2

Functional integral representation of the partition function

The customary approach to nonrelativistic many-body theory is to proceed with the method of second quantization begun in the first chapter. There is another approach, the method of functional integrals, which we shall follow here. Of course, what can be done in one formalism can always be done in another. Nevertheless, functional integrals seem to be the method of choice for most elementary particle theorists these days, and they seem to lend themselves more readily to nonperturbative phenomena such as tunneling, instantons, lattice gauge theory, etc. For gauge theories they are practically indispensable. However, there is a certain amount of formalism that must be developed before we can start to discuss physical applications. In this chapter, we shall derive the functional integral representation of the partition function for interacting relativistic non-gauge field theories. As a check on the formalism, as well as to obtain some feeling for how functional integrals work, we shall then rederive some well-known results on relativistic ideal gases for bosons and fermions.

2.1 Transition amplitude for bosons

Let $\hat{\phi}(\mathbf{x}, 0)$ be a Schrödinger-picture field operator at time $t = 0$ and let $\hat{\pi}(\mathbf{x}, 0)$ be its conjugate momentum operator. The eigenstates of the field operator are labeled $|\phi\rangle$ and satisfy

$$\hat{\phi}(\mathbf{x}, 0)|\phi\rangle = \phi(\mathbf{x})|\phi\rangle \tag{2.1}$$

where $\phi(\mathbf{x})$ is the eigenvalue, as indicated, a function of $\mathbf{x}$. We also have the usual completeness and orthogonality conditions,

$$\int d\phi(\mathbf{x})|\phi\rangle\langle\phi| = 1 \tag{2.2}$$

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$$\langle\phi_a|\phi_b\rangle \;=\; \prod_{\mathbf{x}} \delta(\phi_a(\mathbf{x}) - \phi_b(\mathbf{x})) \tag{2.3}$$

Similarly, the eigenstates of the conjugate momentum field operator satisfy

$$\hat{\pi}(\mathbf{x}, 0)|\pi\rangle = \pi(\mathbf{x})|\pi\rangle \tag{2.4}$$

The completeness and orthogonality conditions are

$$\int \frac{d\pi(\mathbf{x})}{2\pi} \, |\pi\rangle\langle\pi| = 1 \tag{2.5}$$

$$\langle\pi_a|\pi_b\rangle \;=\; \prod_{\mathbf{x}} \delta(\pi_a(\mathbf{x}) - \pi_b(\mathbf{x})) \tag{2.6}$$

The practical meaning of the formal expressions (2.2), (2.3), (2.5), and (2.6) is elucidated in Section 2.6.

Just as in quantum mechanics one may work in coordinate space or in momentum space, one may work here in the field space or in the conjugate momentum space. In quantum mechanics, one goes from one to the other by using

$$\langle x|p\rangle = \mathrm{e}^{ipx} \tag{2.7}$$

In field theory one has the overlap

$$\langle\phi|\pi\rangle \;=\; \exp\left(i \int d^3x \, \pi(\mathbf{x})\phi(\mathbf{x}) \right) \tag{2.8}$$

In a natural generalization one goes from a denumerably finite number of degrees of freedom N in quantum mechanics to a continuously infinite number of degrees of freedom in quantum field theory: $\sum_{i=1}^{N} p_i x_i \to \int d^3x \, \pi(\mathbf{x})\phi(\mathbf{x})$.

For the dynamics one requires a Hamiltonian, which is now a functional of the field and of its conjugate momentum:

$$H = \int d^3x \, \mathcal{H}(\hat{\pi}, \hat{\phi}) \tag{2.9}$$

Now suppose that a system is in a state $|\phi_a\rangle$ at a time $t = 0$. After a time t_f it evolves into $\mathrm{e}^{-iHt_\mathrm{f}}|\phi_a\rangle$, assuming that the Hamiltonian has no explicit time dependence. The transition amplitude for going from a state $|\phi_a\rangle$ to a state $|\phi_b\rangle$ after a time t_f is thus $\langle\phi_b|\mathrm{e}^{-iHt_\mathrm{f}}|\phi_a\rangle$.

For statistical mechanical purposes we will be interested in cases where the system returns to its original state after the time t_f. To obtain a practical definition of the transition amplitude we use the following prescription: we divide the time interval $(0, t_\mathrm{f})$ into N equal steps of duration

$\Delta t = t_f/N$. Then, at each time interval we insert a complete set of states, alternating between (2.2) and (2.5):

$$
\begin{aligned}
\langle \phi_a | \mathrm{e}^{-iHt_f} | \phi_a \rangle = \lim_{N \to \infty} \int & \left(\prod_{i=1}^{N} d\pi_i \, d\phi_i / 2\pi \right) \\
& \times \langle \phi_a | \pi_N \rangle \langle \pi_N | \mathrm{e}^{-iH\Delta t} | \phi_N \rangle \langle \phi_N | \pi_{N-1} \rangle \\
& \times \langle \pi_{N-1} | \mathrm{e}^{-iH\Delta t} | \phi_{N-1} \rangle \cdots \\
& \times \langle \phi_2 | \pi_1 \rangle \langle \pi_1 | \mathrm{e}^{-iH\Delta t} | \phi_1 \rangle \langle \phi_1 | \phi_a \rangle
\end{aligned}
\tag{2.10}
$$

We know that

$$
\langle \phi_1 | \phi_a \rangle = \delta(\phi_1 - \phi_a)
\tag{2.11}
$$

(as a shorthand for (2.3)) and that

$$
\langle \phi_{i+1} | \pi_i \rangle = \exp\left(i \int d^3 x \, \pi_i(\mathbf{x}) \phi_{i+1}(\mathbf{x}) \right)
\tag{2.12}
$$

Since $\Delta t \to 0$, we can expand as follows, keeping terms up to first order:

$$
\begin{aligned}
\langle \pi_i | \mathrm{e}^{-iH_i \Delta t} | \phi_i \rangle & \approx \langle \pi_i | (1 - iH_i \Delta t) | \phi_i \rangle \\
& = \langle \pi_i | \phi_i \rangle (1 - iH_i \Delta t) \\
& = (1 - iH_i \Delta t) \exp\left(-i \int d^3 x \, \pi_i(\mathbf{x}) \phi_i(\mathbf{x}) \right)
\end{aligned}
\tag{2.13}
$$

where

$$
H_i = \int d^3 x \, \mathcal{H}\left(\pi_i(\mathbf{x}), \phi_i(\mathbf{x}) \right)
\tag{2.14}
$$

Putting it all together we get

$$
\begin{aligned}
\langle \phi_a | \mathrm{e}^{-iHt_f} | \phi_a \rangle = \lim_{N \to \infty} \int & \left(\prod_{i=1}^{N} d\pi_i \, d\phi_i / 2\pi \right) \delta(\phi_1 - \phi_a) \\
& \times \exp\left\{ -i\Delta t \sum_{j=1}^{N} \int d^3 x \, [\mathcal{H}(\pi_j, \phi_j) - \pi_j(\phi_{j+1} - \phi_j)/\Delta t] \right\}
\end{aligned}
\tag{2.15}
$$

where $\phi_{N+1} = \phi_a = \phi_1$. The advantage of alternating between π and ϕ for the insertion of a complete set of states is that the Hamiltonian in (2.13) and (2.15) is evaluated at a single point in time.

Taking the continuum limit of (2.15), we finally arrive at the important result

$$
\begin{aligned}
\langle \phi_a | & \mathrm{e}^{-iHt_\mathrm{f}} | \phi_a \rangle \\
&= \int [d\pi] \int_{\phi(\mathbf{x},0)=\phi_a(\mathbf{x})}^{\phi(\mathbf{x},t_\mathrm{f})=\phi_a(\mathbf{x})} [d\phi] \\
&\quad \times \exp\left[i \int_0^{t_\mathrm{f}} dt \int d^3x \left(\pi(\mathbf{x},t) \frac{\partial \phi(\mathbf{x},t)}{\partial t} - \mathcal{H}\left(\pi(\mathbf{x},t), \phi(\mathbf{x},t) \right) \right) \right]
\end{aligned} \tag{2.16}
$$

The symbols $[d\pi]$ and $[d\phi]$ denote functional integration as defined in (2.15). The integration over $\pi(\mathbf{x},t)$ is unrestricted, but the integration over $\phi(\mathbf{x},t)$ is such that the field starts at $\phi_a(\mathbf{x})$ at $t=0$ and ends at $\phi_a(\mathbf{x})$ at $t=t_\mathrm{f}$. Note that all references to operators have gone.

2.2 Partition function for bosons

Recall that

$$
Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(H-\mu_i \hat{N}_i)} = \sum_a \int d\phi_a \, \langle \phi_a | \mathrm{e}^{-\beta(H-\mu_i \hat{N}_i)} | \phi_a \rangle \tag{2.17}
$$

where the sum runs over all states. This expression is very similar to that for the transition amplitude defined in the previous section. In fact we can express Z as an integral over fields and their conjugate momenta by making use of (2.16). In order to make that connection, we switch to an imaginary time variable $\tau = it$. The trace operator in (2.17) simply means that we must integrate over all ϕ_a. Finally, if the system admits a conserved charge then we must make the replacement

$$
\mathcal{H}(\pi,\phi) \to \mathcal{H}(\pi,\phi) - \mu \mathcal{N}(\pi,\phi) \tag{2.18}
$$

where $\mathcal{N}(\pi,\phi)$ is the conserved charge density. We finally arrive at the fundamental formula

$$
\begin{aligned}
Z = \int [d\pi] \int_{\mathrm{periodic}} [d\phi] \\
\times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi,\phi) + \mu \mathcal{N}(\pi,\phi) \right) \right]
\end{aligned} \tag{2.19}
$$

The term "periodic" means that the integration over the field is constrained in such a way that $\phi(\mathbf{x},0) = \phi(\mathbf{x},\beta)$. This follows from the trace operation, setting $\phi_a(\mathbf{x}) = \phi(\mathbf{x},0) = \phi(\mathbf{x},\beta)$. There is no restriction over the π integration. The expression for the partition function (2.19) can readily be generalized to an arbitrary number of fields and conserved charges.

2.3 Neutral scalar field

The most general renormalizable Lagrangian for a neutral scalar field ϕ is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \mathrm{U}(\phi) \tag{2.20}$$

where the potential is

$$\mathrm{U}(\phi) = g\phi^3 + \lambda\phi^4 \tag{2.21}$$

and $\lambda \geq 0$ for the stability of the vacuum. The momentum conjugate to this field is

$$\pi \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \;=\; \frac{\partial\phi}{\partial t} \tag{2.22}$$

and the Hamiltonian is obtained through the usual Legendre transformation

$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \mathrm{U}(\phi) \tag{2.23}$$

There is no conserved charge.

We shall evaluate the partition function by returning to the discretized version:

$$Z = \lim_{N\to\infty}\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{d\pi_i}{2\pi}\int_{\text{periodic}}d\phi_i\right)$$

$$\times \exp\left(\sum_{j=1}^{N}\int d^3x\Big\{i\pi_j(\phi_{j+1}-\phi_j)\right.$$

$$\left.- \Delta\tau\left[\tfrac{1}{2}\pi_j^2 + \tfrac{1}{2}(\nabla\phi_j)^2 + \tfrac{1}{2}m^2\phi_j^2 + \mathrm{U}(\phi)\right]\Big\}\right) \tag{2.24}$$

The momentum integrals can be evaluated immediately since they are simply products of Gaussian integrals. We divide position space into M^3 small cubes with $V = L^3$, $L = aM$, $a \to 0$, $M \to \infty$, M being an integer. For convenience and to make sure that Z remains explicitly dimensionless at each stage of the calculation, we write $\pi_j = A_j/(a^3\Delta\tau)^{1/2}$ and integrate A_j from $-\infty$ to ∞. We get

$$\int_{-\infty}^{\infty}\frac{dA_j}{2\pi}\,\exp\left[-\tfrac{1}{2}A_j^2 + i\left(\frac{a^3}{\Delta\tau}\right)^{1/2}(\phi_{j+1}-\phi_j)A_j\right]$$

$$= (2\pi)^{-1/2}\,\exp\left(\frac{-a^3(\phi_{j+1}-\phi_j)^2}{2\Delta\tau}\right) \tag{2.25}$$

for each cube. Thus far we have

$$Z = \lim_{M,N\to\infty} (2\pi)^{-M^3 N/2} \int \left(\prod_{i=1}^{N} d\phi_i \right)$$
$$\times \exp\left\{ \Delta\tau \sum_{j=1}^{N} \int d^3x \left[-\frac{1}{2}\left(\frac{\phi_{j+1} - \phi_j}{\Delta\tau}\right)^2 \right.\right.$$
$$\left.\left. -\frac{1}{2}(\nabla\phi_j)^2 - \frac{1}{2}m^2\phi_j^2 - U(\phi_j) \right]\right\}$$
(2.26)

Taking the continuum limit, we obtain

$$Z = N' \int_{\text{periodic}} [d\phi] \exp\left(\int_0^\beta d\tau \int d^3\mathcal{L} \right) \tag{2.27}$$

The Lagrangian is expressed as a functional of ϕ and of its first derivatives. The formula (2.27) expresses the partition function Z as a functional integral over ϕ of the exponential of the action in imaginary time. The overall normalization constant N' is irrelevant, since multiplication of Z by any constant will not change the thermodynamics.

Next, we turn to the case of noninteracting fields by letting $U(\phi) = 0$. Interactions will be discussed in a later chapter. We define

$$S = \int_0^\beta d\tau \int d^3x\, \mathcal{L} = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2 \right] \tag{2.28}$$

Integrating by parts, and using the periodicity of ϕ, we obtain

$$S = -\frac{1}{2} \int_0^\beta d\tau \int d^3x\, \phi \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 \right) \phi \tag{2.29}$$

The field admits a Fourier expansion:

$$\phi(\mathbf{x},\tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)} \phi_n(\mathbf{p}) \tag{2.30}$$

where $\omega_n = 2\pi n T$, owing to the constraint of periodicity that $\phi(\mathbf{x},\beta) = \phi(\mathbf{x},0)$ for all $\mathbf{x}$. The normalization in (2.30) is chosen such that each Fourier amplitude is dimensionless. Substituting (2.30) into (2.29) and recalling that the field is real, we find that

$$S = -\tfrac{1}{2}\beta^2 \sum_n \sum_{\mathbf{p}} (\omega_n^2 + \omega^2)\phi_n(\mathbf{p})\phi_n^*(\mathbf{p}) \tag{2.31}$$

with $\omega = \sqrt{\mathbf{p}^2 + m^2}$. The integrand depends only on the magnitude of the field, $A_n(\boldsymbol{p}) = |\phi_n(\boldsymbol{p})|$. Integrating out the phases, we get

$$
\begin{aligned}
Z &= N' \prod_n \prod_{\mathbf{p}} \left\{ \int_{-\infty}^{\infty} dA_n(\mathbf{p}) \, \exp\left[-\tfrac{1}{2}\beta^2(\omega_n^2 + \omega^2)A_n^2(\mathbf{p})\right] \right\} \\
&= N' \prod_n \prod_{\mathbf{p}} (2\pi)^{1/2} \left[\beta^2(\omega_n^2 + \omega^2)\right]^{-1/2}
\end{aligned}
\tag{2.32}
$$

From the treatment above, we know that a factor of $(2\pi)^{-1/2}$ appears for each momentum integration. Thus, ignoring an overall factor that is independent of volume and temperature,

$$
Z = \prod_n \prod_{\mathbf{p}} \left[\beta^2(\omega_n^2 + \omega^2)\right]^{-1/2}
\tag{2.33}
$$

The partition function can be formally written as

$$
Z = N' \int [d\phi] \exp\left[-\tfrac{1}{2}(\phi, D\phi)\right] = N''(\det D)^{-1/2}
\tag{2.34}
$$

where N'' is a constant. Here D equals $\beta^2(-\partial^2/\partial\tau^2 - \nabla^2 + m^2)$ in $(\mathbf{x}, \tau)$ space and $\beta^2(\omega_n^2 + \omega^2)$ in $(\mathbf{p}, \omega_n)$ space, and $(\phi, D\phi)$ is the inner product on the function space. The expression (2.34) follows from the formula for Riemann integrals with a constant matrix D:

$$
\int_{-\infty}^{\infty} dx_1 \cdots dx_n \, \mathrm{e}^{-x_i D_{ij} x_j} = \pi^{n/2} (\det D)^{-1/2}
\tag{2.35}
$$

One may also derive (2.33) using (2.34).

We now have

$$
\ln Z = -\tfrac{1}{2} \sum_n \sum_{\mathbf{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right]
\tag{2.36}
$$

Using the following identities,

$$
\ln\left[(2\pi n)^2 + \beta^2\omega^2\right] = \int_1^{\beta^2\omega^2} \frac{d\theta^2}{\theta^2 + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right]
\tag{2.37}
$$

and

$$
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + (\theta/2\pi)^2} = \frac{2\pi^2}{\theta}\left(1 + \frac{2}{\mathrm{e}^\theta - 1}\right)
\tag{2.38}
$$

and dropping a temperature-independent term, we can write

$$
\ln Z = -\sum_{\mathbf{p}} \int_1^{\beta\omega} d\theta \left(\frac{1}{2} + \frac{1}{\mathrm{e}^\theta - 1}\right)
\tag{2.39}
$$

Carrying out the integral and dropping terms that are independent of temperature and volume, we finally get

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[-\tfrac{1}{2}\beta\omega - \ln(1 - \mathrm{e}^{-\beta\omega}) \right] \tag{2.40}$$

This expression is identical to the bosonic version of (1.31) with $\mu = 0$, except that (2.40) includes the zero-point energy. Both

$$E_0 = -\frac{\partial}{\partial\beta}\ln Z_0 = \frac{1}{2}V \int \frac{d^3p}{(2\pi)^3}\,\omega \tag{2.41}$$

and

$$P_0 = T\frac{\partial}{\partial V}\ln Z_0 \; = -\frac{E_0}{V} \tag{2.42}$$

should be subtracted, since the vacuum is a state with zero energy and pressure.

2.4 Bose–Einstein condensation

An interesting system is obtained by considering a theory with a charged scalar field $\boldsymbol{\Phi}$. The field $\boldsymbol{\Phi}$ is then complex and describes bosons of positive and negative charge, i.e., they are each other's antiparticle. The Lagrangian density in this case is

$$\mathcal{L} = \partial_\mu\boldsymbol{\Phi}^*\partial^\mu\boldsymbol{\Phi} - m^2\boldsymbol{\Phi}^*\boldsymbol{\Phi} - \lambda(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2 \tag{2.43}$$

This expression has an obvious U(1) symmetry:

$$\boldsymbol{\Phi} \to \boldsymbol{\Phi}' = \boldsymbol{\Phi}\mathrm{e}^{-i\alpha} \tag{2.44}$$

where α is a real constant. This is a global symmetry since the multiplying phase factor is independent of spacetime location.

By Noether's theorem, there is a conserved current associated with each continuous symmetry of the Lagrangian. We can find this current by letting the phase factor α depend on the spacetime coordinate for a moment. In this case the U(1) transformation is

$$\begin{aligned}
\mathcal{L} \to \mathcal{L}' &= \partial_\mu(\boldsymbol{\Phi}^*\mathrm{e}^{i\alpha(x)})\partial^\mu(\boldsymbol{\Phi}\mathrm{e}^{-i\alpha(x)}) - m^2\boldsymbol{\Phi}^*\boldsymbol{\Phi} - \lambda(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2 \\
&= \mathcal{L} + \boldsymbol{\Phi}^*\boldsymbol{\Phi}\partial_\mu\alpha\,\partial^\mu\alpha + i\partial_\mu\alpha(\boldsymbol{\Phi}^*\partial^\mu\boldsymbol{\Phi} - \boldsymbol{\Phi}\partial^\mu\boldsymbol{\Phi}^*)
\end{aligned} \tag{2.45}$$

The equation of motion for the "field" $\alpha(x)$ is

$$\partial^\mu\frac{\partial\mathcal{L}'}{\partial(\partial^\mu\alpha)} = \frac{\partial\mathcal{L}'}{\partial\alpha} \tag{2.46}$$

Since $\partial\mathcal{L}'/\partial\alpha = 0$, it follows that the "current" $\partial\mathcal{L}'/\partial(\partial^\mu\alpha) = \boldsymbol{\Phi}^*\boldsymbol{\Phi}\partial_\mu\alpha + i\boldsymbol{\Phi}^*\partial_\mu\boldsymbol{\Phi} - i\boldsymbol{\Phi}\partial_\mu\boldsymbol{\Phi}^*$ is conserved. We recover our original theory by letting

$\alpha(x) = $ constant. The conserved current density is then

$$j_\mu = i(\mathbf{\Phi}^* \partial_\mu \mathbf{\Phi} - \mathbf{\Phi} \partial_\mu \mathbf{\Phi}^*) \qquad (2.47)$$

with $\partial^\mu j_\mu = 0$. The conservation law may be verified independently using the equation of motion for $\mathbf{\Phi}$. The full current and density are $J_\mu = \int d^3x\, j_\mu(x)$ and $Q = \int d^3x\, j_0(x)$.

It is convenient to decompose $\mathbf{\Phi}$ into real and imaginary parts using the real fields ϕ_1 and ϕ_2, $\mathbf{\Phi} = (\phi_1 + i\phi_2)/\sqrt{2}$. In terms of the conjugate momenta $\pi_1 = \partial\phi_1/\partial t$, $\pi_2 = \partial\phi_2/\partial t$, the Hamiltonian density and charge are

$$\mathcal{H} = \tfrac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\right] + \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2 \qquad (2.48)$$

and

$$Q = \int d^3x (\phi_2\pi_1 - \phi_1\pi_2) \qquad (2.49)$$

The partition function is

$$Z = \int [d\pi_1][d\pi_2] \int_{\text{periodic}} [d\phi_1][d\phi_2] \times \exp\left[\int_0^\beta d\tau \int d^3x\right.$$

$$\left. \times \left(i\pi_1\frac{\partial\phi}{\partial\tau} + i\pi_2\frac{\partial\phi_2}{\partial\tau} - \mathcal{H}(\pi_1, \pi_2, \phi_1, \phi_2) + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\right] \quad (2.50)$$

where we have used a chemical potential associated with the conserved charge Q. Integrating out the conjugate momenta, we get

$$Z = (N')^2 \int_{\text{periodic}} [d\phi_1][d\phi_2]$$

$$\times \exp\left\{\int_0^\beta d\tau \int d^3x \left[-\tfrac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - \tfrac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} + i\mu\phi_1\right)^2\right.\right.$$

$$\left.\left. -\tfrac{1}{2}(\nabla\phi_1)^2 - \tfrac{1}{2}(\nabla\phi_2)^2 - \tfrac{1}{2}m^2\phi_1^2 - \tfrac{1}{2}m^2\phi_2^2 - \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2\right]\right\} \quad (2.51)$$

where N' is the same divergent normalization factor as before. Notice that the argument of the exponential in (2.51) differs from one's naive expectation of

$$\mathcal{L}(\phi_1, \phi_2, \partial_\mu\phi_1, \partial_\mu\phi_2; \mu = 0) + \mu j_0(\phi_1, \phi_2, i\partial\phi_1/\partial\tau, i\partial\phi_2/\partial\tau)$$

by an amount $\mu^2\mathbf{\Phi}^*\mathbf{\Phi}$, owing to the momentum dependence of j_0.

The expression (2.51) cannot be evaluated in closed form unless $\lambda = 0$. In this case, the functional integral becomes Gaussian and can then be worked out analogously to that for the free scalar field.

The components of $\mathbf{\Phi}$ can be Fourier-expanded:

$$\phi_1 = \sqrt{2}\zeta\cos\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}}e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\phi_{1;n}(\mathbf{p})$$

$$\phi_2 = \sqrt{2}\zeta\sin\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}}e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\phi_{2;n}(\mathbf{p})$$

(2.52)

Here ζ and θ are independent of $(\mathbf{x}, \tau)$ and determine the full infrared behavior of the field; that is, $\phi_{1;0}(\mathbf{p}=\mathbf{0}) = \phi_{2;0}(\mathbf{p}=\mathbf{0}) = 0$. This allows for the possibility of condensation of the bosons into the zero-momentum state. Condensation means that in the infinite-volume limit a finite fraction of the particles resides in the $n = 0$, $\mathbf{p} = \mathbf{0}$ state.

Setting $\lambda = 0$ and substituting (2.52) into (2.51) after an integration by parts, see (2.29), we find

$$Z = (N')^2\left(\prod_n\prod_{\mathbf{p}}\int d\phi_{1;n}(\mathbf{p})\,d\phi_{2;n}(\mathbf{p})\right)e^S \qquad (2.53)$$

where

$$S = \beta V(\mu^2 - m^2)\zeta^2 - \tfrac{1}{2}\sum_n\sum_{\mathbf{p}}\Big(\phi_{1;-n}(-\mathbf{p}),\,\phi_{2;-n}(-\mathbf{p})\Big)D\begin{pmatrix}\phi_{1;n}(\mathbf{p})\\\phi_{2;n}(\mathbf{p})\end{pmatrix}$$

and

$$D = \beta^2\begin{pmatrix}\omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n\\2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2\end{pmatrix}$$

Carrying out the integrations,

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \ln(\det D)^{-1/2} \qquad (2.54)$$

The second term can be handled as follows:

$$\ln\det D = \ln\left\{\prod_n\prod_{\mathbf{p}}\beta^4\big[(\omega_n^2 + \omega^2 - \mu^2)^2 + 4\mu^2\omega_n^2\big]\right\}$$

$$= \ln\left\{\prod_n\prod_{\mathbf{p}}\beta^2\big[\omega_n^2 + (\omega - \mu)^2\big]\right\} + \ln\left\{\prod_n\prod_{\mathbf{p}}\beta^2\big[\omega_n^2 + (\omega + \mu)^2\big]\right\}$$

(2.55)

Putting all this together,

$$\ln Z = \beta V (\mu^2 - m^2)\zeta^2 - \tfrac{1}{2} \sum_n \sum_{\mathbf{p}} \ln \left\{ \beta^2 \left[\omega_n^2 + (\omega - \mu)^2 \right] \right\}$$

$$- \tfrac{1}{2} \sum_n \sum_{\mathbf{p}} \ln \left\{ \beta^2 \left[\omega_n^2 + (\omega + \mu)^2 \right] \right\} \qquad (2.56)$$

The last two terms in (2.56) are precisely of the form (2.36). All we need to do is recall (2.40) and make the substitutions $\omega \to \omega - \mu$ and $\omega \to \omega + \mu$, respectively, for the two terms in (2.56). We obtain

$$\ln Z = \beta V (\mu^2 - m^2)\zeta^2 - V \int \frac{d^3 p}{(2\pi)^3}$$

$$\times \left[\beta \omega + \ln \left(1 - e^{-\beta(\omega - \mu)} \right) + \ln \left(1 - e^{-\beta(\omega + \mu)} \right) \right] \quad (2.57)$$

There are several observations we can make about (2.57). The momentum integral is convergent only if $|\mu| \le m$. The parameter ζ appears in the final expression but θ does not, as expected from the U(1) symmetry of the Lagrangian. In this context, since the parameter ζ is not determined *a priori*, it should be treated as a variational parameter that is related to the charge carried by the condensed particles. At fixed β and μ, $\ln Z$ is an extremum with respect to variations of such a free parameter. Thus

$$\frac{\partial \ln Z}{\partial \zeta} = 2\beta V (\mu^2 - m^2)\zeta = 0 \qquad (2.58)$$

which implies that $\zeta = 0$ unless $|\mu| = m$, in which case ζ is undetermined by this variational condition. When $|\mu| < m$ we simply recover the results obtained in Chapter 1, namely (1.31).

To determine ζ when $|\mu| = m$, note that the charge density $\rho = Q/V$ is given by

$$\rho = \frac{T}{V} \left(\frac{\partial \ln Z}{\partial \mu} \right)_{\mu=m} = 2m\zeta^2 + \rho^*(\beta, \mu = m) \qquad (2.59)$$

where

$$\rho^* = \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{e^{\beta(\omega - m)} - 1} - \frac{1}{e^{\beta(\omega + m)} - 1} \right)$$

(The case $\mu = -m$ is handled analogously.) Here the separate contributions from the condensate (the zero-momentum mode) and the thermal excitations are manifest. If the density ρ is kept fixed and the temperature is lowered, μ will decrease until the point $\mu = m$ is reached. If the temperature is lowered even further then $\rho^*(\beta, \mu = m)$ will be less than

ρ. Therefore ζ is given by

$$\zeta^2 = \frac{\rho - \rho^*(\beta,\, \mu = m)}{2m} \tag{2.60}$$

when $\mu = m$ and $T < T_c$. The critical temperature is determined implicitly by the equation

$$\rho = \rho^*(\beta_c,\, \mu = m) \tag{2.61}$$

In the nonrelativistic limit, one obtains

$$T_c = \frac{2\pi}{m} \left(\frac{\rho}{\zeta(3/2)} \right)^{2/3} \qquad \rho \ll m^3 \tag{2.62}$$

In the ultrarelativistic limit, one finds

$$T_c = \left(\frac{3\rho}{m} \right)^{1/2} \qquad \rho \gg m^3 \tag{2.63}$$

In the limit $m \to 0$, we have $|\mu| \to 0$ and $T_c \to \infty$. When $m = 0$, all the charge resides in the condensate, at all temperatures, and none is carried by the thermal excitations.

There is a second-order phase transition at T_c. This can be shown rigorously by a careful examination of the behavior of the chemical potential $\mu(\rho, T)$ as a function of T near T_c with ρ fixed. This analysis is left as an exercise. A more intuitive way to see this involves the general Landau theory of phase transitions [1]. The order parameter ζ drops continuously to zero as T_c is approached from below and remains zero above T_c. Physically, the reason for a phase transition is the following. At $T = 0$, all the conserved charge can reside in the zero-momentum mode on account of the bosonic character of the particles. (This would not be the case for fermions.) As the temperature is raised, some of the charge is excited out of the condensate. Eventually, the temperature becomes great enough to completely melt, or thermally disorder, the condensate. There is no reason for ζ to drop to zero discontinuously; hence the transition is second order.

2.5 Fermions

We now turn our attention to (Dirac) fermions. In relativistic quantum mechanics, we know that electrons or muons are described by a four-component spinor ψ. The components are identified as ψ_α, with α running from 1 to 4. The motion of a free electron is characterized by a

wavefunction

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{s} \sqrt{\frac{M}{E}} \left[b(p, s)u(p, s)\,\mathrm{e}^{-ip\cdot x} + d^*(p, s)v(p, s)\,\mathrm{e}^{ip\cdot x} \right]$$

(2.64)

Here u and v are positive- and negative-energy plane-wave spinors, respectively. The sum on s runs over the two possible spin orientations for a spin-1/2 Dirac fermion. The expansion coefficients $b(p, s)$ and $d^*(p, s)$ are complex functions in relativistic quantum mechanics but become operators in a field theory. As usual, $p \cdot x = p^\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x}$. Equation (2.64) is normalized as

$$\int d^3x\,\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) = \sum_{\mathbf{p}} \sum_{s} \left[|b(p, s)|^2 + |d(p, s)|^2 \right] = 1 \qquad (2.65)$$

In the absence of interactions, the Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m)\psi \qquad (2.66)$$

The Dirac matrices γ^μ, which are defined by the anticommutators $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, are in the standard convention

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

(2.67)

Each of these is a 4×4 matrix: "1" denotes the unit 2×2 matrix and $\boldsymbol{\sigma}$ denotes the triplet of Pauli matrices. In (2.66), $\bar{\psi} = \psi^\dagger\gamma^0$ and $\slashed{\partial} \equiv \gamma^\mu\partial_\mu = \gamma^\mu\partial/\partial x^\mu$. Written out explicitly,

$$\mathcal{L} = \psi^\dagger\gamma^0 \left(i\gamma^0\frac{\partial}{\partial t} + i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m \right) \psi \qquad (2.68)$$

The Lagrangian has a global U(1) symmetry, so that $\psi \to \psi\mathrm{e}^{-i\alpha}$ and $\psi^\dagger \to \psi^\dagger\mathrm{e}^{i\alpha}$. Following Noether's theorem, there is a conserved current associated with this symmetry. To find it, we proceed in the same way as we did for the charged scalar field theory. We allow α to depend on x, treating it as an independent field. Under the above phase transformation, $\mathcal{L} \to \mathcal{L} + \bar{\psi}\,[\slashed{\partial}\alpha(x)]\psi$. Using the equation of motion for $\alpha(x)$, namely $\partial_\mu(\partial\mathcal{L}/\partial[\partial_\mu\alpha(x)]) - \partial\mathcal{L}/\partial\alpha(x) = 0$, we find the conservation law

$$\partial_\mu j^\mu = 0$$
$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad (2.69)$$

Now we set $\alpha = $ constant to recover our original theory. The total conserved charge is

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger \psi \tag{2.70}$$

For relativistic quantum mechanics in the absence of interactions this is a trivial result because of (2.65).

In the field theory we treat ψ as a basic field. The momentum conjugate to this field is, from (2.68),

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial \psi/\partial t)} = i\psi^\dagger \tag{2.71}$$

because $\gamma^0\gamma^0 = 1$. Thus, somewhat paradoxically, ψ and $\psi^\dagger$ must be treated independently in the Hamiltonian formalism. The Hamiltonian density is found using the standard procedure:

$$\mathcal{H} = \Pi\frac{\partial \psi}{\partial t} - \mathcal{L} = \psi^\dagger\left(i\frac{\partial}{\partial t}\right)\psi - \mathcal{L} = \bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi \tag{2.72}$$

The partition function is

$$Z = \mathrm{Tr}^\dagger \mathrm{e}^{-\beta(H - \mu\hat{Q})} \tag{2.73}$$

Apart from two differences, which could be lost in the formalism if we are not careful, we can follow the steps leading up to (2.19) and write

$$Z = \int [id\psi^\dagger][d\psi]\exp\left[\int_0^\beta d\tau \int d^3x\, \bar{\psi}\left(-\gamma^0\frac{\partial}{\partial\tau} + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m + \mu\gamma^0\right)\psi\right] \tag{2.74}$$

Recall that ψ and $\psi^\dagger$ are independent fields, which must be integrated independently. In contrast with boson fields, there is no advantage in attempting to integrate the conjugate momentum separately from the field. The two differences mentioned above have to do with the periodicity of the field in imaginary time τ and with the nature of the "classical" (in the path-integral formulation) fields $\psi(\mathbf{x},\tau)$ and $\psi^\dagger(\mathbf{x},\tau)$ over which we integrate.

The canonical commutation relations for bosons are

$$\left[\hat{\phi}(\mathbf{x},t),\, \hat{\pi}(\mathbf{y},t)\right] = i\hbar\delta(\mathbf{x} - \mathbf{y})$$
$$\left[\hat{\phi}(\mathbf{x},t),\, \hat{\phi}(\mathbf{y},t)\right] = [\hat{\pi}(\mathbf{x},t),\, \hat{\pi}(\mathbf{y},t)] = 0 \tag{2.75}$$

and for fermions

$$\left\{\hat{\psi}_\alpha(\mathbf{x},t),\, \hat{\psi}^\dagger_\beta(\mathbf{y},t)\right\} = \hbar\delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}) \tag{2.76}$$
$$\left\{\hat{\psi}_\alpha(\mathbf{x},t),\, \hat{\psi}_\beta(\mathbf{y},t)\right\} = \left\{\hat{\psi}^\dagger_\alpha(\mathbf{x},t),\, \hat{\psi}^\dagger_\beta(\mathbf{y},t)\right\} = 0$$

These commutation relations are the only ones allowed by the fundamental spin-statistics theorem in relativistic quantum field theory. In the limit $\hbar \to 0$ the field operators are replaced by their eigenvalues. For the case of bosons, those eigenvalues are actually c-number functions, as illustrated in (2.1). We have expressed the partition function as a functional integral over these c-number functions, or "classical fields". For the case of fermions, the $\hbar \to 0$ limit is rather peculiar since the eigenvalues replacing the field operators anticommute with each other! This is of course connected with the Pauli exclusion principle and with the famous spin-statistics theorem. Note that (2.74) instructs us to integrate over these "classical" but anticommuting functions. The mathematics necessary to handle this situation was studied by Grassmann. There are Grassmann variables, Grassmann algebra, and Grassmann calculus.

For a single Grassmann variable η, there is only one anticommutator to define the algebra,

$$\{\eta, \eta\} = 0 \tag{2.77}$$

Because of this, the most general function of η is (using a Taylor series expansion) $f(\eta) = a + b\eta$, where a and b are c-numbers. Integration is defined by

$$\int d\eta = 0$$
$$\int d\eta\, \eta = 1 \tag{2.78}$$

The first of these says that the integral is invariant under the shift $\eta \to \eta + a$, and the second is just a convenient normalization.

In a more general setting, we may have a set of Grassmann variables $\eta_i, i = 1, 2, \ldots N$, and a paired set $\eta_i^\dagger$. The algebra is defined by

$$\{\eta_i, \eta_j\} = \{\eta_i, \eta_j^\dagger\} = \{\eta_i^\dagger, \eta_j^\dagger\} = 0 \tag{2.79}$$

The most general function of these variables may be written as

$$f = a + \sum_i a_i \eta_i + \sum_i b_i \eta_i^\dagger + \sum_{i,j} a_{ij} \eta_i \eta_j + \sum_{i,j} b_{ij} \eta_i^\dagger \eta_j^\dagger$$
$$+ \sum_{i,j} c_{ij} \eta_i^\dagger \eta_j + \cdots + d\, \eta_1^\dagger \eta_1 \eta_2^\dagger \eta_2 \cdots \eta_N^\dagger \eta_N \tag{2.80}$$

Integration over all variables of (2.80) is defined by

$$\int d\eta_1^\dagger d\eta_1 \cdots d\eta_N^\dagger d\eta_N\, f = d \tag{2.81}$$

Integrals over Grassmann variables were introduced for the explicit purpose of dealing with path integrals over fermionic coordinates. The

bibliography at the end of this chapter refers the interested reader to more detailed treatments.

For our purposes, the only integral we need is

$$\int d\eta_1^\dagger d\eta_1 \cdots d\eta_N^\dagger d\eta_N \, e^{\eta^\dagger D\eta} = \det D \tag{2.82}$$

where D is an $N \times N$ matrix. This formula is simple to prove if $N = 1$ or 2. The general case is left as an exercise for the reader.

As with bosons, it is most convenient to work in $(\mathbf{p}, \omega_n)$ space instead of $(\mathbf{x}, \tau)$ space. In imaginary time we can write

$$\psi_\alpha(\mathbf{x}, \tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)} \tilde{\psi}_{\alpha;n}(\mathbf{p}) \tag{2.83}$$

where both n and $\mathbf{p}$ run over negative and positive values. For an arbitrary function defined over the interval $0 \leq \tau \leq \beta$, the discrete frequency ω_n can take on the values $n\pi T$. For bosons we argued that we must take $\omega_n = 2\pi nT$ in order that $\phi(\mathbf{x}, \tau)$ be periodic, which followed from the trace operation in the partition function. This can be verified by examining the properties of the thermal Green's function for bosons defined by

$$G_B(\mathbf{x}, \mathbf{y}; \tau_1, \tau_2) = Z^{-1} \text{Tr} \left\{ \hat{\rho} T_\tau \left[\hat{\phi}(\mathbf{x}, \tau_1) \hat{\phi}(\mathbf{y}, \tau_2) \right] \right\} \tag{2.84}$$

Here T_τ is the imaginary time ordering operator, which for bosons acts as follows:

$$T_\tau \left[\hat{\phi}(\mathbf{x}, \tau_1) \hat{\phi}(\mathbf{y}, \tau_2) \right] = \hat{\phi}(\tau_1) \hat{\phi}(\tau_2) \theta(\tau_1 - \tau_2) + \hat{\phi}(\tau_2) \hat{\phi}(\tau_1) \theta(\tau_2 - \tau_1) \tag{2.85}$$

where θ is the step-function. Using the fact that T_τ commutes with $\hat{\rho} = e^{-\beta K}$, where $K \equiv H - \mu \hat{Q}$, and the cyclic property of the trace we find that

$$\begin{aligned}
G_B(\mathbf{x}, \mathbf{y}; \tau, 0) &= Z^{-1} \text{Tr} \left[e^{-\beta K} \hat{\phi}(\mathbf{x}, \tau) \hat{\phi}(\mathbf{y}, 0) \right] \\
&= Z^{-1} \text{Tr} \left[\hat{\phi}(\mathbf{y}, 0) \, e^{-\beta K} \hat{\phi}(\mathbf{x}, \tau) \right] \\
&= Z^{-1} \text{Tr} \left[e^{-\beta K} e^{\beta K} \hat{\phi}(\mathbf{y}, 0) \, e^{-\beta K} \hat{\phi}(\mathbf{x}, \tau) \right] \\
&= Z^{-1} \text{Tr} \left[e^{-\beta K} \hat{\phi}(\mathbf{y}, \beta) \hat{\phi}(\mathbf{x}, \tau) \right] \\
&= Z^{-1} \text{Tr} \left\{ \hat{\rho} T_\tau \left[\hat{\phi}(\mathbf{x}, \tau) \hat{\phi}(\mathbf{y}, \beta) \right] \right\} \\
&= G_B(\mathbf{x}, \mathbf{y}; \tau, \beta) \tag{2.86}
\end{aligned}$$

(Notice that $\hat{\phi}(\mathbf{y}, \beta) = e^{\beta K} \hat{\phi}(\mathbf{y}, 0) \, e^{-\beta K}$, in analogy with the real time Heisenberg time-evolution expression $\hat{\phi}(\mathbf{y}, t) = e^{iHt} \hat{\phi}(\mathbf{y}, 0) \, e^{-iHt}$.) The result (2.86) implies that $\phi(\mathbf{y}, 0) = \phi(\mathbf{y}, \beta)$ and hence $\omega_n = 2\pi nT$.

For fermions, however, instead of (2.85) one has (in direct analogy with the real time Green's functions)

$$T_\tau\left[\hat\psi(\tau_1)\hat\psi(\tau_2)\right] = \hat\psi(\tau_1)\hat\psi(\tau_2)\theta(\tau_1-\tau_2) - \hat\psi(\tau_2)\hat\psi(\tau_1)\theta(\tau_2-\tau_1) \quad (2.87)$$

Following the same steps as in (2.86), one is led to

$$G_\mathrm{F}(\mathbf{x},\mathbf{y};\tau,0) = -G_\mathrm{F}(\mathbf{x},\mathbf{y};\tau,\beta) \quad (2.88)$$

This implies that

$$\psi(\mathbf{x},0) = -\psi(\mathbf{x},\beta) \quad (2.89)$$

and hence

$$\omega_n = (2n+1)\pi T \quad (2.90)$$

This antiperiodicity required of fermion fields is in no way inconsistent with the trace operation in the partition function. The trace only means that the system returns to its original state after a "time" β. Since the sign of ψ is just an overall phase and hence is not observable, the right-hand side of (2.89) describes the same physical state as the left-hand side.

Now we are ready to evaluate (2.74). Inserting (2.83) and using (2.82) we get

$$Z = \left[\prod_n\prod_\mathbf{p}\prod_\alpha \int id\tilde\psi^\dagger_{\alpha;n}(\mathbf{p})d\tilde\psi_{\alpha;n}(\mathbf{p})\right]\mathrm{e}^S \quad (2.91)$$

where

$$S = \sum_n\sum_\mathbf{p} i\tilde\psi^\dagger_{\alpha;n}(\mathbf{p})D_{\alpha\rho}\tilde\psi_{\rho;n}(\mathbf{p})$$

$$D = -i\beta\left[(-i\omega_n+\mu) - \gamma^0\gamma\cdot\mathbf{p} - m\gamma^0\right]$$

and so

$$Z = \det D \quad (2.92)$$

In (2.92) the determinantal operation is carried out over both Dirac indices (thus with 4×4 matrices) and in frequency–momentum space. Using

$$\ln\det D = \mathrm{Tr}\ln D \quad (2.93)$$

and (2.67), one finds that

$$\ln Z = 2\sum_n\sum_\mathbf{p}\ln\left\{\beta^2\left[(\omega_n+i\mu)^2+\omega^2\right]\right\} \quad (2.94)$$

Since the summation is over both negative and positive frequencies (2.94) can be put into a form analogous to (2.55),

$$\ln Z = \sum_n \sum_{\mathbf{p}} \left\{ \ln \left[\beta^2 (\omega_n^2 (\omega - \mu)^2) \right] + \ln \left[\beta^2 (\omega_n^2 (\omega + \mu)^2) \right] \right\} \quad (2.95)$$

Following (2.37), we write

$$\ln \left[(2n+1)^2 \pi^2 + \beta^2 (\omega \pm \mu)^2 \right] = \int_1^{\beta^2 (\omega \pm \mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2 \pi^2} \\ + \ln \left[1 + (2n+1)^2 \pi^2 \right] \quad (2.96)$$

The sum over n may be carried out by using the summation formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)(n-y)} = \frac{\pi (\cot \pi x - \cot \pi y)}{y - x} \quad (2.97)$$

This gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2 \pi^2 + \theta^2} = \frac{1}{\theta} \left(\frac{1}{2} - \frac{1}{e^\theta + 1} \right) \quad (2.98)$$

Integrating over θ and dropping terms that are independent of β and μ, we finally obtain

$$\ln Z = 2V \int \frac{d^3 p}{(2\pi)^3} \left[\beta\omega + \ln \left(1 + e^{-\beta(\omega - \mu)} \right) + \ln \left(1 + e^{-\beta(\omega + \mu)} \right) \right] \quad (2.99)$$

This result agrees with that derived in Chapter 1 using completely different methods.

Notice the factor 2 in (2.99). This factor comes out automatically and owes its existence to the spin-1/2 nature of the fermions. Separate contributions from particles (μ) and antiparticles ($-\mu$) are evident. Finally, this formula also contains a contribution from the zero-point energy.

To recapitulate, the difference between fermions and bosons in the functional integral approach to the partition function is essentially twofold. First, for fermions we must integrate over Grassmann variables instead of c-number variables. Contrast the result (2.92), $Z = \det D$, for fermions with the result (2.34), $Z = (\det D)^{-1/2}$, for bosons. Integration over c-number variables would have led to a factor -1 in (2.99) instead of the factor 2. Second, and this is related to the first, is the fact that the fermion fields are actually antiperiodic in imaginary time, with period β, instead of periodic as is the case for bosons. The consequence is that $\omega_n = (2n+1)\pi T$ for fermions whereas $\omega_n = 2\pi n T$ for bosons. These two points account for the difference between (2.57) (with $\zeta = 0$, of course) and (2.99).

2.6 Remarks on functional integrals

The notation used for functional integration (and differentiation!) is deceptively simple. It must be kept simple, for if we think back on the tremendous progress made in mechanics and electromagnetism in the nineteenth century, it was certainly made easier by the introduction of compact notation for differentiation, integration, and vectors. This also seems to be the case with functional methods in modern quantum physics. However, it is also clear that the mathematical symbols we are using represent rather exotic entities. For example, (2.6) uses a Dirac delta function whose argument is a difference between two functions. A less formal and compact, but more practical, way to view these objects is to start with a complete orthonormal set of real functions for the physical problem of interest. Call this set $w_n(x)$, with n any positive integer. Then any function may be written as

$$a(x) = \sum_{n=1}^{\infty} a_n w_n(x) \tag{2.100}$$

Another function may be expressed as

$$b(x) = \sum_{n=1}^{\infty} b_n w_n(x) \tag{2.101}$$

Then

$$\delta\left(a(x) - b(x)\right) = \prod_{n=1}^{\infty} \delta(a_n - b_n) \tag{2.102}$$

and

$$\int [da(x)] = \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} da_n \tag{2.103}$$

and so on. Most physical problems are defined on the space of a continuous variable, such as position. For such problems it is intuitively obvious that the functional integral ought to be divergent in general since the possible functional configurations form an uncountably infinite set. Indeed, it seems that the extent to which mathematical rigor can be applied to functional integrals is still uncertain. This should be no surprise since they are just a means of phrasing the physical content of relativistic quantum field theory. The extent to which mathematical rigor can be applied in the operator formalism is probably no more certain, because of the highly singular nature of the products of field operators at a point. For physical problems defined on a space of discrete variables, some mathematical

rigor can be applied. This is one reason why certain spacetime theories are defined on a spacetime lattice. This will be studied in Chapter 10.

2.7 Exercises

2.1 For the charged scalar field show, by direct application of the equation of motion for $\mathbf{\Phi}$, that $j_\mu = i(\mathbf{\Phi}^* \partial_\mu \mathbf{\Phi} - \mathbf{\Phi} \partial_\mu \mathbf{\Phi}^*)$ is conserved.

2.2 If j_μ is conserved show that

$$\dot{Q} = \frac{d}{dt} \int d^3x \, j_0(\mathbf{x}, t) = 0$$

2.3 Obtain (2.62) and (2.63), starting from (2.59) to (2.61).

2.4 For Bose–Einstein condensation, consider μ as a function of ρ and T. If ρ is held fixed, show that μ and $\partial\mu/\partial T$ are continuous but $\partial^2\mu/\partial T^2$ is discontinuous at $T_{\rm c}$.

2.5 Prove (2.82).

2.6 Fill in the steps leading from (2.91)–(2.93) to (2.94).

2.7 When $m = 0$ show that (2.99) can be evaluated in closed form, leading to $P = T \ln Z/V = \mu^4/12\pi^2 + \mu^2 T^2/6 + 7\pi^2 T^4/180$.

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