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A NOTE ON SPACES $C_p(X)$ K-ANALYTIC-FRAMED IN $\mathbb{R}^X$

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Abstract

This paper characterizes the K-analyticity-framedness in $\mathbb{R}^X$ for $C_p(X)$ (the space of real-valued continuous functions on X with pointwise topology) in terms of $C_p(X)$. This is used to extend Tkachuk's result about the K-analyticity of spaces $C_p(X)$ and to supplement the Arkhangel′skiĭ–Calbrix characterization of σ-compact cosmic spaces. A partial answer to an Arkhangel′skiĭ–Calbrix problem is also provided.

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1. Preliminaries

Christensen [12] proved that a metric and separable space X is σ-compact if and only if $C_p(X)$ is analytic, that is, a continuous image of the Polish space $\mathbb{N}^{\mathbb{N}}$. Calbrix [11] showed that a completely regular Hausdorff space X is σ-compact if $C_p(X)$ is analytic. The converse does not hold in general; for if $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $X = \mathbb{N} \cup \{\xi\}$, where the set of natural numbers $\mathbb{N}$ is considered with the discrete topology, then $C_p(X)$ is a metrizable Baire space [17] but not even K-analytic by [22, p. 64]. A closely related result is given in [4]: *A regular cosmic space X is σ-compact if and only if $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$*, that is, there exists a K-analytic space Y such that $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, although it was already known [18] that if X is σ-bounded (that is, a countable union of functionally bounded sets), then $C_p(X)$ is $K_{\sigma\delta}$-framed in $\mathbb{R}^X$.

In this note we prove: (a) that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if $C_p(X)$ has a bounded resolution, that is a family $\{A_\alpha \mid \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets covering $C_p(X)$ with $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$ such that each A_α is pointwise bounded; and (b) $C_p(X)$ with

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a bounded resolution is an angelic space. Then [4, Theorem 3.4] combined with (a) yields that a regular *cosmic* space X (that is, a continuous image of a metric separable space) is σ-compact if and only if $C_p(X)$ has a bounded resolution. Hence, for metric separable X the space $C_p(X)$ is analytic if and only if it has a bounded resolution. Part (b) implies that for any topology ξ on $C(X)$ stronger than the pointwise one the space $(C(X), \xi)$ is K-analytic if and only if it is quasi-Souslin if and only if it admits a (relatively countably) compact resolution. This extends a recent result of Tkachuk [21] and answers a question of Bierstedt (personal communication): *What about Tkachuk's theorem for topologies on $C(X)$ different from the pointwise one*? We apply Proposition 1 (and Corollary 2) to give a partial answer to [4, Problem 1].

A topological Hausdorff space (or *space* for short) X is called:

(i) *analytic*, if X is a continuous image of the space $\mathbb{N}^{\mathbb{N}}$;

(ii) *K-analytic*, if there is an upper semi-continuous (usc) set-valued map from $\mathbb{N}^{\mathbb{N}}$ with compact values in X whose union is X;

(iii) *quasi-Souslin*, if there exists a set-valued map T from $\mathbb{N}^{\mathbb{N}}$ covering X such that if $(\alpha_n)_n$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to α in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has an adherent point in X belonging to $T(\alpha)$;

(iv) *Lindelöf Σ* (also called K-countably determined) if there exists a usc set-valued map from a subspace of $\mathbb{N}^{\mathbb{N}}$ with compact values in X covering X.

It is known that a space X is Lindelöf Σ if and only if it has a countable network modulo some compact cover of X; see [1]. Recall that analytic $\Rightarrow$ K-analytic $\Rightarrow$ quasi-Souslin and K-analytic $\Rightarrow$ Lindelöf Σ.

By Talagrand [20] every K-analytic space admits a compact resolution, although the converse does not hold in general. Talagrand [20] showed that for a compact space X the space $C_p(X)$ is K-analytic if and only if $C_p(X)$ has a compact resolution. Canela [5] extended this result to paracompact and locally compact spaces X. Finally, Tkachuk [21] extended Talagrand's result to any completely regular Hausdorff space X.

A space X is *angelic* if every relatively countably compact set A in X is relatively compact and each $x \in \overline{A}$ is the limit of a sequence of A. In angelic spaces (relative) compact sets, (relative) countable compact sets and (relative) sequential compact sets are the same; see [16].

2. Bounded resolutions in $C_p(X)$ and K-analytic-framedness of $C_p(X)$ in $\mathbb{R}^X$

We start with the following, where $\overline{B}^{\mathbb{R}^X}$ denotes the closure of B in the space $\mathbb{R}^X$.

LEMMA 1. *Let X be a nonempty set and let Z be a subspace of $\mathbb{R}^X$. If Z has a countable network modulo a cover $\mathcal{B}$ of Z by pointwise bounded subsets, then $Y = \bigcup\{\overline{B}^{\mathbb{R}^X} \mid B \in \mathcal{B}\}$ is a Lindelöf Σ-space such that $Z \subseteq Y \subseteq \mathbb{R}^X$.*

PROOF. Let $\mathcal{N} = \{T_n \mid n \in \mathbb{N}\}$ be a countable network modulo a cover $\mathcal{B}$ of Z consisting of pointwise bounded sets. Set $\mathcal{N}_1 = \{\overline{T_n}^{\mathbb{R}^X} \mid n \in \mathbb{N}\}$, $\mathcal{B}_1 = \{\overline{B}^{\mathbb{R}^X} \mid B \in \mathcal{B}\}$

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and $Y = \cup\mathcal{B}_1$. Clearly every element of $\mathcal{B}_1$ is a compact subset of $\mathbb{R}^X$. We show that $\mathcal{N}_1$ is a network in Y modulo the compact cover $\mathcal{B}_1$ of Y. In fact, if U is a neighborhood in $\mathbb{R}^X$ of $\overline{B}^{\mathbb{R}^X}$, the regularity of $\mathbb{R}^X$ and compactness of $\overline{B}^{\mathbb{R}^X}$ are used to obtain a closed neighborhood V of $\overline{B}^{\mathbb{R}^X}$ in $\mathbb{R}^X$ contained in U. Since $\mathcal{N}$ is a network modulo $\mathcal{B}$ in Z, there exists $n \in \mathbb{N}$ with $B \subseteq T_n \subseteq V \cap Z$, which implies that $\overline{B}^{\mathbb{R}^X} \subseteq \overline{T_n}^{\mathbb{R}^X} \subseteq U$. According to Nagami's criterion [1, Proposition IV.9.1], Y is a Lindelöf Σ-space which clearly satisfies $Z \subseteq Y \subseteq \mathbb{R}^X$. □

PROPOSITION 1. *The following are equivalent:*
(i) $C_p(X)$ *admits a bounded resolution.*
(ii) $C_p(X)$ *is K-analytic-framed in* $\mathbb{R}^X$ *and* $C_p(X)$ *is angelic.*
(iii) $C_p(X)$ *is K-analytic-framed in* $\mathbb{R}^X$.
(iv) *For any topological vector space (tvs) Y containing $C_p(X)$ there exists a space Z such that $C_p(X) \subseteq Z \subseteq Y$ and Z admits a resolution consisting of Y-bounded sets.*

PROOF. (i) implies (ii). Let $\{A_\alpha \mid \alpha \in \mathbb{N}^\mathbb{N}\}$ be a bounded resolution for $C_p(X)$. Denote by B_α the closure of A_α in $\mathbb{R}^X$ and put $Z = \bigcup\{B_\alpha \mid \alpha \in \mathbb{N}^\mathbb{N}\}$. Clearly each B_α is a compact subset of $\mathbb{R}^X$ and Z is a quasi-Souslin space (see [6, Proposition 1]) such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$.

Since each quasi-Souslin space Z has a countable network modulo a resolution $\mathcal{B}$ of Z consisting of countably compact sets (see, for instance, [14, proof of Theorem 8]) and every countable compact subset of $\mathbb{R}^X$ is pointwise bounded, then Lemma 1 ensures that $Y = \bigcup\{\overline{B}^{\mathbb{R}^X} \mid B \in \mathcal{B}\}$ is a Lindelöf Σ-space, hence Lindelöf, such that $Z \subseteq Y \subseteq \mathbb{R}^X$. Given that every Lindelöf quasi-Souslin space Y is K-analytic and $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, then $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. Hence, by [18] the space υX is Lindelöf Σ. Since each Lindelöf Σ space is web-compact in the sense of Orihuela, then [19, Theorem 3] is used to deduce that $C_p(\upsilon X)$ is angelic. Hence, $C_p(X)$ is also angelic [8, Note 4].

(iii) implies (iv). If L is a space with a compact resolution $\{A_\alpha \mid \alpha \in \mathbb{N}^\mathbb{N}\}$ and $C_p(X) \subseteq L \subseteq \mathbb{R}^X$, then $\{A_\alpha \cap C_p(X) \mid \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution in $Z := C_p(X)$ consisting of bounded sets in any tvs Y topologically containing $C_p(X)$.

That (iv) implies (i) is obvious. □

The next theorem extends the main result in Tkachuk [21] and answers the question of [14].

THEOREM 1. *Let ξ be a topology on $C(X)$ stronger than the pointwise one. The following assertions are equivalent.*
(i) $(C(X), \xi)$ *is K-analytic.*
(ii) $(C(X), \xi)$ *is quasi-Souslin.*
(iii) $(C(X), \xi)$ *admits a (relatively countably) compact resolution.*

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PROOF. Any condition mentioned above implies (by Proposition 1) the angelicity of $C_p(X)$. Therefore (by the angelic lemma; see [16, p. 29]) the space $(C(X), \xi)$ is angelic as well. But for angelic spaces all three conditions mentioned above are equivalent by [6, Corollary 1.1]. □

It is easy to see that if X is σ-bounded, then $C_p(X)$ has a bounded resolution. Indeed, if X is covered by a sequence $(C_n)_n$ of functionally bounded sets, then $\{A_\alpha \mid \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $A_\alpha = \{f \in C(X) : \sup_{x\in C_n} |f(x)| \le \alpha(n), n \in \mathbb{N}\}$ is a bounded resolution for $C_p(X)$. If X is a locally compact group, then X is σ-compact if and only if $C_p(X)$ has a bounded resolution. This easily follows from the fact that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$, where D is a discrete space and G is a compact subgroup of X; see [13, Theorem 1 and Remark (ii)]. Proposition 1 combined with [4, Theorem 2.4] yields the following result.

COROLLARY 1. *Let X be a regular cosmic space. Then X is σ-compact if and only if $C_p(X)$ has a bounded resolution.*

The corresponding variant of Corollary 1 for the weak* dual of Banach spaces does not hold in general. Let E be an infinite-dimensional separable non-reflexive Banach space. Then the weak topology $\sigma(E, E')$ is cosmic and not σ-compact but the weak* dual $(E', \sigma(E', E))$ is even analytic.

If $C_p(C_p(X))$ has a bounded resolution, then X is angelic by Proposition 1. If $C_p(C_p(X))$ is K-analytic, then X is finite [1, IV.9.21]. We note the following result.

COROLLARY 2. *For a realcompact space X the space $C_p(C_p(X))$ has a bounded resolution if and only if X is finite.*

PROOF. If $C_p(C_p(X))$ has a bounded resolution, it is K-analytic-framed in $\mathbb{R}^{C(X)}$. Consequently there is a K-analytic space Y such that $C_p(C_p(X)) \subseteq Y \subseteq \mathbb{R}^{C(X)}$. By [4, Corollary 3.4] every compact subset of X is finite. Since $X \subseteq Y \subseteq \mathbb{R}^{C(X)}$ and X is realcompact, then X is a closed subspace of Y. Hence, X is a K-analytic space whose compact sets are finite; so it must be countable [1, Proposition IV.6.15]. Consequently, $C_p(X)$ is a separable metric space, hence a cosmic space. Again [4, Theorem 2.4] is used to deduce that $C_p(X)$ is σ-compact and [2, Theorem 6.1] concludes that X is finite. □

REMARK 1. Corollary 2 does not hold in general. By [3, Proposition 9.31] (see also [4, Remark]) there exists an infinite space X such that $C_p(X)$ is σ-bounded; hence $C_p(C_p(X))$ has a bounded resolution. Recall also that [4, Corollary 2.6] shows that $C_p(\mathbb{N}^{\mathbb{N}})$ is not K-analytic-framed in $\mathbb{R}^X$. In [4, Problem 1] Arkhangel′skiĭ and Calbrix ask if there exists a regular analytic space Z containing $C_p(\mathbb{N}^{\mathbb{N}})$ $(C_p(C_p(\mathbb{N}^{\mathbb{N}})))$. Proposition 1 (Corollary 2) provides a partial answer. Indeed, if Y is a tvs containing $C_p(\mathbb{N}^{\mathbb{N}})$ $(C_p(C_p(\mathbb{N}^{\mathbb{N}})))$, then there does not exist a space Z with $C_p(\mathbb{N}^{\mathbb{N}}) \subseteq Z \subseteq Y$ $(C_p(C_p(\mathbb{N}^{\mathbb{N}})) \subseteq Z \subseteq Y)$ admitting a resolution consisting of Y-bounded sets.

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REMARK 2. Cascales and Orihuela [8] introduced the class $\mathfrak{G}$ of locally convex spaces (lcs) E for which there is a family $\{A_\alpha \mid \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of the topological dual E' of E covering E' such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, and sequences are equicontinuous in each A_α. Class $\mathfrak{G}$ includes (DF)-spaces, (LM)-spaces (hence metrizable lcs), the space of distributions $D'(\Omega)$ and the space $A(\Omega)$ of real analytic functions for open $\Omega \subseteq \mathbb{R}^\mathbb{N}$, and so on. From [7, Theorem 11] it follows that the weak topology $\sigma(E, E')$ of an lcs E in class $\mathfrak{G}$ is angelic. Now applying the argument used in the proof of Theorem 1 one concludes that *if* $E \in \mathfrak{G}$ *and* ξ *is a topology on* E *stronger than* $\sigma(E, E')$, *then* (E, ξ) *is quasi-Souslin if and only if it is* K*-analytic if and only if it admits a* (*relatively countably*) *compact resolution*. A similar result fails to hold for the weak* topology $\sigma(E', E)$ of the dual E' of an lcs $E \in \mathfrak{G}$. Indeed, in [15] we proved that $(E', \sigma(E', E))$ is quasi-Souslin for each $E \in \mathfrak{G}$ but in [9] we provided spaces $E \in \mathfrak{G}$ such that $(E', \sigma(E', E))$ is not K-analytic. On the other hand, by [10, Corollary 2.8] the space $C_p(X)$ belongs to class $\mathfrak{G}$ only if and only if X is countable; so the angelicity of $C_p(X)$ (which we used in Theorem 1) cannot be automatically deduced from Cascales and Orihuela's result [7, Theorem 11] mentioned above.

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