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THE SPACE GROUPS OF TWO DIMENSIONAL MINKOWSKI SPACE

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Introduction. Let E be an n-dimensional real affine space, V its vector space of translations and $A(E)$ the affine group of E. Suppose that $(.,.)$ is a nondegenerate symmetric bilinear form on V of signature $(n - 1, 1)$, $O(V)$ its orthogonal group and $S(V)$ its group of similarities.

A subgroup S of $A(E)$ is called a *Minkowski space group* if $\Lambda = S \cap V$ is a lattice in V and the projection K of S on $GL(V)$ is contained in $O(V)$. The lattice Λ is invariant under K. If $\{(t(g), g)\}$ is a system of representatives of elements of K in S, the induced function $\bar{t} : K \to V/\Lambda$ is a cocycle. The trio $\{\Lambda, K, \bar{t}\}$ completely determines S, since

$$S = \{(a + t(g), g) | a \in \Lambda, g \in K\}.$$

Space groups are considered equivalent if they are conjugate in $A(E)$. In particular, conjugation by a translation preserves Λ and K, but replaces $\bar{t}$ with a cohomologous cocycle; it therefore suffices to study the cohomology group $H^1(K, V/\Lambda)$ for given K and Λ.

The conjugacy classes of space groups in $A(E)$ can be determined as follows. Call a subgroup K of $O(V)$ *crystallographic* if it leaves invariant a lattice in V and find the conjugacy classes of such groups in $GL(V)$. For each class $\{K\}$, determine the set of lattices invariant under K. The normaliser $N(K)$ of K in $GL(V)$ acts naturally on $L(K)$. For each orbit $\{\Lambda\}$ in $L(K)$ under this action, calculate the cohomology group $H^1(K, V/\Lambda)$. The subgroup $N(K, \Lambda)$ consisting of all elements of $N(K)$ leaving Λ invariant acts on $H^1(K, V/\Lambda)$ by the rule $(h.\bar{t})(g) = h\bar{t}(h^{-1}gh)$. The orbits of this action correspond to inequivalent space groups with 'point group' K and 'lattice' Λ.

The principal purpose of this paper is to carry out this procedure in the case $n = 2$. Earlier, Janner and Ascher [**4**; **5**] had studied possibilities for K and Λ by applying the theory of binary integral quadratic forms. We prefer, however, to give a more self-contained geometric discussion of the problem which, perhaps, throws some light on the theory of quadratic forms. In particular, we can derive a complete set of inequivalent *ambiguous* primitive quadratic forms for a given discriminant D. On counting them, we rediscover the classical result [**3**; **6**] that their number is a certain power of 2, which must divide the total class number.

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Further discussion of space groups can be found in [**7**; **8**].

1. Point groups. Recall that a *similarity* $f : V — V$ is a linear map such that $(f(v_1), f(v_2)) = m(v_1, v_2)$ for some constant $m \neq 0$, called the *multiplier* of f, and all $v_1, v_2 \in V$. There exists a basis $\{e_1, e_2\}$ of V such that $(e_1, e_1) = (e_2, e_2) = 0$ and $(e_1, e_2) = 1$. The matrices of similarities with respect to this basis have one of the forms

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix},$$

where $ab \neq 0$ is the multiplier m, and are accordingly called 'direct' or 'opposite'. In particular, the elements of $O(V)$ have matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix},$$

according to whether they are 'rotations' or 'reflections'. The fixed line of such a reflection is spanned by the vector $e_1 + ae_2$; the line spanned by $e_1 - ae_2$ is orthogonal to this line and will be called the *normal line* of the reflection.

Let W be an abstract group with two generators w_1 and w_2, subject to the relations $w_1^2 = w_2^2 = 1$; W is a 'Coxeter group' with the graph

$$\bullet\overset{\infty}{\text{———}}\bullet.$$

Consider those representations $\rho : W \to O(V)$ for which $\rho(w_i) = s_i$ are reflections for $i = 1, 2$. Suppose

$$s_1 = \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & b^{-1} \\ b & 0 \end{bmatrix}$$

and define $n(\rho) = \operatorname{tr}(s_1s_2) = a^{-1}b + ab^{-1}$. Since $n(\rho)$ is a real number of the form $x + x^{-1}$, we have $|n(\rho)| \geqq 2$. The element s_1s_2 is of finite order if and only if $b = \pm a$, in which case $n(\rho) = \pm 2$; otherwise, the representation ρ is faithful. Conversely, given a real number n such that $|n| \geqq 2$, we can construct a representation $\rho : w \to O(V)$ for which $n(\rho) = n$ by defining

$$\rho(w_1) = \begin{bmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{bmatrix}, \quad \rho(w_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where a_n is the root of $a_n + a_n^{-1} = n$ for which $|a_n| \geqq 1$. The image of this representation in $O(V)$ will be denoted by $W(n)$.

1.1 PROPOSITION. *If representations $\rho_i : W \to O(V)$ $(i = 1, 2)$ are conjugate by an element of $GL(V)$, then $n(\rho_1) = n(\rho_2)$. Conversely, if $n(\rho_1) = n(\rho_2)$, then ρ_1 and ρ_2 are conjugate by an element of $S(V)$.*

Proof. Since $n(\rho)$ can be characterised as the numerically least trace of the elements in $\rho(W)$, the first assertion is clear. Conversely, suppose $n(\rho_1) =$

$n(\rho_2)$. If the fixed lines of $\rho_1(w_1)$ and $\rho_2(w_1)$ are spanned, respectively, by $e_1 + ae_2$ and $e_1 + be_2$, the similarity

$$g = \begin{bmatrix} 1 & 0 \\ 0 & ba^{-1} \end{bmatrix}$$

has the property that $g\rho_1(w)g^{-1} = \rho_2(w_1)$. Replacing ρ_2 by the conjugate representation $g^{-1}\rho_2 g$, we may assume that

$$\rho_1(w_1) = \rho_2(w_1) = \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix}.$$

Suppose

$$\rho_1(w_2) = \begin{bmatrix} 0 & c^{-1} \\ c & 0 \end{bmatrix}, \quad \rho_2(w_2) = \begin{bmatrix} 0 & d^{-1} \\ d & 0 \end{bmatrix}.$$

Since $n(\rho_1) = n(\rho_2)$, we have $a^{-1}c + ac^{-1} = a^{-1}d + ad^{-1}$ or $(a^2 - cd)(c - d) = 0$, so that either $d = c$ or $d = a^2c^{-1}$. In the first case, $\rho_2 = \rho_1$, while in the second ρ_2 is the conjugate of ρ_1 by $\rho_1(w_1)$, which is still a similarity.

The group $W(n)$ and its rotation subgroup $W(n)^+$ do not contain, for $|n| > 2$, the element -1_V; adjoining it, we obtain groups denoted by $\pm W(n)$ and $\pm W(n)^+$. Since $\pm W(-n) = \pm W(n)$, the latter groups will only be considered for *positive* values of $n > 2$.

1.2 PROPOSITION. *A crystallographic subgroup K of $O(V)$ is conjugate in $S(V)$ to one of the groups $W(n)$, $W(n)^+$, $\pm W(n)$, $\pm W(n)^+$ for some integer n such that $|n| \geqq 2$. No two of these groups are themselves conjugate in $GL(V)$.*

Proof. Consider the rotation subgroup K^+ of K. If $\theta \in K^+$, then $\operatorname{tr}(\theta) \in \mathbf{Z}$ since K is crystallographic. If K^+ is not contained in $\{\pm 1_V\}$, there exists an element

$$\theta = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \neq \pm 1_V$$

in K^+ for which $|\operatorname{tr}(\theta)|$ is least among all such elements. By replacing θ with θ^{-1}, we may assume that $|a| > 1$. Suppose

$$\psi = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$$

is another element of K^+ different from $\pm 1_V$ and for which $|b| > 1$. Then $|\operatorname{tr}(\psi)| \geqq |\operatorname{tr}(\theta)|$, which is equivalent to $|b| \geqq |a|$, with equality holding only if $\psi = \pm\theta$. Let $k > 0$ be an integer such that $|a|^k \leqq |b| < |a|^{k+1}$, i.e. $1 \leqq |ba^{-k}| < |a|$. Since

$$\psi\theta^{-k} = \begin{bmatrix} ba^{-k} & 0 \\ 0 & b^{-1}a^k \end{bmatrix} \in K^+,$$

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the choice of θ implies that $\psi = \pm\theta^k$. Thus K^+ is either equal to $\{1_V\}$, $\{\pm 1_V\}$, generated by θ or generated by θ and -1_V. In other words, it equals $W(2)^+$, $W(-2)^+$, $W(n)^+$ or $\pm W(n)^+$, where $n = \operatorname{tr}(\theta) \in \mathbf{Z}$; in the last case, we can ensure that $n > 0$ by replacing θ with $-\theta$.

If $K \neq K^+$, then K must contain a reflection s. In the first three of the above cases, K is the image of a representation which maps w_1 and w_2 to, respectively, s and s, s and $-s$, s and θs. By Proposition 1.1, K is conjugate in $S(V)$ to the group $W(n)$. In the last case, it is clear that K is conjugate to $\pm W(n)$.

The final statement follows from the fact that conjugation by an element of $GL(V)$ preserves determinants, traces and the element -1_V.

We have not yet shown that the groups in Proposition 1.2 are actually crystallographic. This will be done below when we determine *all* the lattices left invariant by them.

1.3 PROPOSITION. *For all crystallographic subgroups K of $O(V)$, we have $H^1(K, V) = 0$.*

Proof. If $n = \pm 2$, this is true for the groups $W(n)$ and $W(n)^+$ since they are finite. For any $n > 2$, this holds for $\pm W(n)$ and $\pm W(n)^+$ since they contain -1_V (if $t : K \to V$ is a cocycle, we have $t(-1_V) = t(g(-1_V)g^{-1}) = 2t(g) + gt(-1_V)$, so that t is the coboundary corresponding to $t(-1_V)/2$). For the infinite cyclic groups $W(n)^+$, $n > 2$, this follows from the invertibility of $1 - s_1s_2$. There remain the groups $W(n)$, $|n| > 2$. If $t : K \to V$ is a cocycle, we must have $t(s_i^2) = (1 + s_i)t(s_i) = 0$ for $i = 1, 2$, so that $t(s_1) = x_1(e_1 - a_ne_2)$, $t(s_2) = x_2(e_1 - e_2)$ for some $x_1, x_2 \in \mathbf{R}$. This is the coboundary corresponding to the vector

$$(1 - a_n^{-1})^{-1}((x_1 - a_n^{-1}x_2)e_1 + (x_1 - x_2)e_2).$$

1.4 COROLLARY. *Two space groups are isomorphic if and only if they are conjugate in $A(E)$.*

Proof. This follows from Proposition 1.3 of [**7**].

For $n = \pm 2$, the finite groups $W(n)$ and $W(n)^+$ leave invariant a positive definite symmetric bilinear form on V. The corresponding space groups can therefore be also viewed as Euclidean space groups and are well known [**2**; **7**]. *We shall assume henceforth that $|n| > 2$.*

2. Normalisers. If H is the group of all homotheties of V, HK is clearly a normal subgroup of the normaliser $N(K)$ of K in $GL(V)$. Let μ denote the similarity

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

of V with multiplier -1; we have $\mu s \mu^{-1} = -s$ for any reflection s in $O(V)$.

For $n > 0$, let σ_n denote the reflection

$$\begin{bmatrix} 0 & a_n^{-1/2} \\ a_n^{1/2} & 0 \end{bmatrix}.$$

2.1 Proposition. (a) *If* $K = W(n)$, *we have* $N(K) = HK \cup HK\sigma_n$ *if* $n > 0$ *and* $N(K) = HK \cup HK\, \sigma_{-n}\mu$ *if* $n < 0$.

(b) *If* $K = \pm W(n)$, *then* $N(K) = HK \cup HK\, \sigma_n \cup HK\mu \cup HK\, \sigma_n\mu$.

(c) *If* $K = W(n)^+$ *or* $\pm W(n)^+$, *then* $N(K) = S(V)$.

Proof. It is easily verified that conjugation by σ_n or σ_{-n}, according to whether $n > 0$ or $n < 0$, interchanges s_1 and s_2; conjugation by μ reverses the signs of s_1 and s_2.

In case (a), consider an element $g \in N(K)$; then gs_1g^{-1} is a reflection in K. Since all reflections in K are conjugate within K to one of s_1, s_2, we may suppose that $gs_1g^{-1} = s_1$ by multiplying g by an element of K and possibly σ_n or $\sigma_{-n}\mu$. Since s_1s_2 and s_2s_1 are the only elements of trace n in $W(n)$, we must have $gs_1s_2g^{-1} = s_1s_2$ or s_2s_1, so that $gs_2g^{-1} = s_2$ or $s_1s_2s_1$. The second case can be reduced to the first by multiplying g by s_1, which does not affect the assumption $gs_1g^{-1} = s_1$. Now g commutes with both s_1 and s_2 and is consequently a homothety [**1**, Chapter V, § 2, Proposition 1].

In case (b), the argument is similar, except that there are now 4 conjugacy classes of reflections in K, represented by $\pm s_1$, $\pm s_2$, so that g may have to be multiplied by any one of σ_n, μ or $\sigma_n\mu$ to achieve the equation $gs_1g^{-1} = s_1$.

In case (c), if $g \in N(K)$, we must have $gs_1s_2g^{-1} = s_1s_2$ or s_2s_1 for the same reason as above. In the second case, we can multiply g by s_1 and reduce it to the first. It is easily verified that an element of $GL(V)$ commuting with s_1s_2 is a direct similarity and, conversely, that any similarity normalises K.

Note that in all cases $N(K)$ is contained in $S(V)$.

3. Lattices. Since -1_V leaves every lattice invariant, the groups $W(n)$, $W(-n)$ and $\pm W(n)$ have precisely the same invariant lattices. It is therefore sufficient to consider only the group $W(n)$, for $n > 0$. This is a 'linear Coxeter group' in the sense of [**8**], so that we may apply the results of that paper.

Let $\alpha_1 = a_n^{-1/2}(e_1 - a_ne_2)$, $\alpha_2 = -(n+2)^{1/2}(e_1 - e_2)$ and $\hat{\alpha}_i$ the functional $\langle v, \hat{\alpha}_i\rangle = 2(v, \alpha_i)/(\alpha_i, \alpha_i)$ on V for $i = 1, 2$. Then $s_i(v) = v - \langle v, \hat{\alpha}_i\rangle\alpha_i$. Let $n_{ij} = \langle \alpha_j, \hat{\alpha}_i\rangle$ and define N to be the matrix (n_{ij}); we find that

$$N = \begin{bmatrix} 2 & -(n+2) \\ -1 & 2 \end{bmatrix}.$$

A 'basic system' [**8**] of $W(n)$ is a set $B = \{b_1\alpha_1, b_2\alpha_2\}$, where $b_i > 0$ for $i = 1, 2$, and $n_{ij}b_j \in \mathbf{Z}b_i$ for $i, j = 1, 2$. This amounts to requiring B to be a positive multiple of a set of the form $B_p = \{p\alpha_1, \alpha_2\}$, where $p > 0$ is an integer dividing $n + 2$. The lattice spanned by B_p is called the *root lattice* of B_p and denoted by $Q(B_p)$. The *fundamental weights* of B_p are the elements $\{p\omega_1, \omega_2\}$, where

$$\omega_1 = (2-n)^{-1}(2\alpha_1 + \alpha_2)$$
$$\omega_2 = (2-n)^{-1}((n+2)\alpha_1 + 2\alpha_2)$$

satisfy $\langle \omega_i, \hat{\alpha}_j \rangle = \delta_{ij}$, the Kronecker delta.

If Λ is a lattice invariant under K, we have earlier defined [**7**] Λ^* to be the additive group of all $v \in V$ such that $v - g(v) \in \Lambda$ for all $g \in K$. It suffices to fulfil this condition for a set of generators of K, in view of the equation $v - g_1g_2(v) = (v - g_1(v)) + g_1(v - g_2(v))$. Clearly Λ^* contains Λ and every lattice between Λ and Λ^* is also invariant under K. In our case, it is easily seen [**8**] that $Q(B_p)^* = P(B_p)$ is the lattice spanned by $\{p\omega_1, \omega_2\}$ and called the *weight lattice* of B_p. The quotient group $P(B_p)/Q(B_p)$ is of order $|\det N| = n - 2$ for every p.

3.1 PROPOSITION. *If Λ is a lattice invariant under $W(n)$, $n > 0$, then there exists a constant $c > 0$ and a unique divisor $p > 0$ of $n + 2$ such that $Q(B_p) \subset c\Lambda \subset P(B_p)$ and $(c\Lambda)^* = P(B_p)$. Conversely, every lattice of this form is invariant under $W(n)$.*

Proof. This follows from Proposition 1.3 of [**8**].

The condition $\Lambda^* = P(B_p)$ is equivalent to saying that

(1) $\quad \Lambda \cap \mathbf{R}\alpha_1 = \mathbf{Z}p\alpha_1, \quad \Lambda \cap \mathbf{R}\alpha_2 = \mathbf{Z}\alpha_2,$

or that the elements $p\alpha_1/2$ and $\alpha_2/2$ do not belong to Λ whenever they happen to be weights (the first for p even, the second for $(n+2)/p$ even).

If $p > 0$ is a divisor of $n + 2$ and $q > 0$ a divisor of $n - 2$, we shall denote $(n+2)/p$ and $(n-2)/q$ by $\bar{p}$ and $\bar{q}$ respectively. Note that the *GCD* of p and q must divide 4.

3.2 *Definition.* (a) If p and q are both odd, let $\Lambda_{p,q}$ be the lattice spanned by $p\alpha_1$, α_2 and $\bar{q}p\omega_1$.

(b) If p and $\bar{q}$ are both even and $4|p$ only if q is odd, let $\Lambda_{p,q}$ be the lattice spanned by $p\alpha_1$, α_2 and $\bar{q}p\omega_1/2$.

(c) If p, $\bar{p}$, q and $\bar{q}$ are all even, $4|q$ if $n = 2 \bmod 8$ and $4|p$ if $n = 6 \bmod 8$, let $\Lambda_{p,q}'$ be the lattice spanned by $p\alpha_1$, α_2 and $(\bar{q}p\omega_1 + \alpha_2)/2$.

It is easy to verify that a lattice Λ of the form $\Lambda_{p,q}$ or $\Lambda_{p,q}'$ lies between $Q(B_p)$ and $P(B_p)$ and satisfies (1). Furthermore, we have

(2) $\quad [\Lambda : Q(B_p)] = q.$

Since $\Lambda_{p,q}' \neq \Lambda_{p,q}$, for otherwise both would contain $\alpha_2/2$, this shows that the lattices $\Lambda_{p,q}$ and $\Lambda_{p,q}'$ are all distinct and in fact not even related by a homothety.

3.3 *Remark.* The only case in which *both* $\Lambda_{p,q}$ and $\Lambda_{p,q}'$ are defined is when p, $\bar{p}$, q and $\bar{q}$ are all even, $n = 2 \bmod 8$ and $4|q$.

3.4 PROPOSITION. *Every lattice invariant under $W(n)$, $W(-n)$ or $\pm W(n)$, for $n > 0$, is a multiple of one of the lattices $\Lambda_{p,q}$ or $\Lambda_{p,q}'$ for unique divisors p of $n + 2$ and q of $n - 2$.*

Proof. In view of Proposition 3.2, it is sufficient to consider a lattice Λ between $Q(B_p)$ and $P(B_p)$ which satisfies (1).

If p is odd, the class of $p\omega_1$ generates the group $P(B_p)/Q(B_p)$. Therefore Λ is spanned by $p\alpha_1$, α_2 and $\bar{q}p\omega_1$ for some $q|n - 2$. If q is even, Λ contains $q\bar{q}p\omega_1/2 = \alpha_2/2 \bmod Q(B_p)$, contradicting (1). If q is odd, $\Lambda = \Lambda_{p,q}$.

When p is even, but $\bar{p}$ is odd, $P(B_p)/Q(B_p)$ is still cyclic (the class of ω_2 is a generator) but the class of $p\omega_1$ is only of order $\frac{1}{2}(n - 2)$. Therefore either $\Lambda = P(B_p)$ or else Λ is spanned by $p\alpha_1$, α_2 and $\bar{q}p\omega_1/2$ for some $q|\frac{1}{2}(n - 2)$, so that $\bar{q}$ is even. The first case is excluded since $P(B_p)$ contains $\frac{1}{2}(n - 2)\omega_2 = p\alpha_1/2 \bmod Q(B_p)$. In the other cases, q must be odd if $4|p$, since otherwise Λ would contain $q\bar{q}p\omega_1/4 = p\alpha_1/2 \bmod Q(B_p)$. Therefore $\Lambda = \Lambda_{p,q}$.

When both p and $\bar{p}$ are even, so that $n = 2 \bmod 4$ the group $P(B_p)/Q(B_p)$ is no longer cyclic, but rather generated by the class of $p\omega_1$, of order $\frac{1}{2}(n - 2)$, and the class of $\alpha_2/2$, of order 2. Since Λ cannot contain $\alpha_2/2$, it follows readily that Λ is spanned by $p\alpha_1$, α_2 and one of the elements $\bar{q}p\omega_1/2$, $(\bar{q}p\omega_1 + \alpha_2)/2$ for some $q|\frac{1}{2}(n - 2)$, so that $\bar{q}$ is even. In the first case, one sees as above that $4|p$ only if q is odd, so that $\Lambda = \Lambda_{p,q}$. In the second, q must be even since otherwise Λ would contain $q(\bar{q}p\omega_1 + \alpha_2)/2 = \alpha_2/2 \bmod Q(B_p)$. Furthermore, since $q(\bar{q}p\omega_1 + \alpha_2)/4 = -p\alpha_1/2 + (q/2 - p/2)\alpha_2/2$, we must have $q/2 \not\equiv p/2 \bmod 2$. Thus, if $n = 2 \bmod 8$, $p/2$ is odd so that $4|q$, whereas if $n = 6 \bmod 8$, $q/2$ is odd so that $4|p$. We have proved that $\Lambda = \Lambda_{p,q}'$.

The next step is to sort out the action of the normaliser on these lattices. Write $\Lambda_1 \sim \Lambda_2$ if Λ_1 and Λ_2 are related by a homothety.

3.5 PROPOSITION. *We have $\sigma_n(\Lambda_{p,q}) \sim \Lambda_{\bar{p},q}$ and $\sigma_n(\Lambda_{p,q}') \sim \Lambda_{\bar{p},q}'$ unless one of the following holds:*

(i) $n = 2 \bmod 8$ *and* $4 \nmid \bar{q}$, *when* $\sigma_n(\Lambda_{p,q}) \sim \Lambda_{\bar{p},q}'$ *and* $\sigma_n(\Lambda_{p,q}') \sim \Lambda_{\bar{p},q}$;

(ii) $n = 6 \bmod 8$ *and* $2|q$, *when* $\sigma_n(\Lambda_{p,q}) \sim \Lambda_{\bar{p},q}'$, *or* $4 \nmid \bar{p}$, *when* $\sigma_n(\Lambda_{p,q}') \sim \Lambda_{p,q}$.

Proof. For a given $p|n + 2$, let ϕ be the composite of σ_n and the homothety with respect to $(n + 2)^{1/2}/p$. We find that

$$\phi(p\alpha_1) = \alpha_2, \quad \phi(\alpha_2) = \bar{p}\alpha_1, \quad \phi(p\omega_1) = \omega_2, \quad \phi(\omega_2) = \bar{p}\omega_1.$$

Therefore ϕ maps lattices between $Q(B_p)$ and $P(B_p)$ to those between $Q(B_{\bar{p}})$ and $P(B_{\bar{p}})$. Since it preserves indices and condition (1), the assertion follows from (2) and Remark 3.3, except possibly in the case specified there. To settle this case, we note that $\phi(\Lambda_{p,q})$ is then spanned by $\bar{p}\alpha_1$, α_2 and $\bar{q}\omega_2/2$. Since $2\omega_2 = p\bar{p}\omega_1 + \alpha_2$ and $p/2$ is relatively prime to $n - 2$, $\bar{q}\omega_2/2$ can be replaced by $\bar{q}p\omega_1/2$ if $4|\bar{q}$ and by $(\bar{q}p\omega_1 + \alpha_2)/2$ if $4 \nmid \bar{q}$. Correspondingly, we have $\phi(\Lambda_{p,q}) = \Lambda_{\bar{p},q}$ or $\Lambda_{\bar{p},q}'$ and therefore $\phi(\Lambda_{p,q}') = \Lambda_{\bar{p},q}'$ or $\Lambda_{\bar{p},q}$.

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3.6 Proposition. (a) *If n is odd,* $\mu(\Lambda_{p,q}) \sim \Lambda_{\bar{p},\bar{q}}$.

(b) *If* $4|n$, $\mu(\Lambda_{p,q}) \sim \Lambda_{\bar{p}/2,\bar{q}/2}$ *if p is odd and* $\mu(\Lambda_{\bar{p},q}) \sim \Lambda_{2\bar{p},\bar{q}/2}$ *if p is even.*

(c) *If* $n = 2 \bmod 4$, *then* $\mu(\Lambda_{p,q}) \sim \Lambda_{\bar{p}/2,\bar{q}/2}'$ *if p is odd,* $\mu(\Lambda_{p,q}) \sim \Lambda_{\bar{p},\bar{q}/4}$ *if p and $\bar{p}$ are even and* $4|\bar{q}$, *while* $\mu(\Lambda_{p,q}) \sim \Lambda_{2\bar{p},\bar{q}/2}$ *otherwise. Furthermore,* $\mu(\Lambda_{p,q}') \sim \Lambda_{\bar{p},\bar{q}}'$ *if* $4|\bar{p}$ *or* $4|\bar{q}$ *and* $\mu(\Lambda_{p,q}') \sim \Lambda_{\bar{p}/2,\bar{q}/2}$ *otherwise.*

Proof. For a given $p|n + 2$, let ψ be the composite of μ and the homothety with respect to $(n + 2)^{1/2}/p(n - 2)^{1/2}$. Then

$$\psi(p\alpha_1) = \omega_2, \quad \psi(\alpha_2) = -\bar{p}\omega_1, \quad \psi(p\omega_1) = -\alpha_2/n - 2,$$
$$\psi(\omega_2) = -\bar{p}\alpha_1/n - 2.$$

Consider the lattice $\Lambda = \psi(\Lambda_{p,q})$, where p and q are odd; it is spanned by $\bar{p}\omega_1$, ω_2 and α_2/q. Suppose $\Lambda \cap \mathbf{R}\alpha_2 = \mathbf{Z}c\alpha_2$ for some $c > 0$; then $c\alpha_2 - x\alpha_2/q \in P(B_{\bar{p}})$ for some $x \in \mathbf{Z}$, which implies that $c - x/q \in \mathbf{Z}$ and hence $qc \in \mathbf{Z}$, since p is odd. As $\alpha_2/q \in \Lambda$, we have $c = 1/q$. Secondly, let $\Lambda \cap \mathbf{R}\alpha_1 = \mathbf{Z}d\alpha_1$ for some $d > 0$; then $d\alpha_1 - x\alpha_2/q \in P(B_{\bar{p}})$. Applying $\hat{\alpha}_1$ and $\hat{\alpha}_2$ to this element, we deduce that $2d + (n + 2)x/q \in \bar{p}\mathbf{Z}$ and $-d - 2x/q \in \mathbf{Z}$. This is equivalent to saying that $d = \bar{p}(qy - px)/2q$ for some $y \notin \mathbf{Z}$ such that

(3) $\quad -py + qx \in 2\mathbf{Z}.$

If $\bar{p}$ is even, so is $\bar{q}$ and (3) is always satisfied. Since $GCD(p, q) = 1$ there exist y and x such that $qy - px = 1$, showing that $d = \bar{p}/2q$. If $\bar{p}$ is odd, so is $\bar{q}$, and (3) requires that $y = x \bmod 2$, so that $qy - px$ is even and $d = \bar{p}/q$. In view of condition (1), we conclude that $q\Lambda$ is one of the standard lattices associated with the divisor $\bar{p}$ of $n + 2$, if $\bar{p}$ is odd, and $\bar{p}/2$ if $\bar{p}$ is even.

To determine the index of the corresponding root lattice in $q\Lambda$, we note that Λ contains $\psi(\Lambda_{p,1}) = P(B_{\bar{p}})$ as a sublattice of index q. The diagram of inclusions

$$\begin{array}{ccc} P(B_{\bar{p}}) & \longrightarrow & \Lambda \\ \uparrow & & \uparrow \\ Q(B_{\bar{p}}) & \longrightarrow & q^{-1}Q(B_{\bar{p}}) \end{array}$$

shows that $[q\Lambda\colon Q(B_{\bar{p}})] = [\Lambda\colon q^{-1}Q(B_{\bar{p}})] = q(n - 2)/q^2 = \bar{q}$, so that $[q\Lambda\colon Q(B_{\bar{p}/2})] = \bar{q}/2$. In view of Remark 3.3, $q\Lambda$ is uniquely determined, except if $n = 2 \bmod 8$. Since we then have

$$(q\bar{p}\omega_1 + \alpha_2)/2 = -\tfrac{1}{2}(p - 1)q\bar{p}\omega_1 + q\omega_2 - \tfrac{1}{2}(q - 1)\alpha_2 \in q\Lambda,$$

$q\Lambda$ must equal $\Lambda_{\bar{p}/2,\bar{q}/2}'$.

The argument is similar in the remaining cases, and leads to the stated results.

In view of Proposition 2.1, one can deduce

3.7 Proposition. *If $n > 0$, a complete set of representative lattices for the groups* $K = W(n)$, $W(-n)$ *and* $\pm W(n)$, *with respect to the action of the nor-*

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maliser $N(K)$, *is given in Table* 1. (The notation '$p \approx \bar{p}$' means that only *one* of the values p, $\bar{p}$ is to be used for the first index, etc.)

TABLE 1

n	Λ	p, q	$K = W(n)$	$K = W(-n)$	$K = \pm W(n)$
$n = 1 \bmod 2$	$\Lambda_{p,q}$	—	$p \approx \bar{p}$	$q \approx \bar{q}$	$p \approx \bar{p},\ q \approx \bar{q}$
$n = 0 \bmod 4$	$\Lambda_{p,q}$	$2 \mid p$	—	—	$(p, q) \approx (2\bar{p}, \bar{q}/2)$
$n = 2 \bmod 4$	$\Lambda_{p,q}$	$2 \nmid \bar{p}$ or $4 \nmid \bar{q}$	—	—	$2 \nmid \bar{p}$
	$\Lambda_{p,q}$	$2 \mid p,\ 2 \mid \bar{p},\ 4 \mid \bar{q}$	$p \approx \bar{p}$	$q \approx \bar{q}/4$	$p \approx \bar{p},\ q \approx \bar{q}/4$
	$\Lambda_{p,q}'$	$4 \mid \bar{p}$ or $4 \mid \bar{q}$	$p \approx \bar{p}$	$q \approx \bar{q}$	$p \approx \bar{p},\ q \approx \bar{q}$

3.8 *Remark*. It can be seen from Table 1 that the lattices $\Lambda_{p,q}'$ need only be considered if $4|\bar{q}$ or $4|\bar{p}$, according to whether $n = \pm 2 \bmod 16$. Furthermore, if n is even, the lattices $\Lambda_{p,q}$ need only be considered for even values of p.

When $K = W(n)^+$, $W(-n)^+$ or $\pm W(n)^+$ $(n > 0)$, lattices invariant under $W(n)$ are of course still invariant under K However, there may exist further lattices invariant only under K. The first example occurs for $n = 15$. Let Λ be the lattice spanned by α_1 and $\theta = (3\alpha_1 + \alpha_2)/5$; then

$$s_2 s_1(\alpha_1) = 2\alpha_1 - 5\theta, \quad s_2 s_1(\theta) = -5\alpha_1 + 13\theta,$$

so that Λ is invariant under $W(15)^+$. It is not invariant under $W(15)$, but contains the lattice $\Lambda_{1,1}$, spanned by α_1 and α_2, as an invariant sublattice of index 5. Since the normaliser of K is now all of $S(V)$, it may happen that distinct lattices listed in Table 1 for $\pm W(n)$ may become equivalent for K. In fact, the situation in this case is better described by the classical theory of quadratic forms (see Section 6).

4. The groups $H^1(K, V/\Lambda)$.

In accordance with Remark 3.8, we may omit certain possibilities for Λ from the discussion.

4.1 PROPOSITION. *Suppose* $K = W(n)$ *or* $W(-n)$, *with* $n > 0$. *Then:*

(a) $H^1(K, V/\Lambda) = 0$ *if* $\Lambda = \Lambda_{p,q}$, *with* n *odd, or* $\Lambda = \Lambda_{p,q}'$.

(b) $H^1(K, V/\Lambda) \cong \mathbf{Z}/2\mathbf{Z}$ *if* $\Lambda = \Lambda_{p,q}$, *with* n *even and either* $\bar{p}$ *odd or* $4 \nmid \bar{q}$.

(c) $H^1(K, V/\Lambda) \cong (\mathbf{Z}/2\mathbf{Z})^2$ *if* $\Lambda = \Lambda_{p,q}$, *with* $n = 2 \bmod 4$, p *and* $\bar{p}$ *even and* $4|\bar{q}$.

Proof. First consider the case $K = W(n)$. A function t: $\{s_1, s_2\} \to V$ will induce a cocycle $\bar{t} : K \to V/\Lambda$ if and only if

$$(4) \qquad t(s_i^2) = (1 + s_i)t(s_i) \in \Lambda \quad (i = 1, 2).$$

Using the invertibility of N, one can show as in Proposition 3.7 of [7] that $\bar{t}$ is a coboundary if and only if $t(s_i) = x_i\alpha_i \bmod \Lambda$ for some $x_i \in \mathbf{R}$.

By subtracting from t the coboundary inducing function $s_i \to \langle t(s_i), \hat{\alpha}_i\rangle\alpha_i/2$, we can assume that $\langle t(s_i), \hat{\alpha}_i\rangle = 0$. Condition (4) then simply requires that $2t(s_i) \in \Lambda$ for $i = 1, 2$.

Let $t(s_1) = a\bar{q}\omega_2/2$, $t(s_2) = b\bar{q}p\omega_1/2$ for some $a, b \in \mathbf{R}$. We must have

$$(5) \qquad a\bar{q}\omega_2 \in \Lambda, \quad b\bar{q}p\omega_1 \in \Lambda.$$

It is easy to verify that

$$\begin{aligned}
\Lambda \cap \mathbf{R}\omega_1 &= \mathbf{Z}qp\omega_1 \quad \text{if } \Lambda = \Lambda_{p,q}, \text{ with } n \text{ odd, or } \Lambda = \Lambda_{p,q}'.\\
&= \mathbf{Z}qp\omega_1/2 \quad \text{if } \Lambda = \Lambda_{p,q}, \text{ with } n \text{ even.}
\end{aligned}$$

$$\begin{aligned}
(6) \qquad \Lambda \cap \mathbf{R}\omega_2 &= \mathbf{Z}q\omega_2 \quad \text{if } \Lambda = \Lambda_{p,q}, \text{ with } \bar{p} \text{ odd or } p, \bar{p} \text{ even and}\\
&\qquad \text{and } 4 \nmid \bar{q}, \text{ or } \Lambda = \Lambda_{p,q}'.\\
&= \mathbf{Z}q\omega_2/2 \quad \text{if } \Lambda = \Lambda_{p,q}, \text{ with } p, \bar{p} \text{ even and } 4|\bar{q}.
\end{aligned}$$

Furthermore, for all Λ,

$$(7) \qquad \begin{aligned}
&\{m \in \mathbf{R} | m\bar{q}p\omega_1/2 = x\alpha_2 \bmod \Lambda \quad \text{for some } x \in \mathbf{R}\} = \mathbf{Z}.\\
&\{m \in \mathbf{R} | m\bar{q}\omega_2/2 = x\alpha_1 \bmod \Lambda \quad \text{for some } x \in \mathbf{R}\} = \mathbf{Z}.
\end{aligned}$$

Condition (5) therefore requires b and a to be either in $\mathbf{Z}$ or in $\frac{1}{2}\mathbf{Z}$, according to the two cases in (6). In view of (7), we may correspondingly assume that b or a equals 0, or conclude that the choices of 0 and $\frac{1}{2}$ for b or a produce non-cohomologous cocycles. This proves the statement for $K = W(n)$.

With the above notation, the group $K = W(-n)$ is generated by $-s_1$ and s_2. A function t: $\{-s_1, s_2\} \to V$ will induce a cocycle $\bar{t}$: $K \to V/\Lambda$ if and only if $(1 - s_1)t(-s_1) \in \Lambda$ and $(1 + s_2)t(s_2) \in \Lambda$. The cocycle $\bar{t}$ will be a coboundary precisely when $t(-s_1) = x_1\omega_2 \bmod \Lambda$ and $t(s_2) = x_2\alpha_2 \bmod \Lambda$ for some $x_1, x_2 \in \mathbf{R}$. Subtracting a suitable coboundary from a function t allows us to assume that $t(-s_1) = ap\alpha_1$, $t(s_2) = b\bar{q}p\omega_1/2$ for some $a, b \in \mathbf{R}$. The cocycle conditions require that $2t(-s_1) = 2ap\alpha_1 \in \Lambda$, i.e. $2a \in \mathbf{Z}$, and $b\bar{q}p\omega_1 \in \Lambda$. Using the fact that

$$\begin{aligned}
\{m \in \mathbf{R} | mp\alpha_1 &= x\omega_2 \bmod \Lambda \quad \text{for some } x \in \mathbf{R}\}\\
&= \tfrac{1}{2}\mathbf{Z} \quad \text{if } \Lambda = \Lambda_{p,q}, \text{ with } \bar{p} \text{ odd or } p, \bar{p} \text{ even and}\\
&\qquad 4 \nmid \bar{q}, \text{ or } \Lambda = \Lambda_{p,q}'.\\
&= \mathbf{Z} \quad \text{if } \Lambda = \Lambda_{p,q} \text{ with } p, \bar{p} \text{ even and } 4|\bar{q}.
\end{aligned}$$

the conclusion follows.

In cases (a) and (b) of Proposition 5.1, the action of $N(K, \Lambda)$ on $H^1(K, V/\Lambda)$ is necessarily trivial. In case (c), using Propositions 2.1, 3.5 and 3.6, one sees that $N(K, \Lambda) = K$ except for the following cases:

$$(8) \qquad \text{(i) } K = W(n), n + 2 = p^2, \Lambda = \Lambda_{p,q}, \quad \text{when } N(K, \Lambda) = \pm K \cup \pm K\phi,$$
$$\text{where } \phi(p\alpha_1) = \alpha_2, \phi(\alpha_2) = p\alpha_1.$$

$$(9) \qquad \text{(ii) } K = W(-n), n - 2 = 4q^2, \Lambda = \Lambda_{p,q}, \quad \text{when } N(K, \Lambda) = \pm K \cup \pm K\zeta,$$
$$\text{where } \zeta(p\alpha_1) = \bar{q}p\omega_1/2, \zeta(\alpha_2) = -\bar{q}\omega_2/2.$$

In these exceptional cases, the element ϕ or ζ interchanges the values of the coefficients a and b in the above construction of the cocycles, so that there are 3 orbits in $H^1(K, V/\Lambda)$. Otherwise, the action of $N(K, \Lambda)$ is again trivial.

Applying Proposition 1.2 of [**7**], we deduce

4.2 PROPOSITION. *Suppose* $K = \pm W(n)$. *Then*
(a) $H^1(K, V/\Lambda) = 0$ *if* $\Lambda = \Lambda_{p,q}$, *with* n *odd.*
(b) $H^1(K, V/\Lambda) \cong \mathbf{Z}/2\mathbf{Z}$ *if* $\Lambda = \Lambda_{p,q}'$.
(c) $H^1(K, V/\Lambda) \cong (\mathbf{Z}/2\mathbf{Z})^2$ *if* $\Lambda = \Lambda_{p,q}$, *with either* $n = 0 \bmod 4$, *or* $n = 2 \bmod 4$ *and* $\bar{p}$ *odd.*
(d) $H^1(K, V/\Lambda) \cong (\mathbf{Z}/2\mathbf{Z})^4$ *if* $\Lambda = \Lambda_{p,q}$, *with* $n = 2 \bmod 4$, p *and* $\bar{p}$ *even and* $4|\bar{q}$.

By counting the lattices in Table 1 and the inequivalent space groups corresponding to them, we obtain

4.3 PROPOSITION. *Suppose* $n > 0$ *and* $n^2 - 4 = 2^\nu N$, *with* N *odd. Let* A_n *and* B_n *be as defined in Table 2 and also let* C_n^+, C_n^- *be as given there provided that* $n + 2$ *or* $n - 2$ *is, respectively, a square and equal to zero otherwise. Then the number of inequivalent space groups with point group* $W(n)$, $W(-n)$ *or* $\pm W(n)$ *is equal, respectively, to* $A_n\phi(N) + C_n^+$, $A_n\phi(N) + C_n^-$ *or* $B_n\phi(N) + C_n^+ + C_n^-$, *where* ϕ *denotes the Euler* ϕ*-function.*

TABLE 2

n	A_n	B_n	C_n^+	C_n^-
$n = 1 \bmod 2$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}\phi(n-2)$	$\frac{1}{2}\phi(n+2)$
$n = 0 \bmod 4$	2	2	—	—
$n = \pm 6 \bmod 16$	8	12	$\phi((n-2)/4)$	$\phi((n+2)/4)$
$n = 2 \bmod 16$	$(5\nu - 9)/2$	$(9\nu - 21)/2$	$\phi((n-2)/4)$ $+\phi((n-2)/16)$	$3\phi((n+2)/4)/2$
$n = -2 \bmod 16$	$(5\nu - 9)/2$	$(9\nu - 21)/2$	$3\phi((n-2)/4)/2$	$\phi((n+2)/4)$ $+\phi((n+2)/16)$

For the remaining point groups, we have

4.4 PROPOSITION. (a) *If* $K = W(n)^+$ *or* $W(-n)^+$ *and* Λ *is an invariant lattice, then* $H^1(K, V/\Lambda) = 0$.
(b) *If* $K = \pm W(n)^+$ *and* Λ *is an invariant lattice, then* $H^1(K, V/\Lambda) = 0$ *if* $n = 1 \bmod 2$ *and* $H^1(K, V/\Lambda) \cong \mathbf{Z}/2\mathbf{Z}$ *if* $n = 0 \bmod 4$. *If* $n = 2 \bmod 4$, *let* d_1 *and* d_2 *be the invariant factors of the sublattice* $(1 - s_1s_2)(\Lambda)$ *in* Λ, *with* $d_1|d_2$. *Then* $H^1(K, V/\Lambda) \cong \mathbf{Z}/2\mathbf{Z}$ *if* d_1 *is odd and* $H^1(K, V/\Lambda) \cong (\mathbf{Z}/2\mathbf{Z})^2$ *if* d_1 *is even.*

Proof. In case (a), the group K is infinite cyclic with a generator g such that $1 - g$ is invertible; hence $H^1(K, V/\Lambda) = 0$. Applying Proposition 1.2 of [**7**], we deduce that in case (b), $H^1(K, V/\Lambda)$ is isomorphic to the quotient of (Λ^*/Λ) by $2(\Lambda^*/\Lambda)$. Since $\Lambda^* = (1 - s_1s_2)^{-1}(\Lambda)$, we have $\Lambda^*/\Lambda \cong \Lambda/(1 - s_1s_2)(\Lambda) \cong \mathbf{Z}/d_1\mathbf{Z} \oplus \mathbf{Z}/d_2\mathbf{Z}$, where d_1 and d_2 are the invariant factors of $(1 - s_1s_2)(\Lambda)$ in Λ, with $d_1|d_2$ and $d_1d_2 = |\det (1 - s_1s_2)| = n - 2$. If n is odd, both d_1 and d_2 are odd, so that $H^1(K, V/\Lambda) = 0$. If $n = 0 \bmod 4$, d_1 is odd and d_2 even, so that $H^1(K, V/\Lambda) \cong \mathbf{Z}/2\mathbf{Z}$. Finally, if $n = 2 \bmod 4$, $H^1(K, V/\Lambda)$ is as stated in the proposition.

5. Proper equivalence. The discussion in this section applies to general space groups in affine space and is meant also as a supplement to [**2**] and [**7**].

In certain situations, it is appropriate to consider stronger notions of equivalence for space groups. *Proper* equivalence is defined to be conjugacy in the proper affine group $A^+(E)$, consisting of those elements of $A(E)$ whose linear part has positive determinant. If S is a space group and θ an element of $A(E)\backslash A^+(E)$, S and $\theta S\theta^{-1}$ will be conjugate in $A^+(E)$ if and only if the normaliser $N(S)$ of S in $A(E)$ is *not* contained in $A^+(E)$. In other words:

5.1 PROPOSITION. *An equivalence class $\{S\}$ of space groups in $A(E)$ splits into 2 proper equivalence classes if and only if the normaliser $N(S)$ of S in $A(E)$ is contained in $A^+(E)$.*

The structure of $N(S)$ is described by

5.2 PROPOSITION. *Let Λ, K and $\bar{t}$ be the lattice, point group and cocycle of a space group S in $A(E)$. An element $(b, h) \in A(E)$ belongs to $N(S)$ if and only if $h \in N(K, \Lambda)$ and $h.\bar{t} - \bar{t} = \delta_{h(b)}$, where $\delta_{h(b)}$ denotes the coboundary $K \to V/\Lambda$ corresponding to the element $h(b)$.*

Proof. Since $S = \{(a + t(g), g)|a \in \Lambda, g \in K\}$ and

$$(b, h)^{-1}(a + t(g), g)(b, h) = (b - h^{-1}gh(b) + h^{-1}(a) + h^{-1}t(g), h^{-1}gh)$$

the element (b, h) will be in $N(S)$ if and only if $h \in N(K)$ and

$$(10) \quad t(h^{-1}gh) = b - h^{-1}gh + h^{-1}(a) + h^{-1}t(g) \mod \Lambda.$$

Taking $g = 1$, we see that $h \in N(K, \Lambda)$ and, on multiplying by h, (10) simplifies to the statement that $h.\bar{t} - \bar{t} = \delta_{h(b)}$.

5.3 COROLLARY. *The projection of $N(S)$ on $GL(V)$ equals the stabiliser of the cohomology class of $\bar{t}$ in $H^1(K, V/\Lambda)$ under the action of $N(K, \Lambda)$. The intersection of $N(S)$ with V equals Λ^*.*

5.4 COROLLARY. *If $N(S) \subset A^+(E)$, then $K \subset GL^+(V)$ and the cohomology class of $\bar{t}$ is not of order $\leqq 2$ in $H^1(K, V/\Lambda)$ if* dim *V is odd.*

Proof. The first statement is clear from Corollary 6.3, while the second holds since otherwise $-1_V \in N(K, \Lambda)$ would stabilise the class of $\bar{t}$ and have negative determinant.

In the three dimensional Euclidean case, looking at the table in [**2**], one sees that there are 11 cases which satisfy the conditions of Corollary 5.4. Splitting in fact occurs in all of these cases. In the two-dimensional Euclidean case, the stabiliser of the class of $\bar{t}$ for a point group $K \subset GL^+(V)$ is the entire group of orthogonal symmetries of the lattice Λ, which always contains reflections, so that splitting never happens.

In the situation of this paper, splitting can only happen if $K = W(n)^+$, $W(-n)^+$ or $\pm W(n)^+$, in view of Corollary 6.4. The group $N(K, \Lambda)$ is then the group $S(\Lambda)$ of those similarities which leave Λ invariant. From Proposition 5.4, we see that splitting occurs if and only if $S(\Lambda)$ contains only similarities of positive determinant (not to be confused with 'direct' similarities), except possibly in case (d) of that proposition (with d_1 even), if the similarities of negative determinant in $S(\Lambda)$ do not stabilise the class of $\bar{t}$.

6. Quadratic forms. We recall briefly the correspondence between lattices and quadratic forms. Let V be an n-dimensional real vector space and $(.\,,.)$ a nondegenerate symmetric bilinear form on V of signature $(m, n - m)$. Given a lattice Λ in V, we associate to Λ the class $\{Q_B\}$ of (integrally) equivalent real quadratic forms of signature $(m, n - m)$, where $B = \{u_1, \ldots, u_n\}$ varies over the bases of Λ and $Q_B = \sum_{i,j}(u_i, u_j)X_iX_j$. Conversely, given a quadratic form $Q = \sum_{i,j} q_{ij}X_iX_j$ of signature $(m, n - m)$, there exist vectors $u_1, \ldots, u_n \in V$ such that $(u_i, u_j) = q_{ij}$ and we associate the lattice Λ spanned by $u_1, \ldots, u_n$ to Q. A different choice of $u_1, \ldots, u_n$ produces a lattice of the form $\theta(\Lambda)$ for some θ in $O(V)$, the orthogonal group of V. This establishes a bijective correspondence between classes of orthogonally equivalent lattices and classes of equivalent quadratic forms. It is convenient to extend the concept of equivalence for quadratic forms by considering all positive multiples of a form Q to be equivalent to Q. In terms of lattices, this means that one should consider equivalence with respect to the group $H.O(V)$, where H is the group of homotheties of V. When $2m = n$, the form $-Q$ is also of signature $(m, n - m)$; the corresponding lattices are related by a similarity with multiplier -1. Since such similarities exist only when $2m = n$, if we consider $-Q$ to be equivalent to Q in that situation, we obtain in all cases a bijective correspondence between classes of lattices equivalent with respect to the group of similarities $S(V)$ and classes of equivalent quadratic forms in the broader sense. We will adopt this point of view in the following discussion.

Let Aut (Λ) be the group of all orthogonal automorphisms of a lattice Λ in V. If $\Lambda_2 = \theta(\Lambda_1)$ for some $\theta \in S(V)$, then Aut $(\Lambda_2) = \theta$ Aut $(\Lambda_1)\theta^{-1}$, so that the conjugacy class of Aut (Λ) in $S(V)$ is an invariant of Λ. Suppose that we have determined the conjugacy classes in $S(V)$ of subgroups of the form Aut (Λ) for some lattice Λ. It is then sufficient to consider, for each such class $\{K\}$, the set $L(K)$ of lattices Λ in V such that Aut $(\Lambda) = K$. Members Λ_1 and Λ_2 of $L(K)$ will be equivalent if and only if $\Lambda_2 = \theta(\Lambda_1)$ for some θ in $N(K)$, the normaliser of K in $S(V)$. The remaining problem, therefore, is to find the orbits in $L(K)$ under the action of $N(K)$.

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We now return to the special situation of this paper. Denote the quadratic form $Q = AX_1^2 + BX_1X_2 + CX_2^2$ by (A, B, C) and let $D = B^2 - 4AC$ be its discriminant. We must have $D > 0$ in order for Q to be of signature $(1, 1)$. A particular lattice corresponding to Q is spanned by the vectors $\{u_1, u_2\}$, where

$$\sqrt{2}\, u_1 = e_1 + Ae_2, \quad \sqrt{2}\, u_2 = (B + \sqrt{D})/2A\ e_1 + (B - \sqrt{D})/2\ e_2.$$

The matrix of a rotation

(11) $\rho = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \quad (a + a^{-1} = n > 0, a \geqq 1)$

with respect to the basis $\{u_1, u_2\}$ is

(12) $\begin{bmatrix} \frac{1}{2}(n - Bk) & -Ck \\ Ak & \frac{1}{2}(n + Bk) \end{bmatrix},$

where $k \geqq 0$ and

(13) $n^2 - Dk^2 = 4.$

Since -1_V leaves every lattice invariant, Proposition 1.2 shows that a group of the form Aut (Λ) is conjugate in V to either $\pm W(n)$ or $\pm W(n)^+$ for some integer $n \geqq 2$. If $n > 2$, we have $k \neq 0$ in (13) and Ak, Bk and Ck must all be in $\mathbf{Z}$ in order for (12) to represent an automorphism of Λ. Furthermore, Dk^2 cannot be a square in $\mathbf{Z}$, since otherwise (13) would be impossible. Therefore Q is equivalent to a form with *integral* coefficients and a non-square discriminant; we may also suppose that this form is *primitive*, i.e. that its coefficients have no common factor. Conversely, since the Pell equation (13) is solvable (with $k \neq 0$) when D is the discriminant of such a form, a lattice corresponding to the form is invariant under a rotation (11) with $n > 2$. We shall from now on restict our attention to such forms.

Consider the case when Aut $(\Lambda) = \pm W(n)$, with $n > 2$. In the literature (e.g. [**6**]), such lattices, and the forms corresponding to them, are called 'ambiguous'. Proposition 3.7 describes a complete set of inequivalent lattices *invariant* under $\pm W(n)$. In order for $\pm W(n)$ to be the *complete* group of automorphisms of Λ, (n, k) must be the *least* positive solution of (13). Given n, there is a finite number of discriminants D for which this is true (since D divides $n^2 - 4$) and which may be said to 'belong' to n. We can pass from the lattices to the corresponding forms and group them according to the value of D, retraining only those D which 'belong' to n. By doing this for $n = 3, 4, 5, \ldots,$ we shall eventually obtain, without repetition, lists of inequivalent ambiguous forms with any allowable discriminant D. Conversely, given D, we can first find to which n it belongs—i.e. the least solution (n, k) of (13)—and then apply the procedure to n.

More specifically, choose the following bases for the lattices $\Lambda_{p,q}$ and $\Lambda_{p,q}'$:

$\Lambda_{p,q}$ (n odd)	$\Lambda_{p,q}$ (n even)	$\Lambda_{p,q}'$
$u_1 = \bar{q}p\omega_1$	$u_1 = \bar{q}p\omega_1/2$	$u_1 = \alpha_2$
$u_2 = (\bar{q}p\omega_1 + \alpha_2)/2$	$u_2 = \alpha_2$	$u_2 = (\bar{q}p\omega_1 + \alpha_2)/2.$

On multiplying the resulting forms by $q/2p$ we obtain, respectively, the forms

$$(14)\quad \begin{aligned} F_{p,q} &= (p\bar{q}, p\bar{q}, \tfrac{1}{4}(p\bar{q} - \bar{p}q)), \\ G_{p,q} &= (\tfrac{1}{4}p\bar{q}, 0, -\bar{p}q), \\ H_{p,q} &= (-\bar{p}q, -\bar{p}q, \tfrac{1}{4}(p\bar{q} - \bar{p}q)). \end{aligned}$$

All of these forms are integral with discriminant $n^2 - 4$; however, some may not be primitive. If we isolate those for which the *GCD* of the coefficients equals k, then upon dividing them by k we shall obtain the *primitive* forms with discriminant D.

Since $(n + 2)(n - 2) = Dk^2$, we can express D and k in the form

$$(15)\quad \begin{aligned} D &= 2^{\delta}D_+D_- \quad (D_+|n + 2, D_-|n - 2; D_+ \text{ and } D_- \text{ odd}) \\ k &= 2^{1/2(\nu-\delta)}k_+k_- \quad (k_+^2|n + 2, k_-^2|n - 2; k_+ \text{ and } k_- \text{ odd}), \end{aligned}$$

where 2^{ν} is the largest power of 2 dividing $n^2 - 4$.

6.1 PROPOSITION. *A complete set of inequivalent primitive ambiguous integral quadratic forms with a non-square discriminant $D > 0$ is listed in Table 3, where*

$$(16)\quad p = 2^{\alpha}k_+p_0, \quad q = 2^{\beta}k_-q_0,$$

with α and β as shown in the table and p_0, q_0 any divisors of D_+ and D_- respectively such that

$$(17)\quad GCD(p_0, \bar{p}_0) = 1, \quad GCD(q_0, \bar{q}_0) = 1,$$

where $\bar{p}_0 = D_+/p_0$, $\bar{q}_0 = D_-/q_0$. (The notation '$p_0 \approx \bar{p}_0$' means that only *one* of the values p_0, $\bar{p}_0$ is to be used in p, etc.)

Proof. It is easy to see that the *odd* part of the *GCD* of any of the forms (14) is equal to the product of the *odd* parts of the *GCD*'s of p and $\bar{p}$ and of q and $\bar{q}$ respectively. Consequently, if this *GCD* is to be equal to k, we can express p and q in the form (16) for certain exponents $\alpha, \beta \geqq 0$ and divisors p_0, q_0 of D_+ and D_- respectively, which satisfy (17).

One now examines various possibilities for n. Consider, for example, the case when $n = -2 \bmod 16$. Then $2^{\nu-2}$ and 2^2 are the highest powers of 2 dividing $n + 2$ and $n - 2$, respectively, and $\nu \geqq 6$. According to Table 1, we first have the lattices $\Lambda_{p,q}$ with $2 \nmid \bar{p}$, so that $\alpha = \nu - 2$ and $\beta = 0$. Since $\bar{p}q$ is odd, the *GCD* of the form $G_{p,q}$ is odd, so that k must be odd and $\delta = \nu$. Secondly, there are the forms $\Lambda_{p,q}$ with $2|p$, $2|\bar{p}$ and $4|\bar{q}$, so that $\beta = 0$. The highest power of 2 dividing the *GCD* of the form $G_{p,q}$ is

$$\min\ (\alpha, \nu - 2 - \alpha) \leqq \tfrac{1}{2}(\nu - 2),$$

so that $\delta \geqq 2$. This power should equal $\frac{1}{2}(\nu - \delta)$. Since $p \approx \bar{p}$ in this case, we

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can choose $\alpha = \frac{1}{2}(\nu - \delta)$, which eliminates $\bar{p}$ from further discussion, except in the case $\delta = 2$ since $\bar{p}$ is then also divisible by exactly the same power of 2 as p; we must then insist that $p_0 \approx \bar{p}_0$. The condition $q \approx \bar{q}/4$ translates in both cases into $q_0 \approx \bar{q}_0$. Since $2|p$, we have $\frac{1}{2}(\nu - \delta) \geqq 1$, i.e. $\delta \leqq \nu - 2$.

Finally, there are the lattices $\Lambda_{p,q}'$ with $4|p$, $4|\bar{p}$, $2|q$ and $2|\bar{q}$, so that $\beta = 1$. The highest power of 2 dividing the *GCD* of the form $H_{p,q}$ equals

$$\min (\alpha - 1, \nu - \alpha - 3) < \tfrac{1}{2}(\nu - 4)$$

if $\alpha \neq \frac{1}{2}(\nu - 2)$ and to either $\frac{1}{2}(\nu - 2)$ or $\frac{1}{2}\nu$ if $\alpha = \frac{1}{2}(\nu - 2)$, according to whether $D_+D_- = 3 \bmod 4$ of $D_+D_- = 1 \bmod 4$. Correspondingly, we have $\delta \geqq 5$, $\delta = 2$ or $\delta = 0$. In the last two cases, $\bar{p}$ is divisible by the same power of 2 as p, so that we must require that $p_0 \approx \bar{p}_0$ as well as $q_0 \approx \bar{q}_0$. In the first case, we can let $\alpha = \frac{1}{2}(\nu - \delta) + 1$, which eliminates $\bar{p}$, so that only $q_0 \approx \bar{q}_0$ is needed. Since $4|p$, we have $\frac{1}{2}(\nu - \delta) + 1 \geqq 2$, i.e. $\delta \leqq \nu - 2$.

To see that $D_+ > 1$ when $\delta = 0$ or 2, we argue as follows. If $D_+ = 1$, we could write $n + 2 = 2^{\nu-2}k_+^2$ and $n - 2 = D(2k_-)^2$ or Dk_-^2, according to whether $\delta = 0$ or $\delta = 2$. Therefore either $(2^{1/2(\nu-2)}k_+)^2 - D(2k_-)^2 = 4$ or $(2^{1/2(\nu-2)}k_+)^2 - Dk_-^2 = 4$, in both cases contradicting the minimality of n.

Since $n - 2$ is not a square, we always have $D > 1$.

7.2 *Remark*. Let us consider for a moment the more usual concept of *strict* equivalence of lattices (or the corresponding forms), namely equivalence with respect to the group $H.O^+(V)$. If a form Q corresponds to a lattice Λ, then $-Q$ corresponds to the lattice $\mu(\Lambda)$. In view of Proposition 2.1, we can read off from Table 1 the 'negatives' of the lattices $\Lambda_{p,q}$ and $\Lambda_{p,q}'$ by comparing the columns for $K = W(n)$ and $K = \pm W(n)$. Comparing this with Table 3, we deduce that the forms listed there are strictly equivalent to their negatives if and only if '$q_0 \approx \bar{q}_0$' appears in the last column *and* $D_- = 1$. In the other cases, one should add the negative of the form in order to obtain a complete set of representatives for strict equivalence classes. We also observe that $D_+ > 1$ whenever '$p_0 \approx \bar{p}_0$' appears in the last column.

Let $\pi(x)$ denote the number of distinct prime divisors of an integer $x > 0$. The number of divisors d of x such that $GCD(d, x/d) = 1$ is then equal to $2^{\pi(x)}$. Using this fact and Remark 6.2, we deduce the following well-known

6.3 COROLLARY. *Let $\pi(D)$ be the number of distinct prime divisors of D and 2^δ the highest power of 2 dividing D. The number of strictly inequivalent primitive ambiguous integral quadratic forms with a non-square discriminant $D > 0$ is equal to $2^{\pi(D)}$ if $\delta \geqq 5$ and to $2^{\pi(D)-1}$ if $\delta \leqq 4$, unless $\delta = 2$ and $D/4 = 1 \bmod 4$, when this number is $2^{\pi(D)-2}$.*

We can also supplement this with

6.4 COROLLARY. *Either all or none of such forms are strictly equivalent to their negatives. More precisely, the answer is 'none' unless $D_- = 1$ and either $\delta = 0$, $\delta = 2$ and $n = 2 \bmod 16$, or $\delta = 3$ and $n = 6 \bmod 16$. (This is equivalent to the existence of a solution of the equation $x^2 - Dy^2 = -4$.)*

TABLE 3

δ	D	α	β	Q	p_0, q_0
$n = 1 \bmod 2.$					
0	$D_+D_- = 1 \bmod 4, D_+ > 1$	0	0	$k^{-1}F_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
$n = 0 \bmod 4.$					
2	$D_+D_- = 3 \bmod 4$	1	0	$k^{-1}G_{p,q}$	$(p_0, q_0) \approx (\bar{p}_0, \bar{q}_0)$
$n = 6 \bmod 16.$					
5	—	3	0	$k^{-1}G_{p,q}$	—
3	—	2	0	$k^{-1}G_{p,q}$	$q_0 \approx \bar{q}_0$
$n = -6 \bmod 16.$					
5	—	2	0	$k^{-1}G_{p,q}$	—
3	$D_+ > 1$	1	0	$k^{-1}G_{p,q}$	$p_0 \approx \bar{p}_0$
$n = 2 \bmod 16.$					
ν	—	2	0	$k^{-1}G_{p,q}$	—
$3 \leqq \delta \leqq \nu - 2$	$D_+ > 1$	1	$\frac{1}{2}(\nu - \delta) - 1$	$k^{-1}G_{p,q}$	$p_0 \approx \bar{p}_0$
2	$D_+ > 1$	1	$\frac{1}{2}(\nu - 2) - 1$	$k^{-1}G_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
$5 \leqq \delta \leqq \nu - 2$	$D_+ > 1$	1	$\frac{1}{2}(\nu - \delta) + 1$	$k^{-1}H_{p,q}$	$p_0 \approx \bar{p}_0$
2	$D_+D_- = 3 \bmod 4, D_+ > 1$	1	$\frac{1}{2}(\nu - 2)$	$k^{-1}H_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
0	$D_+D_- = 1 \bmod 4, D_+ > 1$	1	$\frac{1}{2}(\nu - 2)$	$k^{-1}H_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
$n = -2 \bmod 16.$	$(D_- > 1$ in all cases)				
ν	—	$\nu - 2$	0	$k^{-1}G_{p,q}$	—
$3 \leqq \delta \leqq \nu - 2$	—	$\frac{1}{2}(\nu - \delta)$	0	$k^{-1}G_{p,q}$	$q_0 \approx \bar{q}_0$
2	$D_+ > 1$	$\frac{1}{2}(\nu - 2)$	0	$k^{-1}G_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
$5 \leqq \delta \leqq \nu - 2$	—	$\frac{1}{2}(\nu - \delta) + 1$	1	$k^{-1}H_{p,q}$	$q_0 \approx \bar{q}_0$
2	$D_+D_- = 3 \bmod 4, D_+ > 1$	$\frac{1}{2}(\nu - 2)$	1	$k^{-1}H_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$
0	$D_+D_- = 1 \bmod 4, D_+ > 1$	$\frac{1}{2}(\nu - 2)$	1	$k^{-1}H_{p,q}$	$p_0 \approx \bar{p}_0, q_0 \approx \bar{q}_0$

Little of a systematic nature is known about lattices for which Aut $(\Lambda) = \pm W(n)^+$ for some $n > 2$. We simply remark that the problem, raised in Section 4, of classifying such lattices for the purpose of finding inequivalent space groups is the same as the problem of classifying them for the needs of the present Section, since the normaliser of $\pm W(n)^+$ is all of $S(V)$. This may not be true for lattices invariant under $\pm W(n)$ if the corresponding primitive form has a discriminant D which divides $n^2 - 4$ but does not 'belong' to n.

Finally, we remark without further elaboration that similar ideas in two dimensional Euclidean space also lead to a complete classification of ambiguous forms with discriminant $D < 0$. Corollary 6.3 remains true for such forms, as is well known.

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