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LARGE CHAOS IN SMOOTH FUNCTIONS OF ZERO TOPOLOGICAL ENTROPY

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For any $\alpha \in [0, 1)$, examples of C^∞ functions $f_\alpha : [0, 1] \to [0, 1]$ with zero topological entropy and possessing a δ-scrambled set of Lebesgue measure α are given. This answers a question posed by Smítal.

1. INTRODUCTION

In 1975, the notion of chaos in the sense of Li and Yorke was introduced [**9**]. An equivalent and simpler formulation of this concept has been given in [**8**].

DEFINITION: Let I denote a compact real interval and let $f : I \to I$ be a continuous function. Suppose that there exist $\delta \geqslant 0$ and $S \subset I$ with at least two elements such that for any x, $y \in S$, $x \neq y$, and any periodic point p of f:

(i) $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta$,
(ii) $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,
(iii) $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > \delta$.

(Here f^n is the nth iterate of f and a point $p \in I$ is said to be periodic if there exists a positive integer r such that $f^r(p) = p$; the least integer satisfying this property is called the period of p.) Then f is said to be chaotic in the sense of Li and Yorke and S is called a scrambled set of f, or (when $\delta > 0$) a δ-scrambled set.

It can be proved (see for example [**6**]) that any function possessing a periodic point with period $\neq 2^n$, $n = 0, 1, 2, \ldots$ is chaotic in the sense of Li and Yorke. On the other hand, recall Šarkovskii's Theorem [**16**]: *Let I be a compact real interval, $f : I \to I$ a continuous function and order the positive integers as follows:* $3 \ll 5 \ll 7 \ldots \ll 2\cdot 3 \ll 2\cdot 5 \ll 2\cdot 7 \ll \ldots \ll 2^2\cdot 3 \ll 2^2\cdot 5 \ll 2^2\cdot 7 \ll \ldots \ll \ldots \ll 2^n \ll \ldots \ll 2^3 \ll 2^2 \ll 2 \ll 1$. *Then if f has a periodic point of period r, it also has a periodic point of period s for any $r \ll s$.* A function with no periodic points of period not a power of 2 (or equivalently with zero topological entropy; see [**1**] for definition of topological entropy and [**3, 11**] for equivalence) can be chaotic or not [**15**].

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I am grateful to Professor Smítal for useful suggestions in order to consider the zero topological entropy case.

A rather natural question about chaotic functions is whether this chaos can be "physically" observed. For example (see [**5**, p.119]) the function $f\colon [-1, 1] \to [-1, 1]$ such that $f(x) = 1 - \alpha x^2$, where $\alpha = 1.75487\ldots$ is the root of the equation $1 - \alpha(1-\alpha)^2 = 0$, satisfies for almost every x

$$\lim_{n\to\infty} |f^n(x) - f^n(0)| = 0,$$

where $f(0) = 1$, $f(1) = 1 - \alpha$, $f(1-\alpha) = 0$.

An obvious way to get large chaos is to construct scrambled sets of positive Lebesgue measure; in fact in the last few years a number of examples with functions having a scrambled set of positive measure [**7, 14**] or even full measure [**4, 12**] have been published. Nevertheless, they all are strongly non-differentiable. In [**14**] the problem of finding differentiable functions with a finite number of pieces of monotonicity and generating a scrambled set of positive measure remained open. The aim of this paper is to give examples of weakly unimodal C^∞ maps (see below) with scrambled sets of positive measure; in fact it is possible to obtain such functions with zero topological entropy. (In [**6**] it is stated that the function from [**14**] can be modified to make it of class C^1, but no proof is given. Moreover, that map has positive topological entropy and is not piecewise monotone.)

Given $I = [a, b]$ and a continuous function $f\colon I \to I$, f is said to be *weakly unimodal* (respectively *unimodal*) if there exists $c \in (a, b)$ such that $f/[a, c]$ is increasing (respectively strictly increasing) and $f/[c, b]$ is decreasing (respectively strictly decreasing). In [**13**] examples of weakly unimodal C^∞ functions with zero topological entropy and chaotic in the sense of Li and Yorke are given, but nothing is said about the Lebesgue measure of their scrambled sets. We are going to prove the following

Theorem. *For any $\alpha \in [0, 1)$ there exists a weakly unimodal C^∞ function f_α : $[0, 1] \to [0, 1]$ with zero topological entropy such that it has a δ-scrambled set of the Cantor type with Lebesgue measure α for a certain $\delta > 0$.*

With respect to the notation, given I, J compact real intervals $\psi(I; J)$ — respectively $\overline{\psi}(I; J)$ — will denote the increasing bijective linear function — respectively decreasing — mapping I onto J. Lebesgue measure will be represented by λ.

2. Some auxiliary C^∞ functions

Some C^∞ functions that will be useful later are given in this section.

Lemma 1. *Let $0 < a, b \leqslant 1$. Then there exists a function $\varphi = \varphi(a; b)\colon [0, a] \to [0, b]$ of class C^∞ and strictly increasing such that*

$$\varphi(0) = 0, \quad \varphi(a) = b,$$
$$D^k\varphi(0) = D^k\varphi(a) = 0 \text{ for any } k \geqslant 1.$$

Moreover, for any $k \geqslant 1$ *there exists* $\overline{\beta}_k > 0$ *depending only on* k *such that*

$$\left|D^j \varphi(x)\right| < \overline{\beta}_k b/a^k \quad \textit{for any} \quad j = 1, 2, \ldots, k.$$

PROOF: Take

$$\varphi_0(x) = \begin{cases} \exp\{-1/(1-x^2)\} & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise}, \end{cases}$$

a C^∞ function, and define

$$\varphi_2(x) = \frac{\varphi_1(2x-1)}{\int_{-1}^{1} \varphi_0(t)dt},$$

where $\varphi_1(x) = \int_{-1}^{x} \varphi_0(t)dt$. Note that $\varphi_3 = \varphi_2/[0, 1]$ is a strictly increasing C^∞ function such that

$$\varphi_3(0) = 0, \quad \varphi_3(1) = 1;$$
$$D^k\varphi_3(0) = D^k\varphi_3(1) = 0 \quad \text{for any} \quad k \geqslant 1.$$

Now $\varphi(x) = b\varphi_3(x/a)$ is the desired function (choose, for any $k \geqslant 1, \overline{\beta}_k > 0$ such that $\left|D^j\varphi_3(x)\right| < \overline{\beta}_k$ when $j = 1, \ldots, k$ and $x \in [0, 1]$). ∎

LEMMA 2. *Let* $0 \leqslant a, b \leqslant 1, u > 0$. *Then there exist constants* $0 < a_0 < a$, $0 < b_0 \leqslant b$ *and a function* $\chi = \chi(a, u; b)\colon [0, a_0] \to [0, b_0]$ *of class* C^∞ *and strictly increasing such that*

$$\chi(0) = 0, \quad \chi(a_0) = b_0,$$
$$D\chi(0) = 0, \quad D\chi(a_0) = u,$$
$$D^k\chi(0) = D^k\chi(a_0) = 0 \quad \textit{for any} \quad k \geqslant 2.$$

Moreover, for any $k \geqslant 1$ *there exists* $\overline{\overline{\beta}}_k > 0$ *depending only on* k *such that*

$$\left|D^j\chi(x)\right| < \overline{\overline{\beta}}_k u/c^k \quad \textit{for any} \quad j = 1, 2, \ldots, k,$$

where $c = \min\{a, b/u\}$.

PROOF: Consider the function

$$\chi_0(x) = \int_0^x \varphi_3(t)dt,$$

where φ_3 is defined in Lemma 1. Denote $\chi_0(1) = \varepsilon$. Observe that

$$\chi_0(0) = 0, \quad \chi_0(1) = \varepsilon;$$
$$D\chi_0(0) = 0, \quad D\chi_0(1) = 1;$$
$$D^k\chi_0(0) = D^k\chi_0(1) = 0 \quad \text{for any} \quad k \geqslant 2.$$

If we take $a_0 = \min\{a, b/(u\varepsilon)\}$ and $b_0 = a_0 u\varepsilon$, obviously $0 < a_0 \leqslant a$, $0 < b_0 \leqslant b$. Now it is sufficient to define for any $x \in [0, a]$

$$\chi(x) = a_0 u \chi_0(x/a_0)$$

(take, for any $k \geqslant 1$, $\overline{\overline{\beta}}_k > 0$ such that $\left|D^j\chi_0(x)\right| < \overline{\overline{\beta}}_k$ for any $j = 1, 2, \dots, k$ and $x \in [0, 1)$. $\square$

Lemma 3. *Let $0 \leqslant a < b \leqslant 1$, $0 \leqslant c < d \leqslant 1$ (respectively $0 \leqslant d < c \leqslant 1$), $u \geqslant 0$, $v \geqslant 0$ (respectively $u \leqslant 0$, $v \leqslant 0$). Then there exists $\phi = \phi(a, b, u, v; c, d)\colon [a, b] \to [c, d]$ (respectively $\overline{\phi} = \overline{\phi}(a, b, u, v; c, d)\colon [a, b] \to [d, c]$) of class C^∞ and strictly increasing (respectively strictly decreasing) such that*

$$\phi(a) = c, \quad \phi(b) = d,$$
$$D\phi(a) = u, \quad D\phi(b) = v,$$
$$D^k\phi(a) = D^k\phi(b) = 0 \quad \text{for} \quad k \geqslant 2$$

(similarly for $\overline{\phi}$). Moreover, for any $k \geqslant 1$ there exists β_k depending only on k such that

$$\left|D^j\phi(x)\right| < \beta_k w/e^k \quad \text{for any} \quad j = 1, 2, \dots, k,$$

where $e = \min\{|b-a|, |d-c|/|u|, |d-c|/|v|\}$, $w = \max\{|u|, |v|, |d-c|\}$ (similarly for $\overline{\phi}$); we define $y/0 = \infty$ and $y < \infty$ for any y.

Proof: We will consider the case $0 \leqslant a < b \leqslant 1$, $0 \leqslant c < d \leqslant 1$, $u > 0$, $v > 0$; in the other situations the proof is similar.

Let $p = (b-a)/3$, $q = (d-c)/3$ and consider the functions

$$\overline{\chi} = \chi(p, u; q)\colon [0, p_0] \to [0, q_0] \quad \text{and} \quad \overline{\overline{\chi}} = \chi(p, v; q)\colon [0, p_1] \to [0, q_1].$$

Take $a_0 = a + p_0$, $b_0 = b - p_1$, $c_0 = c + q_0$, $d_0 = d - q_1$, define $\varphi = \varphi(b_0 - a_0; d_0 - c_0)$ and now construct

$$\phi(x) = \begin{cases} c_0 - \overline{\chi}(a_0 - x) & \text{if } x \in [a, a_0]; \\ c_0 + \varphi(x - a_0) & \text{if } x \in [a_0, b_0]; \\ d_0 + \overline{\overline{\chi}}(x - b_0) & \text{if } x \in [b_0, b]. \end{cases}$$

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Using Lemmas 1 and 2 it is easy to check that ϕ is the desired function (take $\beta_k = 3^k \max\{\overline{\beta}_k, \overline{\overline{\beta}}_k\}$ for any k). ∎

LEMMA 4. *Let* $0 \leqslant a, b \leqslant 1$, $0 < u$. *Then there exists* $\theta = \theta(a, u; b)\colon [0, a] \to [0, b]$ *of class* C^∞ *and strictly increasing such that*

$$\theta(0) = 0, \quad \theta(a) = b,$$
$$D\theta(a) = u \quad \text{and} \quad D\theta(x) > 0 \quad \text{for any} \quad x \in [0, a],$$
$$D^k\theta(a) = 0 \quad \text{for any} \quad k \geqslant 2.$$

PROOF: Modifying suitably the function φ_3 from Lemma 1, it is not difficult to get a strictly positive C^∞ function $\theta_0\colon [0, a] \to \mathbb{R}$ such that $\theta_0(a) = u$, $D^k\theta_0(0) = D^k\theta_0(a) = 0$ for any $k \geqslant 1$ and

$$\int_0^a \theta_0(t)dt = b.$$

Of course, $\theta(x) = \int_0^x \theta_0(t)dt$ is the required function. ∎

LEMMA 5. *Let* $I = [a, b]$ *and* $J = [c, d]$ *be compact subintervals of* $(0, 1)$. *Then there exists a strictly increasing diffeomorphism* $h = h(I; J)\colon [0, 1] \to [0, 1]$ *of class* C^∞, *such that* $h/I = \psi(I; J)$.

PROOF: It is sufficient to take

$$h(x) = \begin{cases} \theta(a, (d-c)/(b-a); c)(x) & \text{if } x \in [0, a]; \\ \psi(I; J)(x) & \text{if } x \in [a, b]; \\ 1 - \theta(1-b, (d-c)/(b-a); 1-d)(1-x) & \text{if } x \in [b, 1]. \end{cases}$$

∎

3. ADMISSIBLE FUNCTIONS

A key role in this proof is played by the so called *admissible functions.*

A continuous function $g\colon [0, 1] \to [0, 1]$ is said to be *admissible* (with *associated parameters* a, b, c, d) if there exist $0 < a < b < c < d < 1$ satisfying the following conditions:

(i) $g(0) = g(1) = 0$, $g(a) = g(b) = c$, $g(c) = b$, $g(d) = a$;

(ii) $g/[0, a]$ is strictly increasing, $g/[a, b]$ is constant, $g/[b, 1]$ is strictly decreasing;

(iii) $g/[c, d]$ is linear.

Lemma 6. *Let g be an admissible function with associated parameters a, b, c, d. Then g has periodic points with periods 1 and 2 but no other periods.*

Proof: Note that $g(0) = 0$, $g(b) = c$, $g(c) = b$ and thus g has periodic points of periods 1 and 2. On the other hand, let p be a periodic point of g. It is obvious that $p \in [0, a) \cup [b, c]$ and since $g/[0, a]: [0, a] \to [0, c]$ is increasing and $g/[b, c]: [b, c] \to [b, c]$ is decreasing, p can only be a periodic point of period 1 or 2. ∎

Let g be an admissible function with associated parameters a, b, c, d and let $\overline{g}: [0, 1] \to [0, 1]$ be a continuous function such that $\overline{g}(0) = \overline{g}(1) = 0$. We define $g * \overline{g}: [0, 1] \to [0, 1]$ such that

$$g * \overline{g}/([0, 1] \setminus (a, b)) = g/([0, 1] \setminus (a, b)) \quad \text{and}$$
$$g * \overline{g}/[a, b] = \psi([0, 1]; [c, d]) \circ \overline{g} \circ \overline{\psi}([a, b]; [0, 1]).$$

Lemma 7. *$g * \overline{g}$ is a well defined continuous function satisfying the following conditions:*

(i) *if there exists a periodic point of $g * \overline{g}$ with period $r > 1$, then r is even and there exists a periodic point of $\overline{g}$ with period $r/2$;*

(ii) *if p is a periodic point of $\overline{g}$ with period s, then $\overline{\psi}([0, 1]; [a, b])(p)$ is periodic point of $g * \overline{g}$ with period $2s$.*

Proof: It is obvious that $g * \overline{g}$ is a well defined continuous function. By definition, $g * \overline{g}([a, b]) \subset [c, d]$ and $g * \overline{g}([c, d]) = [a, b]$; hence, there is no periodic point of $g * \overline{g}$ in $[a, b] \cup [c, d]$ with odd period. Note also that $g * \overline{g}$ does not have periodic points with period greater than 2 in $[0, a) \cup (b, c) \cup (d, 1]$. Moreover

$$\begin{aligned}(g * \overline{g})^2/[a, b] &= (g * \overline{g})/[c, d] \circ (g * \overline{g})/[a, b] \\ &= g/[c, d] \circ \psi([0, 1]; [c, d]) \circ \overline{g} \circ \overline{\psi}([a, b]; [0, 1]) \\ &= \overline{\psi}([c, d]; [a, b]) \circ \overline{\psi}([0, 1]; [c, d]) \circ \overline{g} \circ \overline{\psi}([a, b]; [0, 1]) \\ &= \overline{\psi}([0, 1]; [a, b]) \circ \overline{g} \circ \overline{\psi}([a, b]; [0, 1]),\end{aligned}$$

and then

$$(g * \overline{g})^{2n}/[a, b] = \overline{\psi}([0, 1]; [a, b]) \circ (\overline{g})^n \circ \overline{\psi}([a, b]; [0, 1]) \quad \text{for any} \quad n \geqslant 1. \tag{1}$$

With this it is easy to verify (i) and (ii). ∎

We will now introduce some special admissible functions of class C^∞. So let $a_i = 1/2^i$, $b_i = 1 - 3/2^i$, $c_i = 1 - 1/2^{i-1}$, $d_i = 1 - 1/2^i$ for any $i \geqslant 3$. Also, take $a^i = a_i/3$,

$b^i = 2a_i/3$, $c^i = b_i - 2(b_i - a_i)/(3 \cdot 2^{i+1})$, $d^i = b_i - (b_i - a_i)/(3 \cdot 2^{i+1})$. Until the end of this section we shall consider i fixed. Then for simplicity we shall write $a = a^i$, $b = b^i$, $c = c^i$, $d = d^i$. Finally put $u = (d-c)/(b-a)$, $v = (a_i - b_i)/(d_i - c_i)$. We define an admissible function g_i with associated parameters a_i, b_i, c_i, d_i such that

$$\begin{aligned} g_i/[0,\, a] &= \phi(0,\, a,\, 0,\, u; 0,\, c); \\ g_i/[a,\, b] &= \psi([a,\, b]; [c,\, d]); \\ g_i/[b,\, a_i] &= \phi(b,\, a_i,\, u,\, 0; d,\, c_i); \\ g_i/[b_i,\, c_i] &= \overline{\phi}(b_i,\, c_i,\, 0,\, v; c_i,\, b_i); \\ g_i/[d_i,\, 1] &= \overline{\phi}(d_i,\, 1,\, v,\, 0; a_i,\, 0). \end{aligned}$$

Observe that g_i is a C^∞ function. Moreover, we have the following

Lemma 8. *Let k be a positive integer. Then there exist constants $\kappa_k > 0$, $\vartheta_k = 2 \cdot 4^k$ depending only on k such that*

$$\left|D^j g_i(x)\right| \leqslant \kappa_k (\vartheta_k)^i$$

for any $x \in [0,\, 1]$ and $1 \leqslant j \leqslant k$.

Proof: We will use Lemma 3. Consider $k \geqslant 1$, $1 \leqslant j \leqslant k$ and distinguish the following cases:

(i) Let $x \in [0,\, a]$. Take $e = \min\{a,\, c/u\}$, $w = \max\{c,\, u\}$. Then $e = 1/(3 \cdot 2^i)$ and $w < 1$. Applying Lemma 3 we obtain

$$\left|D^j g_i(x)\right| < \beta_k 3^k (2^k)^i.$$

(ii) Let $x \in [a,\, b]$. Then obviously

$$Dg_i(x) < 1 \quad \text{and} \quad D^j g_i(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(iii) Let $x \in [b,\, a_i]$. Now we take $e = \min\{a_i - b,\, (c_i - d)/u\}$, $w = \max\{c_i - d,\, u\}$. Since we again have $e = 1/(3 \cdot 2^i)$ and $w < 1$,

$$\left|D^j g_i(x)\right| < \beta_k 3^k (2^k)^i.$$

(iv) Let $x \in [a_i,\, b_i]$. Now

$$D^j g_i(x) = 0 \quad \text{for any} \quad j \geqslant 1.$$

(v) Let $x \in [b_i,\, c_i]$. As usual let $e = \min\{c_i - b_i,\, (c_i - b_i)/|v|\}$, $w = \max\{c_i - b_i,\, |v|\}$. We have $e > 1/4^i$ and $w < 2^i$. Therefore,

$$\left|D^j g_i(x)\right| < \beta_k (2 \cdot 4^k)^i.$$

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(vi) Let $x \in [c_i, d_i]$. Then

$$|Dg_i(x)| < 2^i \quad \text{and} \quad D^j g_i(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(vii) Let $x \in [d_i, 1]$. Similarly as in (iv) we have

$$\left|D^j g_i(x)\right| < \beta_k \left(2 \cdot 4^k\right)^i.$$

The lemma follows from (i)-(vii). □

At this point we also fix an open interval P included in $I_3 = [1/24, 1/12]$ and consider $(p_0, p_1) = \psi(I_3; [a, b])(P)$. We define the admissible function $g(i; P)$ as follows (here $q_0 = g_i(p_0)$, $q_1 = g_i(p_1)$, $r_0 = p_0 + 3(p_1 - p_0)/5$, $r_1 = p_1 - (p_1 - p_0)/5$, $y = (r_1 - r_0)/(p_0 - a)$, $z = (q_0 - c)/(r_1 - r_0)$):

$$\begin{aligned}
g(i; P)/[0, a] &= \phi(0, a, 0, y; 0, r_0);\\
g(i; P)/[a, p_0] &= \psi([a, p_0]; [r_0, r_1]);\\
g(i; P)/[p_0, r_0] &= \phi(p_0, r_0, y, z; r_1, c);\\
g(i; P)/[r_0, r_1] &= \psi([r_0, r_1]; [c, q_0]);\\
g(i; P)/[r_1, p_1] &= \phi(r_1, p_1, z, u; q_0, q_1);\\
g(i; P)/[p_1, 1] &= g_i/[p_1, 1].
\end{aligned}$$

Obviously $g(i; P)$ is a $\mathcal{C}^\infty$ function. In addition to this

$$\begin{aligned}
(g(i; P))^2/[a, p_0] &= \psi([a, p_0]; [c, q_0]) \quad \text{and}\\
g(i; P)/[p_1, b] &= \psi([p_1, b]; [q_1, d]).
\end{aligned} \tag{2}$$

With the same notation we can finally construct the admissible $\mathcal{C}^\infty$ function $\overline{g}(i; P)$ in the following manner ($s_0 = p_0 + (p_1 - p_0)/5$, $s_1 = p_0 + 2(p_1 - p_0)/5$):

$$\begin{aligned}
\overline{g}(i; P)/[0, a] &= \phi(0, a, 0, y; 0, s_0);\\
\overline{g}(i; P)/[a, p_0] &= \psi([a, p_0]; [s_0, s_1]);\\
\overline{g}(i; P)/[p_0, s_0] &= \phi(p_0, s_0, y, 1; s_1, r_0);\\
\overline{g}(i, P)/[s_0, s_1] &= \psi([s_0, s_1]; [r_0, r_1]);\\
\overline{g}(i; P)/[s_1, r_0] &= \phi(s_1, r_0, 1, z; r_1, c);\\
\overline{g}(i; P)/[r_0, 1] &= g(i, P)/[r_0, 1].
\end{aligned}$$

Now

$$\begin{aligned}
(\overline{g}(i; P))^3/[a, p_0] &= \psi([a, p_0]; [c, q_0]) \quad \text{and}\\
\overline{g}(i; P)/[p_1, b] &= \psi([p_1, b]; [q_1, d]).
\end{aligned} \tag{3}$$

In a similar way as before we have the following

Lemma 9. *Let k be a positive integer. Then there exists a constant $\kappa(k; P) > 0$ depending only on k and P such that*

$$\left|D^j g(i; P)(x)\right| < \kappa(k; P)(\vartheta_k)^i,$$
$$\left|D^j \overline{g}(i; P)(x)\right| < \kappa(k, P)(\vartheta_k)^i$$

for any $x \in [0, 1]$ and $1 \leqslant j \leqslant k$, where ϑ_k is defined in Lemma 8.

Proof: To simplify the notation we shall write $g = g(i, P)$, $\overline{g} = \overline{g}(i, P)$. Let $\lambda_0 = \lambda(P)/\lambda(I_3)$, $\lambda_1 = \lambda(Q)/\lambda(I_3)$, where Q is the left connected component of $I_3 \setminus P$, and choose $k \geqslant 1$, $1 \leqslant j \leqslant k$. First we shall examine g; we shall use $y < \lambda_0/\lambda_1$, $z < 5/\lambda_0$.

(i) Let $x \in [0, a]$. Take $e = \min\{a, r_0/y\}$, $w = \max\{r_0, y\}$. Note that $e \geqslant \min\{1/(3 \cdot 2^i), \lambda_1/(3\lambda_0 2^i)\}$ and $w < \max\{1, \lambda_0/\lambda_1\}$, Therefore

$$\left|D^j g(x)\right| < \beta_k 3^k \max\{1, \lambda_0/\lambda_1\}^{k+1}(2^k)^i.$$

(ii) Let $x \in [a, p_0]$. Now

$$Dg(x) < \lambda_0/\lambda_1 \quad \text{and} \quad D^j g(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(iii) Let $x \in [p_0, r_0]$. Take $e = \min\{r_0 - p_0, (c - r_1)/y, (c - r_1)/z\}$, $w = \max\{c - r_1, y, z\}$. It is easy to check that $e > 1/(2^i \max\{\lambda_0/\lambda_1, 5/\lambda_0\})$, $w < \max\{\lambda_0/\lambda_1, 5/\lambda_0\}$. Then

$$\left|D^j g(x)\right| < \beta_k \max\{\lambda_0/\lambda_1, 5/\lambda_0\}^{k+1}(2^k)^i.$$

(iv) Let $x \in [r_0, r_1]$. Then

$$Dg(x) < 5/\lambda_0 \quad \text{and} \quad D^j g(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(v) Let $x \in [r_1, p_1]$. We must consider $e = \min\{p_1 - r_1, (q_1 - q_0)/z, (q_1 - q_0)/u\}$, $w = \max\{q_1 - q_0, z, u\}$. Using $q_1 - q_0 > \lambda_0/(3 \cdot 2^{i+2})$ it follows that $e > \lambda_0^2/(5 \cdot 2^{i+2})$ and $w < 5/\lambda_0$, which implies

$$\left|D^j g(x)\right| < \beta_k (5 \cdot 20^k/\lambda_0^{2k+1})(2^k)^i.$$

(vi) Let $x \in [p_1, 1]$. Then $g(x) = g_1(x)$ and we can use here the results (ii)–(v) from Lemma 8.

Now we study $\overline{g}$.

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(vii) Let $x \in [0, a]$. Reasoning as in (i) it is easy to see that

$$\left|D^j\overline{g}(x)\right| < \beta_k 3^k \max\{1, \lambda_0/\lambda_1\}^{k+1}(2^k)^i.$$

(viii) Let $x \in [a, p_0]$. As in (ii)

$$D\overline{g}(x) < \lambda_0/\lambda_1 \quad \text{and} \quad D^j\overline{g}(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(ix) Let $x \in [p_0, s_0]$. Take $e = \min\{s_0 - p_0, (r_0 - s_1)/y, r_0 - s_1\}$, $w = \max\{r_0 - s_1, y, 1\}$. We have $e \geqslant \min\{\lambda_0, \lambda_1\}/(15 \cdot 2^i)$ and $w \leqslant \max\{1, \lambda_0/\lambda_1\}$. Hence

$$\left|D^j\overline{g}(x)\right| < \beta_k \frac{15^k \max\{1, \lambda_0/\lambda_1\}}{\min\{\lambda_0, \lambda_1\}^k}(2^k)^i.$$

(x) Let $x \in [s_0, s_1]$. Then

$$D\overline{g}(x) = 1 \quad \text{and} \quad D^j\overline{g}(x) = 0 \quad \text{for any} \quad j \geqslant 2.$$

(xi) Let $x \in [s_1, r_0]$. Take $e = \min\{r_0 - s_1, c - r_1, (c - r_1)/z\}$, $w = \max\{c - r_1, 1, z\}$. Thus $e > \lambda_0/(5 \cdot 2^i)$ and $w < 5/\lambda_0$. So

$$\left|D^j\overline{g}(x)\right| < \beta_k(5^{k+1}/\lambda_0^{k+1})(2^k)^i.$$

(xii) Let $x \in [r_0, 1]$. Since $\overline{g}(x) = g(x)$, we can use (iv)–(vi).

Using (i)–(xii), Lemma 9 is proved. ∎

4. Construction of f_α

We want to define f_α, where $\alpha \in [0, 1)$ is fixed. For this purpose let $S \subset I_3$ be a Cantor type set of Lebesgue measure $\alpha\xi/24$, with $\xi > 1$ and $\alpha\xi < 1$. The problem will be solved if we define a weakly unimodal C^∞ function $f\colon [0, 1] \to [0, 1]$ with zero topological entropy with S a δ-scrambled set for f, since then it is sufficient to construct (using Lemma 5) an increasing C^∞ diffeomorphism $h\colon [0, 1] \to [0, 1]$ mapping I_3 linearly onto an interval of length $1/\xi$ and consider $f_\alpha = h \circ f \circ h^{-1}$.

Before we construct f we need some notation. First of all consider $J_{i,i-1} = [a_{i-1}, b_{i-1}]$ for any $i > 3$ and by induction let $J_{i,j} = \overline{\psi}([0, 1]; [a_j, b_j])(J_{i,j+1})$ for any $3 \leqslant j < i - 1$; we shall also write $J_{i,3} = J_i$. We complete with $J_3 = [0, 1]$. Observe that

$$(4) \qquad J_{i+1} \subset J_i \quad \text{for any} \quad i$$

and

$$(5) \qquad J_{l,m} \subset [a_m, b_m] \quad \text{for any} \quad l \quad \text{and} \quad m.$$

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In a similar way let $K_{i,\,i-1} = [c_{i-1},\, d_{i-1}]$ for any $i > 3$ and by induction let $K_{i,\,j} = \psi([0,\, 1]; [c_j,\, d_j])(K_{i,\,j+1})$ for any $3 \leqslant j < i-1$. We shall write $K_{i,\,3} = K_i$ and $K_3 = [0,\, 1]$. As before $K_{i+1} \subset K_i$ for any i. It is easy to check (see [**17**], Section 3) that

$$\lambda(J_i) = \frac{1}{2}\frac{3}{4}\cdots\frac{2^{i-3}-1}{2^{i-3}}, \tag{6}$$

$$\lambda(K_i) = 1/2^{(i+2)(i-3)/2} \tag{7}$$

for any $i > 3$. Thus if we put $[\gamma,\, \mu] = \bigcap\limits_{i=3}^{\infty} J_i$ (see (4)), then $\mu - \gamma = \prod\limits_{j=1}^{\infty}\left(1 - 1/2^j\right) > 0$.

Now write S as $I_3 \setminus \bigcup\limits_{n=1}^{\infty} U_n$, where U_n are open intervals pairwise disjoint and included in I_3. Let $(V_k)_{k=1}^{\infty}$ be a sequence containing each interval U_n infinitely many times. $A_k < B_k$ will denote the connected components of $I_3 \setminus V_k$ and $W_k = \overline{\psi}(I_3; I_3)(V_k)$.

Define a strictly increasing sequence of greater than 3 odd numbers $(i_k)_{k=1}^{\infty}$ with each i_k suitably large so that for any $i \geqslant i_k$

$$\frac{\max\{\kappa_k,\, \kappa(k; V_k),\, \kappa(k; W_k)\}\vartheta_k^i}{(\mu-\gamma)^k 2^{(i+2)(i-3)/2}} < \frac{1}{4^i} \tag{8}$$

and take the sequence of admissible functions $(\overline{g}_i)_{i=3}^{\infty}$ such that

(i) $\overline{g}_i = \overline{g}(i; V_k)$ if $i = i_k$ for some k;
(ii) $\overline{g}_i = g(i; W_k)$ if $i = i_k + 1$ for some k;
(iii) $\overline{g}_i = g_i$ otherwise.

Then

LEMMA 10. *Consider the sequence $(f_i)_{i=4}^{\infty}$ such that*

$$f_i = \overline{g}_3 * \left(\overline{g}_4 * \left(\ldots\left(\overline{g}_{i-1} * \overline{g}_i\right)\ldots\right)\right)$$

for any $i \geqslant 4$. Then $(f_i)_{i=4}^{\infty}$ converges uniformly to a function f with zero topological entropy.

PROOF: The arguments are similar to those made in Sections 4 and 5 from [**17**]. For this purpose our Lemmas 6 and 7 are also needed. ∎

5. Proof of the Theorem

In this last section we shall check that f in Lemma 10 is a weakly unimodal C^{∞} function and S is a δ-scrambled set of f for a suitable $\delta > 0$. With this, the reasoning is complete.

Lemma 11. *f is a weakly unimodal function.*

Proof: Note that either

$$f/(J_i \setminus J_{i+1}) = (\psi([0,\,1]; K_i) \circ \overline{g}_i \circ \psi(J_i; [0,\,1]))/(J_i \setminus J_{i+1}) \quad \text{or}$$

$$f/(J_i \setminus J_{i+1}) = \big(\psi([0,\,1]; K_i) \circ \overline{g}_i \circ \overline{\psi}(J_i; [0,\,1])\big)/(J_i \setminus J_{i+1}) \tag{9}$$

according as i is odd or even. Thus $f/[0,\,\gamma]$ is strictly increasing, $f/[\gamma,\,\mu]$ is constant and $f/[\mu,\,1]$ is strictly decreasing, that is, f is weakly unimodal. □

With the same argument as in Lemma 10 we can prove that for any $i > 3$ the sequence $\big(\overline{g}_i * \big(\overline{g}_{i+1} * \big(\ldots\big(\overline{g}_{i+j-1} * \overline{g}_{i+j}\big)\ldots\big)\big)\big)_{j=1}^{\infty}$ converges, say to $\overline{f}_i$. Then it is simple to verify that

$$f = \overline{g}_3 * \big(\overline{g}_4 * \big(\ldots\big(\overline{g}_{i-1} * \overline{f}_i\big)\ldots\big)\big) \quad \text{for any} \quad i > 3. \tag{10}$$

Lemma 12. *For any $i \geqslant 3$ we have*

$$\left(f^{2^{i-3}}\right)/(J_i \setminus J_{i+1}) = (\psi([0,\,1]; J_i) \circ \overline{g}_i \circ \psi(J_i; [0,\,1]))/(J_i \setminus J_{i+1})$$

if i is odd and

$$\left(f^{2^{i-3}}\right)/(J_i \setminus J_{i+1}) = \big(\overline{\psi}([0,\,1]; J_i) \circ \overline{g}_i \circ \overline{\psi}(J_i; [0,\,1])\big)/(J_i \setminus J_{i+1})$$

if i is even.

Proof: Since the lemma is obvious if $i = 3$, we can suppose $i > 3$. Then using repeatedly (1) and (5) it is easy to show that if $g\colon [0,\,1] \to [0,\,1]$ is continuous and such that $g(0) = g(1) = 0$, then

$$\big(\overline{g}_3 * \big(\overline{g}_4 * \big(\ldots\big(\overline{g}_{i-1} * g\big)\ldots\big)\big)\big)^{2^{i+3}}/J_i = \psi([0,\,1]; J_i) \circ g \circ \psi(J_i; [0,\,1])$$

if i is odd and

$$\big(\overline{g}_3 * \big(\overline{g}_4 * \big(\ldots\big(\overline{g}_{i-1} * g\big)\ldots\big)\big)\big)^{2^{i+3}}/J_i = \overline{\psi}([0,\,1]; J_i) \circ g \circ \overline{\psi}(J_i; [0,\,1])$$

if i is even. Now the lemma follows from (10) and the fact that $\overline{f}_i$ is equal to $\overline{g}_i$ in $\psi(J_i; [0,\,1])(J_i \setminus J_{i+1})$ or $\overline{\psi}(J_i; [0,\,1])(J_i \setminus J_{i+1})$, according as i is odd or even. □

Lemma 13. *S is a δ-scrambled set of f for any $\delta < \mu - \gamma$.*

Proof: For any $i \geqslant 3$ let $I_i = \psi([0,\,1]; J_i)\big([a^i,\,b^i]\big)$ if i is odd and $I_i = \overline{\psi}([0,\,1]; J_i)\big([a^i,\,b^i]\big)$ if i is even. Of course, $I_i \subset J_i \setminus J_{i+1}$.

Using Lemma 12 we shall prove that S is a δ-scrambled set of f for any $\delta < \mu - \gamma$. Note that if $i \neq i_k$ and $i \neq i_k + 1$ for any k then from the definition of g_i

$$f^{2^{i-3}}/I_i = \psi(I_i; I_{i+1}). \tag{11}$$

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Now consider the case $i = i_k$ for some k and let $A = \psi(I_3; I_i)(A_k)$, $B = \psi(I_3; I_i)(B_k)$, $C = \psi(I_3; I_{i+1})(A_k)$, $D = \psi(I_3; I_{i+1})(B_k)$, $E = \psi(I_3; I_{i+2})(A_k)$, $F = \psi(I_3; I_{i+2})(B_k)$. By (2), (3) and the definition of $\bar{g}_i$ when $i = i_k$ or $i = i_k + 1$ we have

(12) $$f^{2^{i-3}}(x) < \gamma \quad \text{for any} \quad x \in A \quad \text{and} \quad f^{2^{i-3}}/B = \psi(B; D),$$

and also

(13) $$f^{3\cdot 2^{i-3}}/A = \psi(A; C),\ f^{2^{i-2}}/C = \psi(C; E),\ f^{2^{i-1}}/D = \psi(D; F).$$

From (12) and (13)

(14) $$f^{5\cdot 2^{i-3}}/A = \psi(A; E),\ f^{5\cdot 2^{i-3}}/B = \psi(B, F).$$

Since $\lambda(I_i) \to 0$ and $i \to \infty$, it is easy to check that for any $x \neq y$ points of S,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geqslant \mu - \gamma,$$
$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

Finally from (11)–(14) we see that, given $x \in S$ and $m \geqslant 0$, it is possible to find $r > 0$ and a strictly increasing sequence (l_n) of multiples of 2^m such that $f^{l_n+r}(x) < \gamma$ if n is odd and $f^{l_n+r}(x) > \mu$ if n is even. Therefore, for any periodic point p of f, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \geqslant \mu - \gamma.$$

□

LEMMA 14. *f is a C^∞ function.*

PROOF: We shall prove that f is a C^∞ function by verifying that $D^k f(\gamma)$ and $D^k f(\mu)$ exist for any k, since it is evident that f is of class C^∞ in $[0, 1] \setminus \{\gamma, \mu\}$. In fact we shall examine the situation with γ; for μ the proof is similar.

Firstly, if $x \in J_i \setminus J_{i+1}$, then by (6) and (7)

$$\left|\frac{f(x) - f(\gamma)}{x - \gamma}\right| < \frac{1}{(\mu - \gamma)2^{(i+2)(i-3)/2-i-1}}.$$

Since $f(x) = f(\gamma)$ for any $x \in [\gamma, \mu]$, it is clear that $Df(\gamma)$ exists and that it is equal to zero.

Now suppose that $D^k f(\gamma)$ exists for a certain positive integer k. Obviously, $D^k f(\gamma) = 0$. Applying (6), (7), (8), (9) and Lemmas 8 and 9 we have that for any

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$i_k \leqslant i \leqslant i_{k+1} - 1$ and any $x \in J_i \setminus J_{i+1}$, $|D^j f(x)| < 1/4^i$ for any $j = 1, 2, \ldots, k$. Hence for any $i \geqslant i_k$ and $x \in J_i \setminus J_{i+1}$,

$$\left| \frac{D^k f(x)}{x - \gamma} \right| < \frac{1}{(\mu - \gamma)2^{i-1}}$$

and so $D^{k+1} f(\gamma)$ exists with $D^{k+1} f(\gamma) = 0$. □

The theorem follows from Lemmas 10, 11, 13 and 14.

6. Final remarks

Remark 1. The idea behind the definition of S is taken from [**14**]. In the construction of our function we have used ideas from [**17**].

Remark 2. A similar result can be obtained for functions with positive topological entropy. In fact it suffices to redefine the function f from the Theorem in (γ, μ) in such a way that there exists $c \in (\gamma, \mu)$ with $f(c) = 1$ and f is unimodal. Now f has a periodic point of period three and therefore has positive topological entropy.

Remark 3. For functions with zero topological entropy, the theorem offers the best possible result. Indeed it can be proved (see [**2**]) that

(i) if f is a C^1 function, then it cannot have a scrambled set of full Lebesgue measure;

(ii) if f is an analytic function with zero topological entropy, then it cannot be chaotic.

In connection with (ii) and a theorem from [**10**] about the non-existence of wandering intervals for C^2 functions without flat points, we conjecture that if f is an analytic function then it cannot have scrambled sets with positive Lebesgue measure.

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