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COMPONENT IMPORTANCE IN COHERENT SYSTEMS WITH EXCHANGEABLE COMPONENTS

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Abstract

This paper is concerned with the Birnbaum importance measure of a component in a binary coherent system. A representation for the Birnbaum importance of a component is obtained when the system consists of exchangeable dependent components. The results are closely related to the concept of the signature of a coherent system. Some examples are presented to illustrate the results.

Keywords: Component importance; exchangeability; order statistics; signature

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1. Introduction

Measuring the importance of a component in a system is an important problem in reliability engineering. Various reliability importance measures have been defined and studied in the literature; see Birnbaum (1969), Barlow and Proschan (1975), Butler (1977), Natvig (1979), (1985), Bergman (1985), Xie (1987), Xie and Bergman (1991), Iyer (1992), Boland and El-Neweihi (1995), Hwang (2001), Andrews (2008), Natvig and Gåsemyr (2009), and Natvig (2011). For a lucid review of the topic, see Kuo and Zhu (2012) and Zhu and Kuo (2014).

Consider a binary coherent system consisting of n binary components. Let $X_i(t)$ denote the state of the ith component at time t, $i = 1, \dots, n$, and define

$$X_i(t) = \begin{cases} 1 & \text{if the } i\text{th component functions at time } t, \\ 0 & \text{if the } i\text{th component has failed at time } t. \end{cases}$$

If T_i denotes the lifetime of the ith component, then $\{X_i(t) = 1\} \equiv \{T_i > t\}$, $i = 1, \dots, n$. Let ϕ denote the structure function of the system. Then the state of the system at time t is defined by

$$\phi(X_1(t), \dots, X_n(t)) = \begin{cases} 1 & \text{if the system is functioning at time } t, \\ 0 & \text{if the system has failed at time } t. \end{cases}$$

Let T denote the lifetime of the system. Birnbaum (1969) defined the importance of the ith component at time t by

$$I_i(t) = \mathbb{P}\{T > t \mid T_i > t\} - \mathbb{P}\{T > t \mid T_i \le t\} \quad \text{for } i = 1, \dots, n. \tag{1}$$

The importance measure defined by (1) is included in the class of lifetime importance measures. Such an importance measure is considered when a system and components have long-term or infinite service missions, and depends on both the positions of the components in the system and component lifetime distributions; see Kuo and Zhu (2012).

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In this paper we study the Birnbaum importance measure for coherent systems consisting of exchangeable components. A sequence on lifetimes $T_1, \dots, T_n$ is exchangeable if

$$\mathbb{P}\{T_1 \le t_1, \dots, T_n \le t_n\} = \mathbb{P}\{T_{\pi(1)} \le t_1, \dots, T_{\pi(n)} \le t_n\}$$

for any permutation $\pi = (\pi(1), \dots, \pi(n))$ of $\{1, \dots, n\}$, i.e. the joint distribution (or survival function) of $T_1, \dots, T_n$ is symmetric in $t_1, \dots, t_n$. Exchangeability means that the components have identical distributions but it allows for a certain type of dependence. For example, if the components are positively exchangeable dependent then the failure of one component causes failure of another component to become more likely. For recent discussions on systems with exchangeable components, see Navarro *et al.* (2008), Navarro and Spizzichino (2010), Navarro and Rubio (2011), and Eryilmaz *et al.* (2011).

Our method for studying $I_i(t)$ is closely related to the concept of the system signature. For a coherent system with lifetime $T = \phi(T_1, \dots, T_n)$ the signature is defined by the vector $\boldsymbol{s} = (s_1(n), \dots, s_n(n))$ with

$$s_m(n) = \frac{\text{number of orderings for which the } m\text{th failure causes system failure}}{n!}, \qquad i = 1, \dots, n$$

and $\sum_{m=1}^{n} s_m(n) = 1$; see, e.g. Samaniego (2007). In words, $s_m(n)$ is the proportion of permutations among the $n!$ equally likely permutations of component lifetimes that result in a minimal cut set failure when m components break down. Therefore, we have $s_m(n) = \mathbb{P}\{T = T_{m:n}\}$, $m = 1, \dots, n$, where $T_{m:n}$ is the mth order statistic among $T_1, \dots, T_n$. For a sequence of independent and identical lifetimes $T_1, \dots, T_n$, Samaniego (1985) showed that the survival function of T can be represented as

$$\mathbb{P}\{T > t\} = \sum_{m=1}^{n} s_m(n)\mathbb{P}\{T_{m:n} > t\}.$$

This representation has been extended to the case of exchangeable components by Navarro *et al.* (2005), (2008), and Navarro and Rychlik (2007). For some recent works on signature-based reliability analysis, see Navarro *et al.* (2013), Parvardeh and Balakrishnan (2013), Eryilmaz (2013), Zarazadeh *et al.* (2014), and Triantafyllou and Koutras (2014).

The signature of a coherent system can be computed from

$$s_m(n) = a_{n-m+1}(n) - a_{n-m}(n), \qquad m = 1, 2, \dots, n,$$

where

$$a_m(n) = \frac{r_m(n)}{\binom{n}{m}}, \qquad i = 1, 2, \dots, n,$$

and $r_m(n)$ is the number of path sets of size m; see Boland (2001).

In Section 2 we obtain a representation for $I_i(t)$ when $T_1, \dots, T_n$ are exchangeable dependent. The measure $I_i(t)$ is represented in terms of the conditional probabilities $\mathbb{P}\{T_{m:n} > t \mid T_i > t\}$ and $\mathbb{P}\{T_{m:n} > t \mid T_i \le t\}$ with well-defined coefficients which depend on the structure of a coherent system.

In Theorem 1 we obtain the following equation:

$$I_i(t) = \sum_{m=1}^{n} [s_m^{+i}(n)\mathbb{P}\{T_{m:n} > t \mid T_i > t\} - s_m^{-i}(n)\mathbb{P}\{T_{m:n} > t \mid T_i \le t\}],$$

where the vectors $\boldsymbol{s}^{+i} = (s_1^{+i}(n), \dots, s_n^{+i}(n))$ and $\boldsymbol{s}^{-i} = (s_1^{-i}(n), \dots, s_n^{-i}(n))$ of coefficients depend only on the structure of the system. Some properties of these coefficients are revealed and they are computed for all coherent systems with $n = 3$ and $n = 4$ exchangeable components. In Section 3 we consider the series system whose lifetime is expressed as $T_* = \min(\phi_1(T_1, \dots, T_{n_1}), \phi_2(T_{n_1+1}, \dots, T_{n_1+n_2}))$. We provide a method for computing the vectors $\boldsymbol{s}^{+i}$ and $\boldsymbol{s}^{-i}$ for the system with lifetime T_* based on the signature vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, and $\boldsymbol{p}^{+i}$ and $\boldsymbol{q}^{+i}$ of the disjoint structures ϕ_1 and ϕ_2.

2. A representation for the Birnbaum measure

Define $r_m^{+i}(n)$ to be the number of path sets of size m including component i in a coherent system of order n. Similarly, let $r_m^{-i}(n)$ denote the number of path sets of size m which do not contain the component i in a coherent system of order n. If $r_m(n)$ is the number of path sets of size m then clearly, we have

$$r_m(n) = r_m^{+i}(n) + r_m^{-i}(n)$$

with $r_0^{+i}(n) = 0$, $r_n^{+i}(n) = 1$, $r_0^{-i}(n) = 0$, and $r_n^{-i}(n) = 0$.

For an illustration, consider the system with structure function

$$\phi(x_1, x_2, x_3) = \max(x_2, \min(x_1, x_3)).$$

The corresponding path sets are $\{2\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. For $i = 1$, we have $r_1^{+1}(n) = 0$, $r_1^{-1}(n) = 1$, $r_2^{+1}(n) = 2$, $r_2^{-1}(n) = 1$, $r_3^{+1}(n) = 1$, $r_3^{-1}(n) = 0$.

In the following theorem, we obtain a representation for the Birnbaum importance of the ith component in an arbitrary coherent system of exchangeable components.

Theorem 1. *For a coherent system consisting of n exchangeable components with lifetimes $T_1, \dots, T_n$,*

$$I_i(t) = \sum_{m=1}^{n} [s_m^{+i}(n)\mathbb{P}\{T_{m:n} > t \mid T_i > t\} - s_m^{-i}(n)\mathbb{P}\{T_{m:n} > t \mid T_i \le t\}], \tag{2}$$

where

$$s_m^{+i}(n) = \frac{r_{n-m+1}^{+i}(n)}{\binom{n-1}{n-m}} - \frac{r_{n-m}^{+i}(n)}{\binom{n-1}{n-m-1}} \quad \textit{for } m = 1, \dots, n, \tag{3}$$

$$s_m^{-i}(n) = \frac{r_{n-m+1}^{-i}(n)}{\binom{n-1}{n-m+1}} - \frac{r_{n-m}^{-i}(n)}{\binom{n-1}{n-m}} \quad \textit{for } m = 2, \dots, n \tag{4}$$

with $s_1^{-i}(n) = 0$.

Proof. Consider first the conditional probability $\mathbb{P}\{T > t \mid T_i > t\}$. Let $S_n(t)$ denote the total number of working components in a system at time t. Then

$$\mathbb{P}\{T > t \mid T_i > t\} = \sum_{m=1}^{n} \mathbb{P}\{T > t \mid T_i > t, S_n(t) = m\}\mathbb{P}\{S_n(t) = m \mid T_i > t\}.$$

Clearly, we have

$$\mathbb{P}\{T > t \mid T_i > t, S_n(t) = m\} = \frac{r_m^{+i}(n)}{\binom{n-1}{m-1}},$$

$$\mathbb{P}\{S_n(t) = m \mid T_i > t\} = \mathbb{P}\{T_{n-m+1:n} > t \mid T_i > t\} - \mathbb{P}\{T_{n-m:n} > t \mid T_i > t\}.$$

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Therefore,

$$P\{T>t\mid T_i>t\}=\sum_{m=1}^{n}\frac{r_m^{+i}(n)}{\binom{n-1}{m-1}}[\mathbb{P}\{T_{n-m+1:n}>t\mid T_i>t\}-\mathbb{P}\{T_{n-m:n}>t\mid T_i>t\}]$$
$$=\sum_{m=1}^{n}\left[\frac{r_{n-m+1}^{+i}(n)}{\binom{n-1}{n-m}}-\frac{r_{n-m}^{+i}(n)}{\binom{n-1}{n-m-1}}\right]\mathbb{P}\{T_{m:n}>t\mid T_i>t\}. \tag{5}$$

On the other hand,

$$\mathbb{P}\{T>t\mid T_i\le t\}=\sum_{m=1}^{n-1}\mathbb{P}\{T>t\mid T_i\le t, S_n(t)=m\}\mathbb{P}\{S_n(t)=m\mid T_i\le t\}.$$

Because

$$\mathbb{P}\{T>t\mid T_i\le t, S_n(t)=m\}=\frac{r_m^{-i}(n)}{\binom{n-1}{m}},$$

we have

$$\mathbb{P}\{T>t\mid T_i\le t\}=\sum_{m=2}^{n}\left[\frac{r_{n-m+1}^{-i}(n)}{\binom{n-1}{n-m+1}}-\frac{r_{n-m}^{-i}(n)}{\binom{n-1}{n-m}}\right]\mathbb{P}\{T_{m:n}>t\mid T_i\le t\}. \tag{6}$$

Thus, the proof is completed by substituting (5) and (6) into (1).

For a sequence of exchangeable lifetimes $T_1,\dots,T_n$, the expressions for the conditional probabilities $\mathbb{P}\{T_{m:n}>t\mid T_i>t\}$ and $\mathbb{P}\{T_{m:n}>t\mid T_i\le t\}$ are presented in the following lemma.

Lemma 1. *For exchangeable lifetimes $T_1,\dots,T_n$,*

$$\mathbb{P}\{T_{m:n}>t\mid T_i>t\}$$
$$=\frac{1}{\mathbb{P}\{T_i>t\}}\sum_{j=n-m}^{n-1}\sum_{s=0}^{n-j-1}(-1)^s\binom{n-1}{j}\binom{n-j-1}{s}\mathbb{P}\{T_{1:s+j+1}>t\} \tag{7}$$

and

$$\mathbb{P}\{T_{m:n}>t\mid T_i\le t\}$$
$$=\frac{1}{\mathbb{P}\{T_i\le t\}}\sum_{j=n-m+1}^{n-1}\sum_{s=0}^{n-j}(-1)^s\binom{n-1}{j}\binom{n-j}{s}\mathbb{P}\{T_{1:s+j}>t\}, \tag{8}$$

where $T_{1:i}=\min(T_1,\dots,T_i)$ *for* $i\ge 1$.

Proof. It is clear that

$$\mathbb{P}\{T_{m:n}>t, T_i>t\}$$
$$=\sum_{j=n-m}^{n-1}\mathbb{P}\{j \text{ of } T_1,\dots,T_{i-1},T_{i+1},\dots,T_n \text{ is greater than } t, T_i>t\}. \tag{9}$$

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Because $T_1, \dots, T_n$ are exchangeable,

$$\begin{aligned}&\mathbb{P}\{j \text{ of } T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n \text{ is greater than } t, T_i > t\}\\ &\quad = \binom{n-1}{j}\mathbb{P}\{T_1 > t, \dots, T_j > t, T_{j+1} \le t, \dots, T_{n-1} \le t, T_i > t\}.\end{aligned}$$

For a sequence of exchangeable binary variables $\xi_1, \dots, \xi_n$, it is known that

$$\begin{aligned}&\mathbb{P}\{\xi_1 = 1, \dots, \xi_k = 1, \xi_{k+1} = 0, \dots, \xi_n = 0\}\\ &\quad = \sum_{s=0}^{n-k}(-1)^s\binom{n-k}{s}\mathbb{P}\{\xi_1 = 1, \dots, \xi_{s+k} = 1\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{P}\{j \text{ of } T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n \text{ is greater than } t, T_i > t\}\\ &\quad = \binom{n-1}{j}\mathbb{P}\{T_1 > t, \dots, T_j > t, T_{j+1} \le t, \dots, T_{n-1} \le t, T_i > t\}\\ &\quad = \binom{n-1}{j}\sum_{s=0}^{n-j-1}(-1)^s\binom{n-j-1}{s}\mathbb{P}\{T_1 > t, \dots, T_{s+j+1} > t\}. \qquad (10)\end{aligned}$$

Thus, the proof of (7) is completed using (10) in (9). The proof of (8) is similar and, hence, omitted.

The following theorem reveals some properties of the vectors of coefficients $\boldsymbol{s}^{+i}$ and $\boldsymbol{s}^{-i}$.

Theorem 2. *The following properties are satisfied for the vectors* $\boldsymbol{s} = (s_1(n), \dots, s_n(n))$, $\boldsymbol{s}^{+i} = (s_1^{+i}(n), \dots, s_n^{+i}(n))$, *and* $\boldsymbol{s}^{-i} = (s_1^{-i}(n), \dots, s_n^{-i}(n))$.

(i) *The arithmetic mean of the mth component of* $\boldsymbol{s}^{+i}$ *over all* $i = 1, \dots, n$ *is equal to the mth element of the signature vector* $\boldsymbol{s} = (s_1(n), \dots, s_n(n))$, *i.e.*

$$\frac{1}{n}\sum_{i=1}^{n} s_m^{+i}(n) = s_m(n) \quad \textit{for all } m = 1, \dots, n.$$

(ii) *For a fixed* i,

$$\sum_{m=1}^{n} s_m^{+i}(n) = 1.$$

(iii) *For a fixed* i, *if there is at least one* m *such that* $s_m^{-i}(n) \neq 0$, *then*

$$\sum_{m=1}^{n} s_m^{-i}(n) = 1.$$

Proof. From (3), we have

$$\begin{aligned}\sum_{i=1}^{n} s_m^{+i}(n) &= \sum_{i=1}^{n}\biggl[\frac{r_{n-m+1}^{+i}(n)}{\binom{n-1}{n-m}} - \frac{r_{n-m}^{+i}(n)}{\binom{n-1}{n-m-1}}\biggr]\\ &= \frac{1}{\binom{n-1}{n-m}}\sum_{i=1}^{n} r_{n-m+1}^{+i}(n) - \frac{1}{\binom{n-1}{n-m-1}}\sum_{i=1}^{n} r_{n-m}^{+i}(n)\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\binom{n-1}{n-m}}(n-m+1)r_{n-m+1}(n) - \frac{1}{\binom{n-1}{n-m-1}}(n-m)r_{n-m}(n) \\
&= n\left[\frac{r_{n-m+1}(n)}{\binom{n}{n-m+1}} - \frac{r_{n-m}(n)}{\binom{n}{n-m}}\right] \\
&= ns_m(n).
\end{aligned}
$$

Thus, Theorem 2(i) is proved. There is only one path set of size n which includes all components, i.e. $r_n^{+i}(n) = 1$. Therefore, the proof of Theorem 2(ii) follows since, from (3), we have

$$\sum_{m=1}^{n} s_m^{+i}(n) = \frac{r_n^{+i}(n)}{\binom{n-1}{n-1}}.$$

If $s_m^{-i}(n) \neq 0$ for at least one m then there is at least one path set which does not include the ith component. Such a path set is always the subset of the path set $\{1, \ldots, i-1, i+1, \ldots, n\}$. Therefore, $r_{n-1}^{-i}(n) = 1$. The proof of Theorem 2(iii) follows from

$$\sum_{m=1}^{n} s_m^{-i}(n) = \frac{r_{n-1}^{-i}(n)}{\binom{n-1}{n-1}}.$$

Proposition 1. *Let* $\mathbf{s}^{+i} = (s_1^{+i}(n), \ldots, s_n^{+i}(n))$ *be the vector associated with the structure* ϕ*, and* $\mathbf{z}^{-i} = (z_1^{-i}(n), \ldots, z_n^{-i}(n))$ *be the vector associated with the dual structure* ϕ^{D}*. Then*

$$z_m^{-i}(n) = s_{n-m+1}^{+i}(n) \quad \textit{for } m = 2, \ldots, n, \; z_1^{-i}(n) = 0.$$

Proof. From (3), we have

$$s_{n-m+1}^{+i}(n) = \frac{r_m^{+i}(n)}{\binom{n-1}{m-1}} - \frac{r_{m-1}^{+i}(n)}{\binom{n-1}{m-2}}.$$

If $d_m^{-i}(n)$ denotes the number of path sets which do not include ith component for the dual structure ϕ^{D} , then

$$r_m^{+i}(n) = \binom{n-1}{m-1} - d_{n-m}^{-i}(n).$$

Thus,

$$s_{n-m+1}^{+i}(n) = \frac{d_{n-m+1}^{-i}(n)}{\binom{n-1}{m-2}} - \frac{d_{n-m}^{-i}(n)}{\binom{n-1}{m-1}} = z_m^{-i}(n) \quad \text{for } m = 2, \ldots, n.$$

By definition $z_1^{-i}(n) = 0$.

In the special case of independent and identical components, the conditional probabilities in (2) become marginal probabilities of order statistics. The result is given in the following corollary.

Corollary 1. *For a coherent system consisting of n independent and identical components,*

$$I_i(t) = \sum_{m=1}^{n} [s_m^{+i}(n)\mathbb{P}\{T_{m\colon n-1} > t\} - s_m^{-i}(n)\mathbb{P}\{T_{m-1\colon n-1} > t\}]$$

with $\mathbb{P}\{T_{n\colon n-1} > t\} = 1$ *and* $\mathbb{P}\{T_{0\colon n-1} > t\} = 0$.

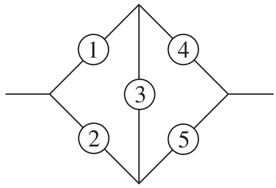

FIGURE 1: Bridge system.

Proof. If $T_1, \dots, T_n$ are independent and identical then

$$\mathbb{P}\{T_{m:\,n} > t \mid T_i > t\} = \mathbb{P}\{T_{m:\,n-1} > t\}, \qquad \mathbb{P}\{T_{m:\,n} > t \mid T_i \le t\} = \mathbb{P}\{T_{m-1:\,n-1} > t\}.$$

Thus, the proof follows from Theorem 1.

Example 1. Consider the bridge system depicted in Figure 1. For this system

$$\boldsymbol{s}^{+i} = \big(0, \tfrac{1}{6}, \tfrac{7}{12}, \tfrac{1}{4}, 0\big), \qquad \boldsymbol{s}^{-i} = \big(0, \tfrac{1}{4}, \tfrac{7}{12}, \tfrac{1}{6}, 0\big) \quad \text{for } i = 1, 2, 4, 5,$$
$$\boldsymbol{s}^{+3} = \big(0, \tfrac{1}{3}, \tfrac{2}{3}, 0, 0\big), \qquad \boldsymbol{s}^{-3} = (0, 0, 1, 0, 0).$$

Using Theorem 2, the signature of this bridge system is $\boldsymbol{s} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$. From Theorem 1,

$$\begin{aligned} I_i(t) = {} & \big[\tfrac{1}{6}\mathbb{P}\{T_{2:\,5} > t \mid T_i > t\} + \tfrac{7}{12}\mathbb{P}\{T_{3:\,5} > t \mid T_i > t\} + \tfrac{1}{4}\mathbb{P}\{T_{4:\,5} > t \mid T_i > t\}\big] \\ & - \big[\tfrac{1}{4}\mathbb{P}\{T_{2:\,5} > t \mid T_i \le t\} + \tfrac{7}{12}\mathbb{P}\{T_{3:\,5} > t \mid T_i \le t\} + \tfrac{1}{6}\mathbb{P}\{T_{4:\,5} > t \mid T_i \le t\}\big] \end{aligned}$$

for $i = 1, 2, 4, 5$, and

$$I_3(t) = \tfrac{1}{3}\mathbb{P}\{T_{2:\,5} > t \mid T_3 > t\} + \tfrac{2}{3}\mathbb{P}\{T_{3:\,5} > t \mid T_3 > t\} - \mathbb{P}\{T_{3:\,5} > t \mid T_3 \le t\}.$$

The computation of the elements of the vectors $\boldsymbol{s}$, $\boldsymbol{s}^{+i}$, and $\boldsymbol{s}^{-i}$ requires the numbers $r_m(n)$, $r_m^{+i}(n)$, and $r_m^{-i}(n)$ for a given structure and i. Although these numbers can be obtained by finding all path sets of the structure for coherent systems with few components, their computation for a system with arbitrary number of components is a well-defined combinatorial problem. In Table 1 we present values for the vectors $\boldsymbol{s}^{+i}$ and $\boldsymbol{s}^{-i}$, $i = 1, \dots, n$ for all coherent systems of order $n = 3$. (See Eryilmaz (2015) for the table of results for the $n = 4$ case.) The elements of these vectors are computed by listing all path sets of coherent systems with $n = 3$ and $n = 4$ components. On the other hand, in the following example we illustrate the computation of the numbers $r_m(n)$, $r_m^{+i}(n)$, and $r_m^{-i}(n)$ for a specific structure using combinatorial arguments.

Example 2. Assume that the system has a linear consecutive-*k-out-of-n*: F structure, i.e. it consists of n linearly ordered components, and fails if and only if at least k consecutive components fail. For this structure, the number of path sets of size m is given by

$$r_m(n) = \sum_{j=0}^{\min(m+1,[(n-m)/k])} (-1)^j \binom{m+1}{j}\binom{n-jk}{m};$$

see, e.g. Eryilmaz (2010). Under the condition that the ith component is in a working state, the linear consecutive-*k-out-of-n*: F system can be decomposed into two modules, where the first

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TABLE 1: The vectors $\boldsymbol{s}$, $\boldsymbol{s}^{+i}$, and $\boldsymbol{s}^{-i}$ for all coherent systems of order $n = 3$.

System type	i	$\boldsymbol{s}^{+i}$	$\boldsymbol{s}^{-i}$
Series. $\boldsymbol{s} = (1, 0, 0)$			
	1	$(1, 0, 0)$	$(0, 0, 0)$
	2	$(1, 0, 0)$	$(0, 0, 0)$
	3	$(1, 0, 0)$	$(0, 0, 0)$
$\min(T_2, \max(T_1, T_3))$. $\boldsymbol{s} = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$			
	1	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 1, 0)$
	2	$(0, 1, 0)$	$(0, 0, 0)$
	3	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 1, 0)$
2-*out-of*-3: F. $\boldsymbol{s} = (0, 1, 0)$			
	1	$(0, 1, 0)$	$(0, 1, 0)$
	2	$(0, 1, 0)$	$(0, 1, 0)$
	3	$(0, 1, 0)$	$(0, 1, 0)$
$\max(T_2, \min(T_1, T_3))$. $\boldsymbol{s} = \left(0, \frac{2}{3}, \frac{1}{3}\right)$			
	1	$(0, 1, 0)$	$(0, \frac{1}{2}, \frac{1}{2})$
	2	$(0, 0, 1)$	$(0, 1, 0)$
	3	$(0, 1, 0)$	$(0, \frac{1}{2}, \frac{1}{2})$
Parallel. $\boldsymbol{s} = (0, 0, 1)$			
	1	$(0, 0, 1)$	$(0, 0, 1)$
	2	$(0, 0, 1)$	$(0, 0, 1)$
	3	$(0, 0, 1)$	$(0, 0, 1)$

(second) module consists of $i-1$ $(n-i)$ components. Assume that the first module includes m_1 working components. Then the second module includes $m - m_1 - 1$ components. Thus,

$$r_m^{+i}(n) = \sum_{m_1=\max(0,m-n+i-1)}^{\min(i-1,m-1)} r_{m_1}(i-1) r_{m-m_1-1}(n-i), \qquad r_m^{-i}(n) = r_m(n) - r_m^{+i}(n).$$

3. Series system with two modules

Consider a coherent system consisting of $n = n_1 + n_2$ components with the index set of components $C = \{1, \dots, n_1 + n_2\}$. Suppose that the system with the component index set C consists of two modules with respective component index sets $\{1, \dots, n_1\}$ and $\{n_1+1, \dots, n_1+n_2\}$ and structure functions ϕ_1 and ϕ_2. If the overall system has a series structure, i.e. the disjoint modules are serially connected, then the system's lifetime is represented as

$$T_* = \min(\phi_1(T_1, \dots, T_{n_1}), \phi_2(T_{n_1+1}, \dots, T_n)). \tag{11}$$

Let k_i denote the minimum number of working components for the functioning of the ith module, $i = 1, 2$. If $h_m(n)$ denotes the number of path sets of size m of the system with the

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lifetime T_* , then

$$h_m(n) = \sum_{j=\max(k_1, m-n_2)}^{\min(m-k_2, n_1)} u_j(n_1) v_{m-j}(n_2) \quad \text{for } m \geq k_1 + k_2, \tag{12}$$

where $u_m(n_1)$ and $v_m(n_2)$ represent respectively the number of path sets of size m of the structuresϕ_1 and ϕ_2; see Eryilmaz (2014).

In this section we aim to derive formulae for the vectors $\boldsymbol{s}^{+i}$ and $\boldsymbol{s}^{-i}$ associated with the system (11) based on the corresponding vectors of the structures ϕ_1 and ϕ_2.

In the following theorem, we obtain an expression for the number of path sets of size m including the ith component for the system with lifetime T_* in terms of the number of path sets of the structures ϕ_1 and ϕ_2. Denote by $u_m^{+i}(n_1)$ $(v_m^{+i}(n_2))$ the number of path sets of size m including the ith component for the system with structure ϕ_1 (ϕ_2).

Theorem 3. *The number of path sets of size m including the i th component for the system with lifetime T_* is*

$$r_m^{+i}(n) = \begin{cases} \displaystyle\sum_{j=\max(k_1^{+i}, m-n_2)}^{\min(n_1, m-k_2)} u_j^{+i}(n_1) v_{m-j}(n_2) & \text{if } i \in \{1, \dots, n_1\}, \\ \displaystyle\sum_{l=\max(k_2^{+i}, m-n_1)}^{\min(n_2, m-k_1)} u_{m-l}(n_1) v_l^{+i}(n_2) & \text{if } i \in \{n_1+1, \dots, n_1+n_2\}, \end{cases}$$

where k_1^{+i} (k_2^{+i}) is the size of the path set which has the smallest size and includes the i th component for the system with structure ϕ_1 (ϕ_2).

Proof. For $i \in \{1, \dots, n_1\}$,

$$r_m^{+i}(n) = \sum_{\substack{j+l=m \\ j \geq k_1^{+i}, l \geq k_2}} \sum u_j^{+i}(n_1) v_l(n_2),$$

which follows from the fact that both modules must work for the series system to function and that the first module works and includes j working components being the ith component in a functioning state, and the second module works and includes l working components such that $j \geq k_1^{+i}$, $l \geq k_2$, and $j + l = m$. Thus, we have

$$r_m^{+i}(n) = \sum_{j=\max(k_1^{+i}, m-n_2)}^{\min(n_1, m-k_2)} u_j^{+i}(n_1) v_{m-j}(n_2) \quad \text{for } i \in \{1, \dots, n_1\}.$$

The case when the ith component belongs to the second module can be proved similarly.

The following result is immediate since for any coherent system $r_m^{+i}(n) + r_m^{-i}(n) = r_m(n)$.

Corollary 2. *The number of path sets of size m which do not include the i th component for the system with lifetime T_* is*

$$r_m^{-i}(n) = \sum_{j=\max(k_1, m-n_2)}^{\min(m-k_2, n_1)} u_j(n_1) v_{m-j}(n_2) - r_m^{+i}(n),$$

where $r_m^{+i}(n)$ is given by Theorem 3 for $i =, \dots, n$.

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Let $\boldsymbol{p} = (p_1(n_1), \dots, p_{n_1}(n_1))$ and $\boldsymbol{q} = (q_1(n_2), \dots, q_2(n_2))$ denote the signatures of the structures ϕ_1 and ϕ_2, respectively. Then

$$u_j(n_1) = \binom{n_1}{j} \sum_{a=n_1-j+1}^{n_1} p_a(n_1), \qquad v_l(n_2) = \binom{n_2}{l} \sum_{b=n_2-l+1}^{n_2} q_b(n_2). \tag{13}$$

On the other hand,

$$u_j^{+i}(n_1) = \binom{n_1-1}{j-1} \sum_{a=n_1-j+1}^{n_1} p_a^{+i}(n_1) \quad \text{for } i \in \{1, \dots, n_1\} \tag{14}$$

and

$$v_l^{+i}(n_2) = \binom{n_2-1}{l-1} \sum_{b=n_2-l+1}^{n_2} q_b^{+i}(n_2) \quad \text{for } i \in \{n_1+1, \dots, n_1+n_2\}. \tag{15}$$

Using (13)–(15) in Theorem 3, we have

$$\begin{aligned} r_m^{+i}(n) = &\sum_{j=\max(k_1^{+i}, m-n_2)}^{\min(n_1, m-k_2)} \binom{n_1-1}{j-1} \sum_{a=n_1-j+1}^{n_1} p_a^{+i}(n_1) \binom{n_2}{m-j} \\ &\times \sum_{b=n_2-m+j+1}^{n_2} q_b(n_2) \quad \text{for } i \in \{1, \dots, n_1\} \end{aligned} \tag{16}$$

and

$$\begin{aligned} r_m^{+i}(n) = &\sum_{l=\max(k_2^{+i}, m-n_1)}^{\min(n_2, m-k_1)} \binom{n_1}{m-l} \sum_{a=n_1-m+l+1}^{n_1} p_a(n_1) \binom{n_2-1}{l-1} \\ &\times \sum_{b=n_2-l+1}^{n_2} q_b^{+i}(n_2) \quad \text{for } i \in \{n_1+1, \dots, n_1+n_2\}. \end{aligned} \tag{17}$$

Therefore, using (3), the vector of coefficients $\boldsymbol{s}^{+i} = (s_1^{+i}(n), \dots, s_n^{+i}(n))$ for the system with lifetime T_* is found to be

$$\begin{aligned} s_m^{+i}(n) = &\frac{1}{\binom{n-1}{n-m}} \sum_{j=\max(k_1^{+i}, n-n_2-m+1)}^{\min(n_1, n-k_2-m+1)} \binom{n_1-1}{j-1} \sum_{a=n_1-j+1}^{n_1} p_a^{+i}(n_1) \\ &\times \binom{n_2}{n-m-j+1} \sum_{b=n_2+m+j-n}^{n_2} q_b(n_2) \\ &- \frac{1}{\binom{n-1}{n-m-1}} \sum_{j=\max(k_1^{+i}, n-n_2-m)}^{\min(n_1, n-k_2-m)} \binom{n_1-1}{j-1} \sum_{a=n_1-j+1}^{n_1} p_a^{+i}(n_1) \\ &\times \binom{n_2}{n-m-j} \sum_{b=n_2+m+j-n+1}^{n_2} q_b(n_2) \quad \text{for } m = 1, \dots, n,\ i \in \{1, \dots, n_1\}. \end{aligned} \tag{18}$$

Similarly, for $i \in \{n_1+1, \ldots, n_1+n_2\}$,

$$\begin{aligned} s_m^{+i}(n) = {} & \frac{1}{\binom{n-1}{n-m}} \sum_{l=\max(k_2^{+i},n-m-n_1+1)}^{\min(n_2,n-m-k_1+1)} \binom{n_1}{n-m-l+1} \sum_{a=n_1-n+m+l}^{n_1} p_a(n_1) \\ & \times \binom{n_2-1}{l-1} \sum_{b=n_2-l+1}^{n_2} q_b^{+i}(n_2) \\ & - \frac{1}{\binom{n-1}{n-m-1}} \sum_{l=\max(k_2^{+i},n-m-n_1)}^{\min(n_2,n-m-k_1)} \binom{n_1}{n-m-l} \sum_{a=n_1-n+m+l+1}^{n_1} p_a(n_1) \\ & \times \binom{n_2-1}{l-1} \sum_{b=n_2-l+1}^{n_2} q_b^{+i}(n_2) \quad \text{for } m=1,\ldots,n. \end{aligned} \tag{19}$$

Thus, the vector of coefficients $\boldsymbol{s}^{+i} = (s_1^{+i}(n), \ldots, s_n^{+i}(n))$ for the system with lifetime T_* can be computed using the signature vectors

$$\begin{aligned} \boldsymbol{p} &= (p_1(n_1), \ldots, p_{n_1}(n_1)), & \boldsymbol{q} &= (q_1(n_2), \ldots, q_2(n_2)), \\ \boldsymbol{p}^{+i} &= (p_1^{+i}(n_1), \ldots, p_{n_1}^{+i}(n_1)), & \boldsymbol{q}^{+i} &= (q_1^{+i}(n_2), \ldots, q_{n_2}^{+i}(n_2)). \end{aligned}$$

Using Corollary 2 with (13), (16), and (17), we obtain

$$\begin{aligned} r_m^{-i}(n) = {} & \sum_{j=\max(k_1,m-n_2)}^{\min(m-k_2,n_1)} \binom{n_1}{j} \sum_{a=n_1-j+1}^{n_1} p_a(n_1) \binom{n_2}{m-j} \sum_{b=n_2-m+j+1}^{n_2} q_b(n_2) \\ & - \sum_{j=\max(k_1^{+i},m-n_2)}^{\min(n_1,m-k_2)} \binom{n_1-1}{j-1} \sum_{a=n_1-j+1}^{n_1} p_a^{+i}(n_1) \binom{n_2}{m-j} \sum_{b=n_2-m+j+1}^{n_2} q_b(n_2) \end{aligned} \tag{20}$$

for $i \in \{1, \ldots, n_1\}$, and

$$\begin{aligned} r_m^{-i}(n) = {} & \sum_{j=\max(k_1,m-n_2)}^{\min(m-k_2,n_1)} \binom{n_1}{j} \sum_{a=n_1-j+1}^{n_1} p_a(n_1) \binom{n_2}{m-j} \sum_{b=n_2-m+j+1}^{n_2} q_b(n_2) \\ & - \sum_{l=\max(k_2^{+i},m-n_1)}^{\min(n_2,m-k_1)} \binom{n_1}{m-l} \sum_{a=n_1-m+l+1}^{n_1} p_a(n_1) \binom{n_2-1}{l-1} \sum_{b=n_2-l+1}^{n_2} q_b^{+i}(n_2) \end{aligned} \tag{21}$$

for $i \in \{n_1+1, \ldots, n_1+n_2\}$.

Thus, using (4), the vector of coefficients $\boldsymbol{s}^{-i} = (s_1^{-i}(n), \ldots, s_n^{-i}(n))$ for the system with lifetime T_* can be computed using the signature vectors

$$\begin{aligned} \boldsymbol{p} &= (p_1(n_1), \ldots, p_{n_1}(n_1)), & \boldsymbol{q} &= (q_1(n_2), \ldots, q_2(n_2)), \\ \boldsymbol{p}^{+i} &= (p_1^{+i}(n_1), \ldots, p_{n_1}^{+i}(n_1)), & \boldsymbol{q}^{+i} &= (q_1^{+i}(n_2), \ldots, q_{n_2}^{+i}(n_2)). \end{aligned}$$

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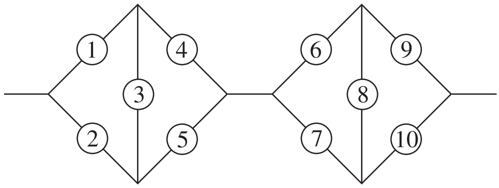

FIGURE 2: Series composition of two bridge structures

Example 3. Consider the series composition of the bridge structures as shown in Figure 2. From Example 1, the signatures of the modules are $\boldsymbol{p} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$ and $\boldsymbol{q} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$, and

$$\boldsymbol{p}^{+i} = \big(0, \tfrac{1}{6}, \tfrac{7}{12}, \tfrac{1}{4}, 0\big), \qquad \boldsymbol{p}^{-i} = \big(0, \tfrac{1}{4}, \tfrac{7}{12}, \tfrac{1}{6}, 0\big) \quad \text{for } i = 1, 2, 4, 5,$$
$$\boldsymbol{p}^{+3} = \big(0, \tfrac{1}{3}, \tfrac{2}{3}, 0, 0\big), \qquad \boldsymbol{p}^{-3} = (0, 0, 1, 0, 0),$$

and

$$\boldsymbol{q}^{+i} = \big(0, \tfrac{1}{6}, \tfrac{7}{12}, \tfrac{1}{4}, 0\big), \qquad \boldsymbol{q}^{-i} = \big(0, \tfrac{1}{4}, \tfrac{7}{12}, \tfrac{1}{6}, 0\big) \quad \text{for } i = 6, 7, 9, 10,$$
$$\boldsymbol{q}^{+8} = \big(0, \tfrac{1}{3}, \tfrac{2}{3}, 0, 0\big), \qquad \boldsymbol{q}^{-8} = (0, 0, 1, 0, 0).$$

By (19) and (20), we obtain

$$\boldsymbol{s}^{+i} = \big(0, \tfrac{1}{12}, \tfrac{17}{84}, \tfrac{37}{126}, \tfrac{5}{18}, \tfrac{5}{42}, \tfrac{1}{42}, 0, 0, 0\big) \quad \text{for } i = 1, 2, 4, 5, 6, 7, 9, 10,$$

and

$$\boldsymbol{s}^{+i} = \big(0, \tfrac{1}{9}, \tfrac{31}{126}, \tfrac{41}{126}, \tfrac{16}{63}, \tfrac{4}{63}, 0, 0, 0, 0\big) \quad \text{for } i = 3, 8.$$

From Theorem 2, the signature of the overall series system is obtained as

$$\boldsymbol{s} = \big(0, \tfrac{4}{45}, \tfrac{19}{90}, \tfrac{3}{10}, \tfrac{86}{315}, \tfrac{34}{315}, \tfrac{2}{105}, 0, 0, 0, \big)$$

which coincides with the result in Da *et al.* (2012). Substituting (20) and (21) into (4), we obtain

$$\boldsymbol{s}^{-i} = \big(0, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{25}{84}, \tfrac{65}{252}, \tfrac{2}{21}, \tfrac{1}{63}, 0, 0, 0\big) \quad \text{for } i = 1, 2, 4, 5, 6, 7, 9, 10$$

and

$$\boldsymbol{s}^{-i} = \big(0, 0, \tfrac{2}{9}, \tfrac{47}{126}, \tfrac{43}{126}, \tfrac{2}{63}, \tfrac{2}{63}, 0, 0, 0\big) \quad \text{for } i = 3, 8.$$

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