\mu}\frac{C}{\tau^m}\\ &\quad + a_{3\mu}\tau^{3m-\mu}\frac{C}{\tau^{2m}} + \dots + a_{\mu\mu}\tau^{\mu m-\mu}\frac{C}{\tau^{(\mu-1)m}}\\ &= C(a_{1\mu} + \dots + a_{\mu\mu})\tau^{m-\mu} \leqslant C(a_{1\mu} + \dots + a_{\mu\mu}).\end{aligned}$$ ∎

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Lemma 2.4 (*cf.* [7, Lemma 1] or [13, Proposition 5.5]) *Let Ω be an open definable subset of R^n and let*

$$f\colon \Omega \times (0,1)^m \ni (x,y) \longmapsto f(x,y) \in R$$

be a definable $\mathcal{C}^1$- function, where $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_m)$. Assume that all the partial derivatives

$$\frac{\partial f}{\partial y_i} \qquad (i = 1,\dots,m)$$

are bounded on $\Omega \times (0,1)^m$.

Then there exists a closed nowhere dense definable subset Σ of Ω such that, for each $x \in \Omega \setminus \Sigma$, the function

$$(0,1)^m \ni y \longmapsto \frac{\partial f}{\partial x_n}(x,y) \in \mathbb{R}$$

is bounded.

Proof First consider the case $m = 1$. In this special case we have the following claim.

Claim ($\mathcal{C}^1$-Extension Theorem, *cf.* [11, Proposition 10]) There exists a closed nowhere dense definable subset Σ of Ω such that f extends to a $\mathcal{C}^1$-function

$$f\colon \Omega \times [0,1) \setminus \Sigma \times \{0\} \ni (x,y) \longmapsto f(x,y) \in R.$$

Indeed, by a dimension argument $\frac{\partial f}{\partial y}$ extends to a continuous function defined on $\Omega \times [0,1) \setminus \Sigma \times \{0\}$ with Σ as above. By the Mean Value Theorem there exists a finite limit $\lim_{y\to 0} f(x,y) \in R$, for each $x \in \Omega$; hence, again by a dimension argument, one can assume that f extends to a continuous function defined on $\Omega \times [0,1) \setminus \Sigma \times \{0\}$. Of course, one can assume that $\Sigma = \varnothing$. Again by removing a small subset of Ω, one can assume that the function $g\colon \Omega \ni x \mapsto f(x,0) \in R$ is of class $\mathcal{C}^1$. Now we check that

$$\lim_{\substack{x\to a\\ y\to 0}} \frac{\partial f}{\partial x_n}(x,y) = \frac{\partial g}{\partial x_n}(a,0)$$

for all $a \in \Omega$, except perhaps for a from a small subset. Of course, one can assume that $g \equiv 0$. Now it suffices to show that for each $a = (a_1,\dots,a_n) \in \Omega$, there exists a definable curve $\lambda\colon (0,1) \to \Omega \times (0,1)$ such that

$$\lim_{t\to 0} \lambda(t) = (a,0) \qquad \text{and} \qquad \lim_{t\to 0} \frac{\partial f}{\partial x_n}(\lambda(t)) = 0.$$

Choose any $\varepsilon, \delta > 0$. There exists $x \in \Omega$ such that $x = (a_1,\dots,a_{n-1},x_n)$, $0 < |x_n - a_n| < \delta$, and $y \in (0,\delta)$ such that $|f(x,y)| < \varepsilon|x-a|$ and $|f(a,y)| < \varepsilon|x-a|$. Then by the Mean Value Theorem, there exists $\theta \in (0,1)$ such that

$$\Big|\frac{\partial f}{\partial x_n}\big(a_1,\dots,a_{n-1},\theta a_n + (1-\theta)x_n, y\big)\Big| = \frac{|f(x,y) - f(a,y)|}{|x-a|} < 2\varepsilon,$$

and by the Curve Selection Lemma, the proof of the claim is complete.

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Now consider the case $m > 1$. Suppose that Lemma 2.4 is not true; *i.e.*, there is an open nonempty subset W of Ω such that for each $x \in W$, there exists $h(x) \in [0,1]^m$ such that

$$\limsup_{y \to h(x)} \Big| \frac{\partial f}{\partial x_n}(x, y) \Big| = \infty.$$

By definable choice and shrinking perhaps W, we can make h definable of class $\mathcal{C}^1$. By a version with a parameter of the Curve Selection Lemma (or the Whitney Wing Lemma), there exists a definable mapping $\alpha\colon (0,1) \times W \to (0,1)^m$ such that for each $x \in W$, $\lim_{t\to 0} \alpha(x,t) = h(x)$ and

$$\lim_{t \to 0} \frac{\partial f}{\partial x_n}(x, \alpha(x,t)) = \pm\infty. \tag{2.2}$$

Perhaps shrinking W and replacing the parameter t by $t' = \rho t$, with ρ small positive, we can assume that $\alpha = (\alpha_1, \dots, \alpha_m)$ is of class $\mathcal{C}^1$ on $W \times (0,1)$ and there is $j \in \{1, \dots, m\}$ such that

$$\Big| \frac{\partial \alpha_j}{\partial t}(x,t) \Big| \geqslant \Big| \frac{\partial \alpha_i}{\partial t}(x,t) \Big| \quad \text{for each} \quad (x,t) \in W \times (0,1), i \in \{1, \dots, m\}.$$

Introducing a new variable $\tau := \alpha_j(x,t)$ in the place of t, we can assume that $\big| \frac{\partial \alpha_i}{\partial t}(x,t) \big| \leqslant 1$, for $i \in \{1, \dots, m\}$. By the $\mathcal{C}^1$-Extension Theorem, shrinking perhaps W, we can assume that α is $\mathcal{C}^1$ on $W \times [0,1)$. The same is true for the function $g(x,t) := f(x, \alpha(x,t))$ and in view of $\mathcal{C}^1$-Extension Theorem, we get a contradiction with (2.2). ■

Proposition 2.5 *Let $f_1, \dots, f_k\colon \Omega \to R$ be any definable bounded functions defined on a definable open bounded subset Ω of R^n. Let $\pi\colon R^{n-1} \times R \ni (x', x_n) \mapsto x' \in R^{n-1}$ be the natural projection. Let p be a fixed positive integer.*

Then there exists a cell decomposition $\{C_\varkappa\}$ of Ω such that for each open cell $C_\varkappa$, there exists a definable $\mathcal{C}^p$-diffeomorphism $\varphi_\varkappa\colon \pi(C_\varkappa) \times (0,1) \to C_\varkappa$ of the form

$$\varphi_\varkappa(x', \xi_n) = (x', \varphi_{\varkappa n}(x', \xi_n)),$$

where $x' \in \pi(C_\varkappa)$, $\xi_n \in (0,1)$ and

(i) *$\big|\frac{\partial^\mu \varphi_{\varkappa n}}{\partial \xi_n^\mu}\big| \leqslant L_p$ for each $\mu \in \{1, \dots, p\}$, with a positive constant $L_p \in \mathbb{N}$ depending only on p;*

(ii) *each of the functions $f_i \circ \varphi_\varkappa$ $(i = 1, \dots, k)$ is of class $\mathcal{C}^p$ on $\pi(C_\varkappa) \times (0,1)$ and*

$$\Big| \frac{\partial^\mu (f_i \circ \varphi_\varkappa)}{\partial \xi_n^\mu} \Big| \leqslant L_p \qquad \textit{for each} \qquad \mu \in \{1, \dots, p\}.$$

Proof By the Cell Decomposition Theorem (see [14, Chapter 3 and Chapter 7, §3]), we reduce the general case to the one where

$$\Omega = \big\{ (x', x_n) : x' \in D, a(x') < x_n < b(x') \big\}$$

is an open bounded $\mathcal{C}^p$-cell in R^n, D is an open bounded cell in R^{n-1}, $a, b\colon D \to R$ are definable $\mathcal{C}^p$-functions, $a < b$ on D, each of the functions f_i is of class $\mathcal{C}^{p+1}$ on Ω,

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and, for each $i \in \{1,\dots,k\}$

$$\text{either} \qquad \Big|\frac{\partial f_i}{\partial x_n}\Big| \leqslant 1 \text{ on } \Omega \qquad \text{or} \qquad \Big|\frac{\partial f_i}{\partial x_n}\Big| \geqslant 1 \text{ on } \Omega.$$

Now the proof splits into two cases.

Case I: $\big|\frac{\partial f_i}{\partial x_n}\big| \leqslant 1$ on Ω, for each $i \in \{1,\dots,k\}$.

Passing perhaps to a finer cell decomposition of Ω, one can assume that

$$\text{(2.3)} \qquad \operatorname{sgn}\Big(\frac{\partial^\nu f_i}{\partial x_n^\nu}\Big) = \text{const on } \Omega, \text{for each } i \in \{1,\dots,k\} \text{ and } \nu \in \{2,\dots,p+1\}.$$

Moreover, one can assume that $b(x') - a(x') \leqslant 2$, for $x' \in D$. Put $c(x') := \frac{1}{2}(a(x') + b(x'))$, for $x' \in D$. Fix an integer $m \geqslant p+1$. Define

$$\varphi_1(x',\xi_n) := \big(x', a(x') + \xi_n^m(c(x') - a(x'))\big),$$
$$\varphi_2(x',\xi_n) := \big(x', b(x') + \xi_n^m(c(x') - b(x'))\big),$$

for each $x' \in D$ and $\xi_n \in (0,1)$. It follows immediately from the assumption of Case I, from (2.3), and from Lemma 2.3 that conditions (i) and (ii) are satisfied in this case.

Case II: there exists $j \in \{1,\dots,k\}$ such that $\big|\frac{\partial f_j}{\partial x_n}\big| \geqslant 1$ on Ω.

Passing perhaps to a finer cell decomposition of Ω, one can assume that

$$\text{(2.4)} \qquad \Big|\frac{\partial f_i}{\partial x_n}\Big| \leqslant \Big|\frac{\partial f_j}{\partial x_n}\Big| \quad \text{for each } i \in \{1,\dots,k\},$$

and $\operatorname{sgn}\big(\frac{\partial f_j}{\partial x_n}\big) = \text{const}$; one can assume without loss of generality that $\frac{\partial f_j}{\partial x_n} \geqslant 1$.

Removing perhaps from D a definable closed nowhere dense subset, one can assume that f_j has a continuous extension defined on

$$\big\{(x',x_n) : x' \in D, a(x') \leqslant x_n \leqslant b(x')\big\}.$$

Now, the main idea is to introduce the following new variable $z_n := f_j(x',y_n)$ in the place of y_n. Then $y_n = \psi(x',z_n)$, for $(x',z_n) \in \widetilde{\Omega}$, where

$$\widetilde{\Omega} := \{(x',z_n) : x' \in D, \widetilde{a}(x') < z_n < \widetilde{b}(x')\},$$

$\widetilde{a}(x') := f_j(x',a(x'))$, and $\widetilde{b}(x') := f_j(x',b(x'))$. Put

$$\widetilde{f}_i(x',z_n) := f_i(x',y_n) = f_i(x',\psi(x',z_n)) \quad \text{for each} \quad (x',z_n) \in \widetilde{\Omega}, i \in \{1,\dots,k\}.$$

Then by the assumption of Case II and by (2.4),

$$\Big|\frac{\partial \psi}{\partial z_n}\Big| = \frac{1}{\big|\frac{\partial f_j}{\partial y_n}\big|} \leqslant 1 \quad \text{and} \quad \Big|\frac{\partial \widetilde{f}_i}{\partial z_n}\Big| = \frac{\big|\frac{\partial f_i}{\partial y_n}\big|}{\big|\frac{\partial f_j}{\partial y_n}\big|} \leqslant 1 \quad \text{for each } i \in \{1,\dots,k\}.$$

Now it suffices to apply Case I to the functions $\widetilde{f}_i$ $(i = 1,\dots,k)$ and ψ to complete the proof. ∎

Remark 2.6 In the semialgebraic case, the number of cells in a cell decomposition in the proof of Proposition 2.5 can be estimated from above by degrees of initial polynomials defining Ω, $f_1,\dots,f_k$, by p and by n (*cf.* [9, Section 20]).

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Proposition 2.7 *Let $F_i\colon \Omega\times(0,1)^m \ni (x,y)\mapsto F_i(x,y)\in R$ $(i=1,\dots,k)$ be a finite number of definable bounded $\mathcal{C}^p$-functions, where Ω is an open definable bounded subset of R^n, $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_m)$, $p\in\mathbb{N}$, $p>0$. Let $q\in\{0,\dots,p-1\}$. Let $\pi\colon R^{n-1}\times R\ni(x',x_n)\mapsto x'\in R^{n-1}$ be the natural projection. Assume that all the partial derivatives*

$$\frac{\partial^{\mu+|\alpha|}F_i}{\partial x_n^{\mu}\partial y^{\alpha}}\quad \textit{with}\quad \mu\in\{0,\dots,q\},\quad 0<\mu+|\alpha|\leqslant p$$

are bounded.

Then there exists a cell decomposition $\{C_\varkappa\}$ of Ω such that for each open cell $C_\varkappa$, there exists a definable $\mathcal{C}^p$-diffeomorphism $\varphi_\varkappa\colon \pi(C_\varkappa)\times(0,1)\to C_\varkappa$ of the form

$$\varphi_\varkappa(x',\xi_n)=(x',\varphi_{\varkappa n}(x',\xi_n)),\qquad \textit{where}\quad x'\in\pi(C_\varkappa),\xi_n\in(0,1)$$

and

(i) $\left|\frac{\partial^{\mu}\varphi_{\varkappa n}}{\partial\xi_n^{\mu}}\right|\leqslant L_p$ *for each $\mu\in\{1,\dots,p\}$, with a positive constant $L_p\in\mathbb{N}$ depending only on p;*

(ii) *for each $i\in\{1,\dots,k\}$, all the partial derivatives*

$$\frac{\partial^{\mu+|\alpha|}}{\partial\xi_n^{\mu}\partial y^{\alpha}}F_i(\varphi_\varkappa(x',\xi_n),y)\quad \textit{with}\quad \mu\in\{0,\dots,q+1\},\quad \mu+|\alpha|\leqslant p,$$

are bounded.

Proof Take any $\alpha\in\mathbb{N}^m$ such that $q+1+|\alpha|\leqslant p$. Then, for each $r\in\{1,\dots,m\}$,

$$\frac{\partial}{\partial y_r}\Big(\frac{\partial^{q+|\alpha|}F_i}{\partial x_n^{q}\partial y^{\alpha}}\Big)=\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q}\partial y^{\alpha+(r)}}$$

is bounded; hence, in view of Lemma 2.4, there exists a closed definable nowhere dense subset Σ of Ω such that for each $x\in\Omega\setminus\Sigma$, the function

$$(0,1)^m\ni y\longmapsto\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^{\alpha}}(x,y)\in R$$

is bounded. By the Definable Choice Theorem (*cf.* [14, Chapter 6, (1.2)]), there exist definable mappings $\delta_{i\alpha}\colon\Omega\setminus\Sigma\to(0,1)^m$ such that

$$\left|\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^{\alpha}}\big(x,\delta_{i\alpha}(x)\big)\right|\geqslant\frac{1}{2}\sup_{y\in(0,1)^m}\left|\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^{\alpha}}(x,y)\right|\quad\text{for any } x\in\Omega\setminus\Sigma. \tag{2.5}$$

Now we apply Proposition 2.5 to all the functions

$$\Omega\setminus\Sigma\ni x\longmapsto\frac{\partial^{|\alpha|}F_i}{\partial y^{\alpha}}\big(x,\delta_{i\alpha}(x)\big)\in R$$

as well as to $\Omega\setminus\Sigma\ni x\mapsto\delta_{i\alpha}(x)\in(0,1)^m$. Thus, there exists a cell decomposition $\{C_\varkappa\}$ of Ω such that for each open cell $C_\varkappa$ there exists a definable $\mathcal{C}^p$-diffeomorphism

$$\varphi_\varkappa\colon\pi(C_\varkappa)\times(0,1)\longrightarrow C_\varkappa$$

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of the form as above, satisfying condition (i) and such that all the functions

$$\delta_{i\alpha}\big(\varphi_\varkappa(x',\xi_n)\big) \quad \text{and} \quad \frac{\partial^{|\alpha|}F_i}{\partial y^\alpha}\big(\varphi_\varkappa(x',\xi_n),(\delta_{i\alpha}\circ\varphi_\varkappa)(x',\xi_n)\big)$$

are $\mathcal{C}^p$ and have all partial derivatives with respect to ξ_n up to order p bounded. Put

$$\widetilde{F}_{i\varkappa}(x',\xi_n,y) := F_i\big(\varphi_\varkappa(x',\xi_n),y\big).$$

Now we have

$$(2.6)\quad \frac{\partial^{q+1}}{\partial\xi_n^{q+1}}\Big(\frac{\partial^{|\alpha|}\widetilde{F}_{i\varkappa}}{\partial y^\alpha}\Big) = \Big(\frac{\partial\varphi_{\varkappa n}}{\partial\xi_n}\Big)^{q+1}\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^\alpha}(\varphi_\varkappa(x',\xi_n),y)+ \text{a polynomial with integral coefficients in } \Big\{\frac{\partial^\nu\varphi_{\varkappa n}}{\partial\xi_n^\nu}(x',\xi_n)\Big\}_{\nu\leqslant p}$$

and

$$\Big\{\frac{\partial^{\mu+|\alpha|}F_i}{\partial x_n^\mu\partial y^\alpha}(\varphi_\varkappa(x',\xi_n),y)\Big\}_{\mu+|\alpha|\leqslant p,\mu\leqslant q} = \Big(\frac{\partial\varphi_{\varkappa n}}{\partial\xi_n}\Big)^{q+1}\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^\alpha}(\varphi_\varkappa(x',\xi_n),y) + \text{a bounded function.}$$

A calculation similar to (2.6) shows that

$$(2.7)\quad \frac{\partial^{q+1}}{\partial\xi_n^{q+1}}\Big(\frac{\partial^{|\alpha|}F_i}{\partial y^\alpha}\big(\varphi_\varkappa(x',\xi_n),(\delta_{i\alpha}\circ\varphi_\varkappa)(x',\xi_n)\big)\Big) = \Big(\frac{\partial\varphi_{\varkappa n}}{\partial\xi_n}\Big)^{q+1}\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^\alpha}(\varphi_\varkappa(x',\xi_n),(\delta_{i\alpha}\circ\varphi_\varkappa)(x',\xi_n)) + \text{a bounded function.}$$

Since (2.7) is a bounded function,

$$\Big(\frac{\partial\varphi_{\varkappa n}}{\partial\xi_n}\Big)^{q+1}\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^\alpha}\big(\varphi_\varkappa(x',\xi_n),(\delta_{i\alpha}\circ\varphi_\varkappa)(x',\xi_n)\big)$$

is bounded too. Hence, by (2.5),

$$\Big(\frac{\partial\varphi_{\varkappa n}}{\partial\xi_n}\Big)^{q+1}\frac{\partial^{q+1+|\alpha|}F_i}{\partial x_n^{q+1}\partial y^\alpha}\big(\varphi_\varkappa(x',\xi_n),y\big)$$

is bounded, and finally by (2.6),

$$\frac{\partial^{q+1}}{\partial\xi_n^{q+1}}\Big(\frac{\partial^{|\alpha|}\widetilde{F}_{i\varkappa}}{\partial y^\alpha}\Big)$$

is bounded, which ends the proof. ∎

Proposition 2.8 *Let $f_1,\dots,f_k\colon \Omega\to R$ be any definable bounded functions defined on an open definable bounded subset Ω of R^n. Let p be any positive integer and let $m\in\{1,\dots,n\}$. Let $\pi\colon R^n\ni(x_1,\dots,x_n)\mapsto(x_1,\dots,x_{n-m})\in R^{n-m}$ denote the natural projection.*

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Then there exists a cell decomposition $\{C_\varkappa\}$ *of* Ω *such that for each open cell* $C_\varkappa$, *there exists a definable* $\mathcal{C}^p$*-diffeomorphism* $\varphi_\varkappa\colon \pi(C_\varkappa)\times(0,1)^m \to C_\varkappa$ *of the form*

$$\varphi_\varkappa(x',\xi) = \big(x', \varphi_{\varkappa 1}(x',\xi_1), \varphi_{\varkappa 2}(x',\xi_1,\xi_2),\dots,\varphi_{\varkappa m}(x',\xi_1,\dots,\xi_m)\big),$$

where $x' = (x_1,\dots,x_{n-m}) \in \pi(C_\varkappa)$, $\xi = (\xi_1,\dots,\xi_m) \in (0,1)^m$, *all the restrictions* $f_i|C_\varkappa$ *are of class* $\mathcal{C}^p$, *and all the partial derivatives*

$$\frac{\partial^{|\alpha|}\varphi_\varkappa}{\partial \xi^\alpha} \quad \text{and} \quad \frac{\partial^{|\alpha|}(f_i\circ\varphi_\varkappa)}{\partial\xi^\alpha} \qquad (i\in\{1,\dots,k\}, \alpha\in\mathbb{N}^m, 0<|\alpha|\leqslant p) \tag{2.8}$$

are bounded.

Proof This is immediate by Propositions 2.5 and 2.7 used repeatedly. ∎

Remark 2.9 It follows from the proof of Proposition 2.7 that there exist bounds on the partial derivatives (2.8) depending only on p and m.

3 Proof of Uniform $\mathcal{C}^p$-Parametrization Theorem

We will argue by induction on $d = \dim T$. By the Cell Decomposition Theorem (see [14, Chapter 3 and Chapter 7, §3]), without any loss of generality, one can assume that T is a $\mathcal{C}^p$-cell of dimension d and, by using an appropriate $\mathcal{C}^p$-diffeomorphism, that T is an open bounded cell in R^d.

By the Good Direction Theorem (*cf.* [14, Chapter 9, (1.4)]), after a linear change of coordinates in R^n and perhaps removing from T a definable subset of dimension $< d$, one can assume that, for any $y \in T$, $(\{y\}\times R^{n-k})\cap X$ is a finite set.

Now by using a cell decomposition of X, we reduce the general case to one such that X is the closure in $T\times R^n$ of the graph of a definable bounded mapping $f = (f_{k+1},\dots,f_n)\colon \Omega\to R^{n-k}$ defined on an open definable bounded subset Ω of $R^d\times R^k$. To finish the proof, it suffices to apply Proposition 2.8 with $p+1$ in the place of p. ∎

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