THE WEAK LAW OF LARGE NUMBERS FOR NONNEGATIVE SUMMANDS

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Abstract

Khintchine's (necessary and sufficient) slowly varying function condition for the weak law of large numbers (WLLN) for the sum of n nonnegative, independent and identically distributed random variables is used as an overarching (sufficient) condition for the case that the number of summands is more generally $[c_n]$, $c_n \to \infty$. Either the norming sequence $\{a_n\}$, $a_n \to \infty$, or the number of summands sequence $\{c_n\}$, can be chosen arbitrarily. This theorem generalizes results from a motivating branching process setting in which Khintchine's sufficient condition is automatically satisfied. A second theorem shows that Khintchine's condition is necessary for the generalized WLLN when it holds with $c_n \to \infty$ and $a_n \to \infty$. Theorem 3, which is known, gives a necessary and sufficient condition for Khintchine's WLLN to hold with $c_n = n$ and a_n a specific function of n; it is extended to general c_n subject to a growth restriction in Theorem 4. Section 6 returns to the branching process setting.

Keywords: Khintchine's condition; nonnegative summand; slowly varying function; arbitrary norming or number of summands; simple branching process

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1. Introduction

1.1. Necessary and sufficient conditions

Let $\{W_i,\ i = 1, 2, \ldots\}$ be independently and identically distributed (i.i.d.) nonnegative random variables with cumulative distribution function $F(x)$, $x \geq 0$. Introduce the notation

$$S_n = \sum_{i=1}^{n} W_i, \qquad \nu(x) = \int_0^x (1 - F(u))\,\mathrm{d}u, \qquad \mu(x) = \int_{[0,x]} u\,\mathrm{d}F(u), \qquad x \geq 0. \tag{1}$$

Khintchine (1936) showed that there exists a sequence $\{a_n,\ a_n > 0\}$, such that

$$\frac{S_n}{a_n} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty$$

(this is known as *relative stability*), if and only if

$$\frac{x(1 - F(x))}{\nu(x)} \to 0 \quad \text{as } x \to \infty. \tag{2}$$

Khintchine's (1936) proof of necessity under the condition that $F(x)$ is continuous, is a proof by contradiction: it is long, technically intricate, and necessarily masterful, since it uses characteristic functions in this setting of nonnegative random variables.

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It was recognized later (see Feller (1966)) that (real-variable) Laplace transform approaches make development simpler in settings of nonnegative random variables.

The condition at (2) implies that $\nu(x)$ is a slowly varying function (SVF) at ∞ as noted by Csörgő and Simons (2008, Section 1), but around 1936 when Khintchine wrote, Karamata's concept of a regularly varying function (RVF) was very new, as was the related and famed Karamata Tauberian theorem which was to be later used effectively in this setting (SVFs are not mentioned in Khintchine (1936)).

In place of (2), Feller (1966, Theorem 3, p. 233; 1971, Theorem 2, p. 236) gave the necessary and sufficient condition

$$\frac{x(1-F(x))}{\mu(x)} \to 0 \quad \text{as } x \to \infty. \tag{3}$$

Feller mentions that this condition is equivalent to $\mu(x)$ being an SVF at ∞.

Feller's (1966) text effectively introduced RVF theory to probability theory. His proof that (3) is necessary and sufficient is based on elegant probability inequalities, and the proof of necessity is partly by a contradiction argument. Feller does not allude to Khintchine (1936), the adaptation of which in Gnedenko and Kolmogorov (1954, p. 139) masks both its original nature, and that it is a necessary and sufficient condition.

The Khintchine weak law of large numbers (WLLN) as expressed by Csörgő and Simons (2008) states that there exists a sequence $\{a_n,\ a_n > 0\}$ such that

$$\frac{S_n}{a_n} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty, \tag{4}$$

if and only if

$$\nu(x) = \int_0^x (1-F(u))\,\mathrm{d}u \quad \text{is SVF as } x \to \infty. \tag{5}$$

For each fixed n, the norming constant a_n can be chosen as the unique solution of $n\nu(a_n) = a_n$. Any choice of constants $\{a_n,\ a_n > 0\}$ satisfying

$$\frac{n\nu(a_n)}{a_n} \to 1$$

will do. For any such set of norming constants, $a_{n+1}/a_n \to 1$ and $a_n \to \infty$.

Csörgő and Simons (2008) call this result the Khintchine–Feller theorem, and, hence, we refer to it as the KF theorem. The necessary and sufficient condition (5) appears to have a more general and more natural form than (2) and (3), although all must be equivalent.

In fact, Rogozin (1971), temporally simultaneous in appearance with Feller (1971), is aware of Khintchine's (1936) result, and in an elegantly and concisely proved Theorem 2 using results from Feller (1966), shows the equivalence of (5) and of relative stability, and of a number of other results. Rogozin's (1971) approach, via Laplace transforms, is centred on the equivalence of (4) and

$$\lim_{n\to\infty} n \log\left(\phi\left(\frac{s}{a_n}\right)\right) = -s, \qquad s \geq 0, \tag{6}$$

where $\phi(s) = \mathbb{E}(\exp(-sW_1))$. This theorem was later expressed as 'Theorem 8.8.1 (Relative Stability Theorem: Rogozin (1971)' in Bingham *et al.* (1987, pp. 372–374), where the emphasis is on SVFs.

We first prove the following 'branching process inspired' extension of the sufficiency part of the Khintchine–Feller–Rogozin results.

Theorem 1. *Assume that (5) holds, and denote $\nu(x)$ by $L(x)$, the classical notation for an SVF.*

(a) *Given a sequence $\{a_n,\ a_n > 0\}$ for which $a_n \to \infty$, define the sequence $\{c_n\},\ c_n \to \infty$, by*

$$c_n = \frac{a_n}{L(a_n)}. \tag{7}$$

Then

$$\frac{S_{[c_n]}}{a_n} \xrightarrow{\mathbb{P}} 1. \tag{8}$$

(b) *Conversely, given a sequence $\{c_n\},\ c_n \to \infty$, a sequence $\{a_n\}$ is defined uniquely by (7). Then (8) holds.*

(c) *In (a), if (8) holds with $a_n \to \infty$ and with $\{c'_n\},\ c'_n \to \infty$, in place of $\{c_n\}$, then $c'_n \sim c_n$, where $c_n = a_n/L(a_n)$. If (8) holds then it holds with $\{c'_n\}$ in place of $\{c_n\}$, where $c'_n \sim c_n$.*

(d) *In (b), if (8) holds with $\{a'_n\},\ a'_n \to \infty$, in place of $\{a_n\}$, then $a'_n \sim a_n$, where $a_n = c_n L(a_n)$. If (8) holds then it holds with $\{a'_n\}$ in place of $\{a_n\}$, where $a'_n \sim a_n$.*

In both the above and the sequel, all asymptotics are as $n \to \infty$.

In our original branching process setting for this theory (see Seneta (1975)), the underlying assumption that (5) holds was automatically fulfilled. The original branching process results are now given in Section 6 as applications of Theorem 1. But, at least to the author, it is a striking feature of this theorem and expressed in part (a), assuming the overarching condition (5), that, for an arbitrarily chosen norming sequence $\{a_n\},\ a_n \to \infty$, there exists a 'balancing' sequence $\{c_n\},\ c_n \to \infty$ to give (8).

1.2. Approach to Theorem 1

The classical (KF) sufficiency result, where $c_n = n$, is a special case of part (b), and so parts (b) and (d), at least, may plausibly be perceived as corollaries of that special case. Thus, presuppose the (apparent, but nontrivial) result (which in any case we need to prove in our Section 3 as one of the keystones of our own approach), that, *for sufficiently large x, the function $u(x) = x/L(x)$ is continuous and strictly increasing to ∞, and so has an inverse function $w(x)$*, where $w(x)$ is strictly increasing to ∞ with x.

Next, replace $\{c_n\},\ c_n \to \infty$, by supposing *a priori* that $[c_n]$ is strictly monotone increasing. Then parts (b) and (d) above may follow by application of the Khintchine–Feller (KF) result (as formulated by Csörgő and Simons (2008)), by the use of subsequences and elementary probabilistic arguments. For example, from KF, there is a unique solution A_n to $n = u(A_n)$, that is, $A_n = w(n)$, and $A_n \uparrow \infty$.

Since, by present supposition, $\{c_n\},\ c_n > 0$, is a sequence such that $[c_n]$ is strictly increasing, it follows that $[c_n] = u(A_{[c_n]})$, and $\{A_{[c_n]}\}$ is clearly a subsequence of $\{A_n\}$, and so, by KF, $S_{[c_n]}/A_{[c_n]} \xrightarrow{\mathbb{P}} 1$. So the above is a version of part (b), with $a_n = A_{[c_n]}$.

To refine such arguments however to encompass properties such as $c_n \to \infty$ as $n \to \infty$, and asymptotic equalities ('$\sim$'), and indeed to cope with parts (a) and (c), we have found it easier to adopt a Laplace transform approach; this proceeds by equating (8) with

$$\lim_{n\to\infty} c_n \log\biggl(\phi\biggl(\frac{s}{a_n}\biggr)\biggr) = \lim_{n\to\infty} [c_n] \log\biggl(\phi\biggl(\frac{s}{a_n}\biggr)\biggr) = -s, \qquad s \geq 0,$$

in imitation of (6), since $c_n \to \infty$. That is, we proceed by generalizing slightly Rogozin's (1971) approach. It will be seen from the brief proofs of parts (a) and (b) in Section 4, that in the sense of this approach, these may be regarded as corollaries of the sufficiency part of the classical KF result.

2. Three more theorems

In Theorem 1, (5) is a sufficient condition for the existence of positive sequences $\{c_n\}$ and $\{a_n\}$, where $c_n \to \infty$ and $a_n \to \infty$ as $n \to \infty$ such that (8) holds.

We need first to say something about necessity for this extension of Khintchine's WLLN.

Theorem 2. *Suppose that (8) holds for some positive sequences $\{c_n\}$ and $\{a_n\}$, where $c_n \to \infty$ and $a_n \to \infty$ as $n \to \infty$, and $a_{n+1}/a_n < C$ for some C in $1 < C < \infty$. Then $\nu(x)$ defined in (1) is an SVF $L(x)$.*

This is enough to cover both the classical case where $c_n = n$ and $a_n \to \infty$, and as we shall see, the original setting for branching processes.

In fact, the proof of our Theorem 2 is a generalization of Rogozin's (1971) elegant proof of necessity in the classical case which uses a sequential criterion for regular variation of Feller (1966, p. 270), which criterion was generalized in Seneta (1971).

An inspiration for this paper was the comprehensive and scholarly paper by Csörgő and Simons (2008), who generalized the classical Khintchine-type results above in a different direction. In their dominating 'Theorem' (p. 34), they gave a necessary and sufficient group of conditions for the existence of a sequence of norming constants $\{d_n,\ d_n > 0\}$ such that

$$\frac{T_{\boldsymbol{p}_n}}{d_n} \xrightarrow{\mathbb{P}} 1,$$

where $T_{\boldsymbol{p}_n} = p_{1,n}W_1 + \cdots + p_{n,n}W_n$, with the $\{W_i\}$ as before, and $\{\boldsymbol{p}_n\} = \{(p_{1,n}, \ldots, p_{n,n})\}$, $n \geq 1$, a sequence of probability distributions, so that $p_{i,n} \geq 0,\ i = 1, \ldots, n$, and $\sum_{i=1}^{n} p_{i,n} = 1$. One of the results of that paper (it occurs as Corollary 5 of their 'Theorem', and is proved in that paper very briefly) fits in neatly with our Theorem 1. We express it here as follows.

Theorem 3. *For the function $\nu(x)$ defined in (1),*

$$\frac{S_n}{n\nu(n)} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty, \tag{9}$$

if and only if

$$\frac{\nu(x\nu(x))}{\nu(x)} \to 1 \quad \text{as } x \to \infty. \tag{10}$$

Csörgő and Simons (2008) call (10) the 'Bojanić–Seneta condition', from its origin in Bojanić and Seneta (1971).

A key step in proving the sufficiency of (9), where $\nu(x)$ is defined in (1), is the immediate deduction of (5) by the classical KF theorem as stated in Section 1. Otherwise, Theorem 3 is a special case of the following result.

Theorem 4. *For the function $\nu(x)$ defined in (1), if $c_n \to \infty$, (10) implies that $\nu(x)$ is SVF, and*

$$\frac{S_{[c_n]}}{c_n\nu(c_n)} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty. \tag{11}$$

If $c_n \to \infty$ and $c_{n+1}/c_n < C < \infty$, where C is a constant, and $\nu(x)$ defined in (1) is SVF, then (11) implies (10).

When I was asked to join in honouring my friend and academic twin Peter Jagers by a contribution to this Festschrift, I needed something which had to do with branching processes:

the probabilistic area (Jagers (1975)) in which we both made our early careers; and had Russian, Austro-Hungarian, and Australian colour, all appropriate to Peter's life. It was time to develop the WLLN themes of Seneta (1975) free of their branching process context, in the light of Csörgő and Simons (2008).

I was pleased to find recognition of the related work of my former student Ross Maller (1978) in Csörgő and Simons (2008); and included citation of Khintchine (1936) and Csörgő and Simons (2008) in a tricentenary review of the WLLN (see Seneta (2013)).

3. Tools

Lemma 1. *For $\nu(x)$ defined in (1) for W a nonnegative random variable with distribution function $F(x) = \mathbb{P}(W \le x)$, $x \ge 0$, and Laplace transform $\phi(s) = \mathbb{E}(\mathrm{e}^{-sW})$, $s \ge 0$,*

$$\int_0^\infty (1 - F(u))\mathrm{e}^{-su}\,\mathrm{d}u = \frac{1 - \phi(s)}{s} \tag{12}$$

and

$$\nu(x) \sim L(x) \quad \text{as } x \to \infty \quad \text{if and only if} \quad \frac{1-\phi(s)}{s} \sim L\left(\frac{1}{s}\right) \quad \text{as } s \to 0+, \tag{13}$$

where $L(x)$ is SVF as $x \to \infty$.

Proof. Equation (12) follows from integration by parts, while (13) is an application of Feller (1966, Theorem 2, p. 421), essentially an application of Karamata's Tauberian theorem. □

The result is well known (see, e.g. Rogozin (1971, p. 577) and Seneta (1974, p. 410)).

Since in Theorem 1 we make the blanket assumption that the left-hand side of (13) is satisfied, we may write, for occasional use,

$$\frac{1-\phi(s)}{s} = \bar{L}\left(\frac{1}{s}\right) \sim L\left(\frac{1}{s}\right), \qquad s \to 0+, \tag{14}$$

where $\bar{L}(x)$ is clearly slowly varying at ∞.

Lemma 2. *If $\nu(x)$ satisfies (5), so we may conveniently write $\nu(x) = L(x)$, for sufficiently large x, the function $u(x) = x/L(x) =: xL_1(x)$ is continuous and strictly increasing (to ∞), so has an inverse function $w(x)$, strictly increasing to ∞. Furthermore, $w(x) = xL_2(x)$, where $L_2(x)$ is SVF as $x \to \infty$.*

Proof. The function $\nu(x)$ is clearly positive and continuous on $x > 0$, and is concave on $x \ge 0$ (see Csörgő and Simons (2008, Lemma 4, p. 50)). Hence, $\nu(x)/x$ is nonincreasing with increasing x, $x > 0$ (see Krasnoselskii and Rutickii (1961, p. 3)). Since $\nu(x) = L(x)$ is SVF, $\nu(x)/x = L(x)/x \to 0$ as $x \to \infty$ from properties of SVFs (see, e.g. Seneta (1976, p. 18)).

For any fixed $x_0 > 0$, write $y_0 = \nu(x_0)/x_0$, so $y_0 > 0$. Since $c_n \to \infty$, for sufficiently large n_0, $y_0 > 1/c_{n_0}$, it follows that

$$\frac{\nu(x_0)}{x_0} - \frac{1}{c_{n_0}} = y_0 - \frac{1}{c_{n_0}} > 0. \tag{15}$$

Since $\nu(x)/x \to 0$ as $x \to \infty$, for sufficiently large x,

$$\frac{\nu(x)}{x} - \frac{1}{c_{n_0}} < 0, \tag{16}$$

so, by continuity, (15), and (16), there is at least one point $a_{n_0} > x_0$ where $\nu(a_{n_0})/a_{n_0} = 1/c_{n_0}$.

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Since $\nu(x)/x$ is nonincreasing with x by the concavity of $\nu(x)$, for $x \geq a_{n_0}$, $\nu(x)/x \leq 1/c_{n_0}$. Now suppose that, for some a, b with $a_{n_0} \leq a < b$,

$$\frac{\nu(a)}{a} = \frac{\nu(b)}{b} = \frac{1}{c}\left(\leq \frac{1}{c_{n_0}}\right). \tag{17}$$

But, from above,

$$0 < \frac{\nu(x_0)}{x_0} - \frac{1}{c_{n_0}} \leq \frac{\nu(x_0)}{x_0} - \frac{1}{c},$$

so $\nu(x_0) > x_0/c$, whence, from (17),

$$\frac{\nu(a) - \nu(x_0)}{a - x_0} < \frac{a/c - x_0/c}{a - x_0} = \frac{1}{c} = \frac{\nu(b) - \nu(a)}{b - a},$$

again a contradiction to the concavity of $\nu(x)$. Thus, for $x \geq a_{n_0}$, $\nu(x)/x$ is strictly decreasing to 0 with increasing x, so $u(x) = x/\nu(x)$ is strictly increasing to ∞ with $x \geq a_{n_0}$, and since it is also continuous in x, it has an inverse function $w(x)$, $x \geq u(a_{n_0})$. In particular, $w(x) \uparrow \infty$ as $x \uparrow \infty$. That $w(x) = xL_2(x)$, where $L_2(x)$ is SVF as $x \to \infty$, follows from Seneta (1976, p. 23). □

Lemma 3. *For any function $\nu(x) > 0$, nondecreasing for sufficiently large x,*

$$\frac{\nu(x\nu(x))}{\nu(x)} \to 1 \quad \text{as } x \to \infty \quad \Longrightarrow \quad \frac{\nu(cx)}{\nu(x)} \to 1 \quad \text{for each } c > 0, \tag{18}$$

so ν is SVF.

Proof. If $\nu(x) \to$ constant $< \infty$ as $x \to \infty$, (18) clearly holds. If $\nu(x) \to \infty$ for fixed $c > 0$ and sufficiently large x, then $\nu(x) > c$, so, by the monotonicity of $\nu(x)$,

$$\frac{\nu(cx)}{\nu(x)} \leq \frac{\nu(x\nu(x))}{\nu(x)}.$$

So, if $c \geq 1$ and x is sufficiently large,

$$1 \leq \frac{\nu(cx)}{\nu(x)} \leq \frac{\nu(x\nu(x))}{\nu(x)},$$

and letting $x \to \infty$, (18) holds.

If $0 < c < 1$, putting $y = cx$,

$$\lim_{x\to\infty} \frac{\nu(cx)}{\nu(x)} = \lim_{y\to\infty} \frac{\nu(y)}{\nu(y/c)} = 1,$$

since $1/c > 1$. □

4. Proof of Theorem 1

Since we are making the blanket assumption that $\nu(x)$ defined by (5) is SVF at ∞, we write $L(x)$ for $\nu(x)$ in this section.

(a) Writing $\phi(s) = \mathbb{E}(\mathrm{e}^{-sW})$, $s > 0$, and noting that $c_n = u(a_n) \to \infty$ by Lemma 2,

$$\begin{aligned}
\ln \mathbb{E}(\mathrm{e}^{-sS_{[c_n]}/a_n}) &= \ln\left(\phi\left(\frac{s}{a_n}\right)\right)^{[c_n]} \\
&= [c_n] \ln\left(\phi\left(\frac{s}{a_n}\right)\right)
\end{aligned}$$

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$$\sim \frac{a_n}{L(a_n)} \ln\left(1 - \left(1 - \phi\left(\frac{s}{a_n}\right)\right)\right) \quad \text{(from (7))}$$
$$\sim -\frac{a_n}{L(a_n)} \left(1 - \phi\left(\frac{s}{a_n}\right)\right) \quad (\text{since } a_n \to \infty)$$
$$= -\frac{s}{L(a_n)} \frac{1 - \phi(s/a_n)}{s/a_n}$$
$$\sim -\frac{s}{L(a_n)} L\left(\frac{a_n}{s}\right) \quad \text{(from (13) of Lemma 1)}$$
$$\sim -s,$$

since $L(a_n/s)/L(a_n) \to 1$ as $n \to \infty$ because L is SVF. Thus, $\mathbb{E}(\mathrm{e}^{-sS_{[c_n]}/a_n}) \to \mathrm{e}^{-s}$, $s > 0$, which is tantamount to (8).

(b) Given $c_n \to \infty$, by Lemma 2, for sufficiently large n, $a_n = w(c_n)$, where $w(x)$ is the inverse function to $u(x) = x/L(x)$, so $a_n = w(c_n) \to \infty$ as $n \to \infty$. Hence, by (a), the result follows.

(c) We have

$$c_n' \left(\frac{s}{a_n}\right) \frac{1 - \phi(s/a_n)}{s/a_n} \to s,$$

and, from Lemma 1,

$$c_n' \left(\frac{s}{a_n}\right) L\left(\frac{a_n}{s}\right) \to s.$$

But, by assumption, $a_n = c_n L(a_n)$, so

$$\frac{c_n'}{c_n} \frac{L(a_n/s)}{L(a_n)} \to 1,$$

so, since $L(x)$ is SVF,

$$\frac{c_n'}{c_n} \to 1, \tag{19}$$

as asserted.

Now suppose that (8) and (19) hold. Then

$$\frac{c_n}{c_n'} c_n' \left(1 - \phi\left(\frac{s}{a_n}\right)\right) \to s,$$

and, by (19), $c_n'(1 - \phi(s/a_n)) \to s$, so (8) holds with c_n' in place of c_n.

(d) We have, as in (c), for $s > 0$,

$$c_n \left(\frac{s}{a_n'}\right) L\left(\frac{a_n'}{s}\right) \to s,$$

that is,

$$\frac{c_n}{a_n'} \frac{L(a_n'/s)}{L(a_n')} L(a_n') \to 1.$$

Thus, since L is SVF,

$$c_n \frac{L(a_n')}{a_n'} \to 1. \tag{20}$$

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Writing $c'_n = a'_n/L(a'_n)$, (20) asserts that $c'_n \sim c_n$. Now, from Lemma 2, recalling that $w(x)$ is the inverse function of $u(x) = x/L(x)$,

$$\begin{aligned} a'_n &= w(c'_n) \\ &= \frac{w(c'_n)}{w(c_n)} w(c_n) \\ &\sim w(c_n) \quad \text{(by the Uniform Convergence theorem since } w \text{ is an RVF of index +1)} \\ &= a_n \\ &= c_n L(a_n), \end{aligned}$$

as asserted.

Suppose now that (8) holds, so, as $n \to \infty$, using the notation in (14),

$$c_n\left(\frac{s}{a_n}\right)\frac{1-\phi(s/a_n)}{s/a_n} = c_n\left(\frac{s}{a_n}\right)\bar{L}\left(\frac{s}{a_n}\right) \to s. \tag{21}$$

Now consider, as $n \to \infty$,

$$c_n\left(\frac{s}{a'_n}\right)\frac{1-\phi(s/a'_n)}{s/a'_n} = c_n\left(\frac{s}{a'_n}\right)\bar{L}\left(\frac{s}{a'_n}\right) = c_n\left(\frac{s}{a_n}\right)\bar{L}\left(\frac{s}{a_n}\right)\frac{a_n}{a'_n}\frac{\bar{L}(s/a'_n)}{\bar{L}(s/a_n)} \to s,$$

using (21), $a'_n \sim a_n$, and the Uniform Convergence theorem of SVFs. Therefore, $c_n(1 - \phi(s/a'_n)) \to s$, so that (8) holds with a'_n in place of a_n, as asserted.

5. Proofs of Theorems 2 and 4

Proof of Theorem 2. In the notation of Lemma 1 and in view of (8), it follows that, for $s \geq 0$ and $n \to \infty$,

$$\left(\phi\left(\frac{s}{a_n}\right)\right)^{[c_n]} \to \mathrm{e}^{-s}.$$

Hence, for $s > 0$, $c_n \ln(\phi(s/a_n) \to -s$ and $c_n(1 - \phi(s/a_n)) \sim s$, so that then

$$\lim_{n\to\infty} c_n U\left(\frac{a_n}{s}\right) = s,$$

where $U(y) = 1 - \phi(1/y),\ y > 0$. Then, putting $t = 1/s$,

$$\frac{U(a_n t)}{U(a_n)} \to \frac{1}{t} \quad \text{as } t \to \infty,$$

where $U(y)$ is monotone in y. Hence, by Seneta (1971, Theorem A, Corollary), $U(y)$ is regularly varying with index -1 as $y \to \infty$, that is, $U(y) = 1 - \phi(1/y) = y^{-1}L_3(y)$, where $L_3(y)$ is an SVF at ∞. Then putting $1/y = \lambda$ gives

$$\frac{1-\phi(\lambda)}{\lambda} = L_3\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to 0+.$$

Hence, from Lemma 1 and (13), $\nu(x) \sim L_3(x)$ as $x \to \infty$, so $\nu(x) = L(x)$, where $L(x)$ is an SVF. □

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In the classical case when we suppose that (8) holds with $c_n = n$ and some sequence $\{a_n\}$, the condition $a_{n+1}/a_n < C$ for some C in $1 < C < \infty$ is implied by (8) itself. Feller (1971, p. 237) remarked that this is 'obvious': to see it, observe that

$$\frac{S_{n+1}}{a_{n+1}} = \frac{a_n}{a_{n+1}}\left(\frac{S_n}{a_n} + \frac{W_{n+1}}{a_n}\right),$$

and since, as $a_n \to \infty$, $W_{n+1}/a_n \xrightarrow{\mathbb{P}} 0$, we have simultaneously

$$\frac{S_n}{a_n} + \frac{W_{n+1}}{a_n} \xrightarrow{\mathbb{P}} 1 \quad \text{and} \quad \frac{S_{n+1}}{a_{n+1}} \xrightarrow{\mathbb{P}} 1,$$

from which a contradiction results if $\limsup_{n\to\infty} a_{n+1}/a_n = \infty$.

Proof of Theorem 4. First suppose that (10) holds. Then, for arbitrary $c_n \to \infty$,

$$\frac{\nu(c_n\nu(c_n))}{\nu(c_n)} \to 1, \tag{22}$$

and, from Lemma 3, $\nu(x)$ is an SVF. Now write $a(n) = c_n\nu(c_n)$, and define

$$c'_n = u(a_n) = \frac{a_n}{\nu(a_n)} = \frac{c_n\nu(c_n)}{\nu(c_n\nu(c_n))}, \tag{23}$$

so that, by Theorem 1(a),

$$\frac{S_{[c'_n]}}{c_n\nu(c_n)} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty. \tag{24}$$

But, from (22) and (23), $c'_n \sim c_n$, so Theorem 1(c) applied to (24) yields (11), as asserted.

Now suppose, conversely, that (11) holds for a sequence $\{c_n\}$ such that $c_{n+1}/c_n < C < \infty$ and $c_n \to \infty$, where $\nu(x)$ satisfies (5), and we can write $L(x)$ in place of $\nu(x)$. First, repeat steps (23) and (24). Since (11) is assumed to hold, by Theorem 1(c), $c'_n \sim c_n$, so, from (23), (22) holds.

Let $\{\theta_{(i)}\}$ be the subsequence of successive maxima of $\{c_n\}$. Then, from the assumptions on $\{c_n\}$,

$$\theta_{(i)} \uparrow \infty \quad \text{and} \quad \frac{\theta_{(i+1)}}{\theta_{(i)}} < C. \tag{25}$$

Now, for $x > 0$ select $r = r(x)$ such that

$$\theta_{(r)} \le x \le \theta_{(r+1)} \; (\le C\theta_{(r)}),$$

using (25), so that $\theta_{(r)} \to \infty$ as $x \to \infty$. Note also that

$$1 \le \delta_{(r)} := \frac{\theta_{(r+1)}}{\theta_{(r)}} \le C. \tag{26}$$

Now, by the monotonicity of ν,

$$\frac{\nu(\theta_{(r)}\nu(\theta_{(r)}))}{\nu(\theta_{(r)})}\frac{\nu(\theta_{(r)})}{\nu(\theta_{(r+1)})} \le \frac{\nu(x\nu(x))}{\nu(x)} \le \frac{\nu(\theta_{(r+1)}\nu(\theta_{(r+1)}))}{\nu(\theta_{(r+1)})}\frac{\nu(\theta_{(r+1)})}{\nu(\theta_{(r)})}. \tag{27}$$

Furthermore,

$$\frac{\nu(\theta_{(r)}\nu(\theta_{(r)}))}{\nu(\theta_{(r)})} \to 1 \quad \text{as } x \to \infty$$

by (11), (22), and

$$\frac{\nu(\theta_{(r+1)})}{\nu(\theta_{(r)})} = \frac{L(\delta_{(r)}\theta_{(r)})}{L(\theta_{(r)})} \to 1 \quad \text{as } x \to \infty,$$

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by the Uniform Convergence theorem of SVFs, on account of (26). Thus, from (27), (10) holds, as asserted. □

6. Branching process applications

Let $\{Z_n\}$ denote an ordinary nondegenerate Bienaymé–Galton–Watson branching process generated by a probability generating function $F(s)$, $s \in [0, 1]$. As usual, put $m = F'(1-)$. It is slightly more convenient for this section to use the notation $\sum_{i=1}^{[c_n]} W_i/a_n$ in place of the earlier notation $S_{[c_n]}/a_n$.

6.1. The supercritical case

In the case $1 < m < \infty$ write $h_n(s)$ for the inverse function of $k_n(s) = -\log \mathbb{E}(\mathrm{e}^{-sZ_n})$.

Theorem 5. *If* $1 < m < \infty$,

$$\frac{Z_n}{c_n} \xrightarrow{\text{a.s.}} W\ (= W(s_0)),$$

with $c_n = 1/h_n(s_0)$ *for some fixed* $s_0 \in (0, -\ln q)$, *where* $0 \le q < 1$ *is the extinction probability of a single ancestor and* $-\ln q = \infty$ *if* $q = 0$, *and*

$$h_n(s_0) = K(m^{-n}K^{-1}(s_0)), \quad \textit{where} \quad K(s) = -\ln \mathbb{E}(\mathrm{e}^{-sW})$$

and $K^{-1}(s)$ *is the inverse function of* $K(s)$. *Furthermore,*

$$c_n \sim \frac{m^n}{L(m^n)}, \quad \textit{where} \quad L(x) = \int_0^x \mathbb{P}(W > y)\,\mathrm{d}y \tag{28}$$

is an SVF as $x \to \infty$.

Conclusion (28) is Theorem 1 of Seneta (1994). It shows that the overarching condition of our present paper, (5), is satisfied. This condition for the supercritical branching process setting was announced in Seneta (1970), and communicated to K. B. Athreya (1971). See Seneta (1994) for details.

Using (28), Seneta (1995, pp. 252–253) proved that

$$\sum_{i=1}^{[c_n]} \frac{W_i}{m^n} \xrightarrow{\mathbb{P}} 1, \tag{29}$$

where W_i, $i = 1, 2, \ldots$, are i.i.d. copies of W.

We show that this as a consequence of our general Theorem 1. Take $a_n = m^n$. Then if we put $c'_n = m^n/L(m^n) = u(m^n)$, by Theorem 1(a), (29) holds with c'_n in place of c_n. But $c'_n \sim c_n$ from (28), so (29) follows from Theorem 1(c).

Let us now take $c_n = m^n$. Then take $a_n = w(c_n)$, where $w(x) = xL_2(x)$ is the inverse function of $u(x) = x/L(x) = xL_1(x)$, by Lemma 2. Hence, by Theorem 1(b),

$$\sum_{i=1}^{[m^n]} \frac{W_i}{w(m^n)} = \sum_{i=1}^{[m^n]} \frac{W_i}{m^n L_2(m^n)} \xrightarrow{\mathbb{P}} 1. \tag{30}$$

6.2. The subcritical case

Theorem 6. *If* $0 < m < 1$ *then, for* $j = 1, 2, \ldots$,

$$\mathbb{P}(Z_n = j \mid Z_n > 0) \to d_j \quad \textit{and} \quad \sum_{j=1}^{\infty} d_j = 1.$$

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Writing $D(s)=\sum_{j=1}^{\infty}d_js^j$, $s\in[0,1]$; $f(s)=1-F(1-s)$, $\phi(s)=1-D(1-s)$, $f_n(s)$ *for the nth functional iterate of* $f(s)$, W *for a random variable with distribution* $\{d_j\}$, *and* W_i, $i=1,2,\ldots$, *for i.i.d. copies of* W, *then*

$$a_n=\frac{1}{f_n(1)}=\frac{1}{\phi^{-1}(m^n)}=m^{-n}L_2'(m^{-n})=w'(m^{-n}), \tag{31}$$

$$\int_0^x \mathbb{P}(W>y)\,\mathrm{d}y\sim\frac{1}{L_1'(x)}. \tag{32}$$

Here $w'(x)=xL_2'(x)=1/\phi^{-1}(1/x)$ *is the inverse function of* $u'(x)=1/\phi(1/x)=xL_1'(x)$, *where* $L_1'(x)$ *is an SVF as* $x\to\infty$, *and* $a_n=1/\mathbb{P}(Z_n>0)$.

The conclusions at (31) and (32) are Theorem 2 of Seneta (1994); (32) shows that the overarching condition (5) of our present paper is satisfied.

Now write, consistent with our previous notation in Lemma 2,

$$L(x)=\int_0^x\mathbb{P}(W>y)\,\mathrm{d}y,\qquad u(x)=\frac{x}{L(x)}=xL_1(x),\qquad w(x)=xL_2(x).$$

We have from (32) that $L_1'(x)\sim L_1(x)$ as $x\to\infty$, so that $u'(x)\sim u(x)$. With $a_n=1/f_n(1)$ as in (31),

$$c_n=u(a_n)\sim u'(a_n)=u'(w'(m^{-n}))=m^{-n}\ (=c_n',\ \text{say}).$$

Hence, from parts (a) and (c) of Theorem 1,

$$\sum_{i=1}^{[c_n']}\frac{W_i}{a_n}=\sum_{i=1}^{[m^{-n}]}\frac{W_i}{1/f_n(1)}\xrightarrow{\mathbb{P}}1.$$

Now put $a_n=m^{-n}$. Then $c_n=u(a_n)=u(m^{-n})=m^{-n}/L(m^{-n})$, so, by Theorem 1(a),

$$\sum_{i=1}^{[c_n]}\frac{W_i}{m^{-n}}\xrightarrow{\mathbb{P}}1,$$

paralleling (29) for $m>1$.

As a final application to the subcritical case, we prove that

$$\sum_{i=1}^{[c_n]}\frac{W_i}{a_n'}\xrightarrow{\mathbb{P}}1, \tag{33}$$

where $c_n=1/f_n(1)$ is now the number of summands, and $a_n'=1/\phi^{-1}(f_n(1))$.

We need to prove that $w(x)\sim w'(x)$ as $x\to\infty$. Now $x=w'(u'(x))\sim w'(u(x))$ since, as above, $u'(x)\sim u(x)$, and w' is an RVF, so the Uniform Convergence theorem applies. Since $w(x)\to\infty$ as $x\to\infty$, substitute $w(x)$ for x in $x\sim w'(u(x))$ to obtain $w(x)\sim w'(x)$. Now put $c_n=1/f_n(1)=w'(m^{-n})$, so that the corresponding $a_n=w(c_n)\sim w'(c_n)=w'(1/f_n(1))=1/\phi^{-1}(f_n(1))=a_n'$. Then (33) follows from parts (b) and (d) of Theorem 1.

The analogy to (33) in the supercritical case, achievable by analogous reasoning to the subcritical case, is to augment (30) by (33) with $c_n=m^n$ and $a_n'=1/K^{-1}(m^{-n})$.

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7. Recollections

Sandor Csörgő mentioned that he was working with Gordon Simons on generalizing Khintchine's theorem in an email to me of September 2005, and that they were using Bojanić and Seneta (1971). I replied that this 'reminded me of a paper published in an obscure place that I wrote long ago about the WLLN for sums of non-negative random variables … Harry Cohn proposed the problem, and by some quirk, I managed to solve it.' The paper was Seneta (1975). Sandor tried to find it digitally. He knew of some of Harry Cohn's work on the WLLN in the context of branching processes. But Seneta (1975) had not been reviewed for MathSciNet. It had a title which had nothing to do with the WLLN, was a book chapter, and remained truly obscure. I have a note that I finally sent a photocopy of it to Sandor by airmail on February 9, 2006. Csörgő and Simons (2008) was ready for publication in May 2006. It was submitted to *Periodica* on October 18, 2007, and accepted March 10, 2008. Sandor Csörgő died February 14, 2008.

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