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EMBEDDING UNCOUNTABLY MANY MUTUALLY EXCLUSIVE CONTINUA INTO EUCLIDEAN SPACE

BY

B. J. BAKER AND MICHAEL LAIDACKER

ABSTRACT. Uncountable collections of continua of dimension m embeddable in E^n are investigated, where the difference between m and n is not restricted to one. Collections of isometric copies of continua equivalent to Menger universal continua and collections of continua analogous to G. S. Young's T_n-sets are the main considerations.

1. **Introduction.** R. L. Moore proved that the plane does not contain uncountably many disjoint triods in [5]. A *triod* is the union of three non-degenerate continua such that the intersection of two of them is exactly one point which is also the intersection of all three of them. G. S. Young, [7], extended this theorem to E^n by proving that there does not exist in E^n an uncountable collection of disjoint T_{n-1}-sets. A T_n-*set* is the union of an n-cell and an arc whose intersection is a relative interior point of the n-cell and an end point of the arc. This paper extends the above investigations to uncountable collections of continua of dimension m embeddable in E^n, where the difference between m and n is not restricted to one. We consider two cases since our results differ in these cases.

The Case Where $n > 2m$. We first consider the *m-dimensional Menger continua* M_m^n in E^n, which are constructed inductively. Let V_0 denote the collection whose only element is the unit cube $I^n = [0,1]^n$. Then for each positive integer i, let H_i denote the collection of all n-cubes obtained by subdividing I^n into 3^{ni} congruent n-cubes, and let V_i denote the collection to which $\underline{v}$ belongs if and only if $\underline{v}$ is a subset of some element of $\underline{w}$ of V_{i-1} and intersects the m-skeleton of $\underline{w}$. If G is a collection of sets, the union of the elements of G will be denoted by G^*. With this notation, $M_m^n = \bigcap_{i=1}^{\infty} V_i^*$.

That M_m^n is a universal space for the class of all compact subspaces of E^n which have dimension $k \leqq m$ was proved by Stan'ko, [6]. The Menger continua M_m^{2m+1} are universal spaces for the class of all separable metric spaces with dimension $k \leqq m$, [3].

THEOREM 1. *There exists an uncountable collection H of disjoint isometric universal compacta of dimension n embedded in E^{2n+1}.*

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PROOF. Let $V_0 = \{I^{2n+1}\}$. For each positive integer i, subdivide I^{2n+1} into $6^{i(2n+1)}$ congruent $(2n+1)$-cubes. Denote by H_i the collection of all these $(2n+1)$-cubes and by V_i the collection of all elements $\underline{h}$ of H_i such that $\underline{h}$ is a subset of some element $\underline{v}$ of V_{i-1} and intersects the n-skeleton of $\underline{v}$. Define $W_n^{2n+1} = \bigcup_{i=1}^{\infty} V_i{}^*$. That W_n^{2n+1} is a universal compactum for the class of all separable metric spaces with covering dimension $k \leqq n$ is a corollary of the following theorem by Bestvina [1]: If X is a k-dimensional $(k-1)$-connected, locally $(k-1)$-connected compact metric space that satisfies DD^kP, then $X \approx M_k^{2k+1}$. Increasing the number of $(2n+1)$-cubes in a subdivision from $3^{i(2n+1)}$ in the construction of M_n^{2n+1} to $6^{i(2n+1)}$ in the construction of W_n^{2n+1} does not affect the properties listed in Bestvina's theorem.

Now we define a subset M of the unit interval I. Let $G_0 = \{I\}$. Then for each positive integer i, let F_i denote the collection of intervals obtained by subdividing I into 6^i congruent intervals, and let G_i denote the collection of all elements of F_i whose left end point is either the left end point or the midpoint of an element of G_{i-1}. Define $M = \bigcap_{i=1}^{\infty} G_i{}^*$. Now for each element α of M, let $V_i(\alpha)$ and $W_n^{2n+1}(\alpha)$ denote the sets obtained by translating V_i and W_n^{2n+1} a distance of $\alpha\sqrt{2n+1}$ along the principal diagonal of I^{2n+1} between the origin and unit point. Obviously $W_n^{2n+1}(\alpha) = \bigcap_{i=1}^{\infty} V_i{}^*(\alpha)$. Denote by H the collection of all sets $W_n^{2n+1}(\alpha)$ for all elements α of M. Clearly H is an uncountable collection of isometric compacta for separable metric spaces with covering dimension $k \leqq n$. It only remains to be shown that the elements of H are disjoint.

For each positive integer i, with $F_0 = \{I\}$ and F_i as previously defined, denote by K_i the collection of all intervals of F_i which contain an end point of an element of F_{i-1}. Letting $N = \{j | j$ is a positive integer and $j \leqq 2n+1\}$, we denote by A the collection of all subsets of N which have exactly n elements. Then for each element α of M, element $\underline{a}$ of A, positive integer $j \leqq 2n+1$, and positive integer i, let $s_{\underline{a}ij}(\alpha)$ denote (1) a translation of I a distance of α to the right if $j \in \underline{a}$ or (2) a translation of $K_i{}^*$ a distance of α to the right if $j \in N\backslash\underline{a}$. Define $b_{\underline{a}i}(\alpha) = \times_{j=1}^{2n+1} s_{\underline{a}ij}(\alpha)$ and $B_i(\alpha) = \{b | b = b_{\underline{a}i}(\alpha)\}$ for some $\underline{a}$ in $A\}$. It follows that $V_1{}^*(\alpha) = B_1{}^*(\alpha)$ while $V_i{}^*(\alpha)$ is a proper subset of $B_i{}^*(\alpha)$ for $i > 1$.

Now suppose α and β are two elements of M such that $W_n^{2n+1}(\alpha)$ and $W_n^{2n+1}(\beta)$ have a point $P(x_1, x_2, \ldots, x_{2n+1})$ in common, and suppose $\alpha < \beta$. Denote by m the smallest positive integer i such that α and β do not belong to the same interval of G_i, and by C the interval of G_{m-1} which contains both α and β. Since $B_m{}^*(\alpha)$ and $B_m{}^*(\beta)$ both contain P, there are elements $\underline{v}$ and $\underline{w}$ of A such that $b_{\underline{v}m}(\alpha)$ and $b_{\underline{w}m}(\beta)$ intersect. There is an element k of N which does not belong to $\underline{v}$ or $\underline{w}$ since $\underline{v} \cup \underline{w}$ contains at most $2n$ elements while N contains $2n+1$ elements. Consequently, x_k is in both $s_{\underline{v}mk}(\alpha)$ and $s_{\underline{w}mk}(\beta)$, where $s_{\underline{v}mk}(\alpha)$ is a translation of $K_m{}^*$ a distance α to the right while $s_{\underline{w}mk}(\beta)$ is a translation of $K_m{}^*$ a distance of β to the right.

For some positive integer q, the interval C is $[6q/6^m, (6q+6)/6^m]$. Since $\alpha < \beta$, α is in the first two-thirds of the first element of F_m which lies in C while β is in the first two-thirds of the fourth element of F_m which lies in C. Then $\alpha \in [6q/6^m, 6q/6^m +$

$2/3(6^m)] = [18q/3(6^m), (18q+2)/3(6^m)]$, and $\beta \in [(6q+3)/6^m, (6q+3)/6^m+2/3(6^m)] = [(18q+9)/3(6^m), (18q+11)/3(6^m)]$. Each component of $K_m{}^*$ is $[(6k-1)/6^m, (6k+1)/6^m]$ for some positive integer k. A translation of a component of $K_m{}^*$ a distance of α to the right is a subset of

$$[(6k-1)/6^m + 18q/3(6^m), (6k+1)/6^m + (18q + 2/3(6^m)] =$$

$$[(18(k+q-1)+15)/3(6^m), (18(k+q)+5)/3(6^m)].$$

A translation of a component $[(6p-1)/6^m, (6p+1)/6^m]$ of $K_m{}^*$ a distance of β to the right is a subset of

$$[(6p-1)/6^m + (18q+9)/3(6^m), (6p+1)/6^m + (18q+11)/3(6^m)] =$$

$$[(18(p+q)+6)/3(6^m), (18(p+q)+14)/3(6^m)].$$

Clearly these intervals are disjoint. Hence $B_m{}^*(\alpha)$ and $B_m{}^*(\beta)$ do not intersect, and the elements of H are disjoint.

Corollary 1. *If G is an uncountable collection of continua of dimension $k \leqq n$, then there exists an uncountable collection H of disjoint continua embedded in E^{2n+1} such that each element of H is homeomorphic to exactly one element of G.*

If X and Y are two closed subsets of E^n, then if there is a homeomorphism f of E^n onto itself such that $f(X)$ is Y, then Y is an *equivalent embedding* of X. If $f(X)$ is a subset of Y, then X is said to be *position-wise embedded* in Y. In [2], Bothe describes a one-dimensional compactum in E^3 which cannot be position-wise embedded in M_1^3.

Corollary 2. *In E^{2n+1}, if M is a continuum of dimension $k \leqq n$ which can be position-wise embedded in W_n^{2n+1}, then there exists an uncountable collection H of disjoint continua in E^{2n+1} each of which is an equivalent embedding of M.*

Question. If M is a continuum of dimension n in E^{2n+1} which cannot be position-wise embedded in W_n^{2n+1}, does there exist in E^{2n+1} an uncountable collection of disjoint continua each of which is an equivalent embedding of M?

The Case Where $n \leqq 2m$. Now let n be a positive integer greater than three and m an integer such that $n/2 \leqq m \leqq n-2$. Let $r = n-m$. For each $i \leqq r$, denote by p_i the point $(x_1, x_2, \ldots, x_n)$ where $x_i = r$ and $x_j = 1/2$ for $j \neq i$. Denote by p_0 the point $(1/2, 1/2, \ldots, 1/2)$. The convex hull of the points $p_0, p_1, \ldots, p_{r-1}$ and p_r will be denoted by $\langle p_0, p_1, \ldots, p_r \rangle$. Define $\underline{t}$ to be $\langle p_0, p_1, \ldots, p_r \rangle$. Then $\underline{t}$ is a closed r-simplex. Each of the straight line intervals p_0p_i intersects exactly one $(n-1)$-dimensional face of I^n. Let F_i denote the face of I^n which is a subset of the hyperplane defined by $x_i = 1$. Designate by G the collection of all F_i for $i \leqq r$. The intersection of the elements of G is an m-dimensional face of I^n. Let $\underline{h}$ denote that face. Then $\underline{h} \cap \underline{t}$ is a unique point which is relatively interior to each of $\underline{h}$ and $\underline{t}$. Define $S_{m,r} = \underline{h} \cup \underline{t}$. We observe that (1) the point p_0 lies in the interior of I^n, which is an open set, (2) for $i \leqq r$, p_i lies in the exterior of I^n, (3) if $Q_0 = \langle p_1, p_2, \ldots p_r \rangle$, then $Q_0 \cap I^n = \emptyset$, (4) if $Q_i = \langle p_0, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_r \rangle$, then $Q_i \cap F_i = \emptyset$ and (5) if B denotes the boundary of I^n, then $\underline{t}$ does not intersect $B \backslash (\bigcup_{i=1}^r F_i)$.

THEOREM 2. *Let n be an integer greater than three, m be an integer such that $n/2 \leqq m \leqq n-2$, $r = n - m$, and G be an uncountable collection of continua in E^n such that g is an element of G implies g is an equivalent embedding of $S_{m,r}$. Then there is an uncountable subcollection G' of G such that every two elements of G' intersect.*

PROOF. For each element g of G, let f_g denote a homeomorphism of E^n onto itself such that $f_g(S_{m,r}) = g$. If X is a subset of E^n, let $X(g)$ denote the set $f_g(X)$. By this notation, $g = \underline{h}(g) \cup \underline{t}(g)$. If R denotes the interior of I^n, then $R(g)$ is a connected open set containing $p_0(g)$ whose boundary contains $F_i(g)$ for $i \leqq r$.

Let W denote a countable basis of connected open sets for E^n. For each element g of G, let $d_{0,g}$ denote an element of W which contains $p_0(g)$ and lies in $R(g)$, and let $H_g = \{d_{0,g}\}$. For each $i \leqq r$, denote by $V_{i,g}$ a finite subcollection of W such that (1) $V_{i,g}$ properly covers $F_i(g)$ and (2) no element of $V_{i,g}$ intersects $Q_i(g) \cup Q_0(g)$. Denote by $V_{0,g}$ a finite subcollection of W such that (1) $V_{0,g}$ properly covers $Q_0(g)$ and (2) every element of $V_{0,g}$ lies in $E^n \backslash \bar{R}(g)$. Let H'_g denote a finite subcollection of W such that (1) H'_g properly convers $\underline{t}(g)$ and (2) no element of H'_g contains a point of $B(g) \backslash [\bigcup_{i=1}^r F_i(g)]$. For each $i \leqq r$, let $U_{i,g}$ denote a finite subcollection of W such that (1) $U_{i,g}$ properly covers the arc $p_0(g)p_i(g)$, and (2) no element of $U_{i,g}$ intersects $F_j(g)$ for $j \neq i$.

Since G is uncountable, there exists an uncountable subcollection G' of G and finite subcollections $H, H', V_0, V_1, V_2, \ldots, V_r, U_1, U_2, \ldots, U_r$ of W such that if g is an element of G', then $H = H_g, H' = H'_g, V_0 = V_{0,g}$, and for each $i \leqq r$, $V_i = V_{i,g}$ and $U_i = U_{i,g}$.

Now let g and q denote two elements of G'. The element d_0 of H contains $p_0(g)$ and $p_0(q)$ and lies in $R(g)$. The collection V_0 covers both $Q_0(g)$ and $Q_0(q)$, and $V_0{}^*$ lies in $E^n \backslash R(g)$. Thus $B(g)$ separates $p_0(q)$ from $Q_0(q)$ in E^n, and consequently $B(g) \cap \underline{t}(q)$ separates $p_0(q)$ from $Q_0(q)$ in $\underline{t}(q)$. Since both $\underline{t}(g)$ and $\underline{t}(q)$ are covered by H', it follows that $B(g) \cap \underline{t}(q)$ is a subset of $\bigcup_{i=1}^r F_i(g)$. For each $i \leqq r$, the arc $p_0(q)p_i(q)$ intersects $B(g)$, and being covered by U_i, it therefore intersects $F_i(g)$. Define $A_i = F_i(g) \cap \underline{t}(q)$. Since $F_i(g)$ is covered by V_i, it follows that $A_i \cap [Q_i(q) \cup Q_0(q)] = \emptyset$. Hence $B(g) \cap \underline{t}(q)$ is $\bigcup_{i=1}^r A_i$.

In the topological simplex $\underline{t}(q)$, for each $i \leqq r$, let $D_{i,1}, D_{i,2}, D_{i,3}, D_{i,4}, \ldots$ denote a sequence of open sets closing down on the compact set A_i such that $D_{i,1}$ does not intersect $Q_i(q) \cup Q_0(q)$. By a theorem from Kuratowski [4, p. 312], since, for each positive integer j, $\{D_{1,j}, D_{2,j}, \ldots D_{r,j}\}$ is a collection of open sets whose union separates $p_0(q)$ from $Q_0(q)$ in $\underline{t}(q)$ and $D_{i,j} \cap Q_i(q) = \emptyset$ for each $i \leqq r$ it follows that there is a point b_j common to all of the sets $D_{i,j}$ for $i \leqq r$. Then the sequence of points $b_1, b_2, b_3, \ldots$ has a limit point of b. Hence b is a point of each A_i, and since

$A_i \subseteq F_i(g)$, b is an element of $F_i(g)$ for each $i \leqq r$. However, $\bigcap_{i=1}^{r} F_i(g) = h(g)$; therefore b is an element of both g and q. Thus G' is an uncountable subcollection of G such that every two elements of G' intersect.

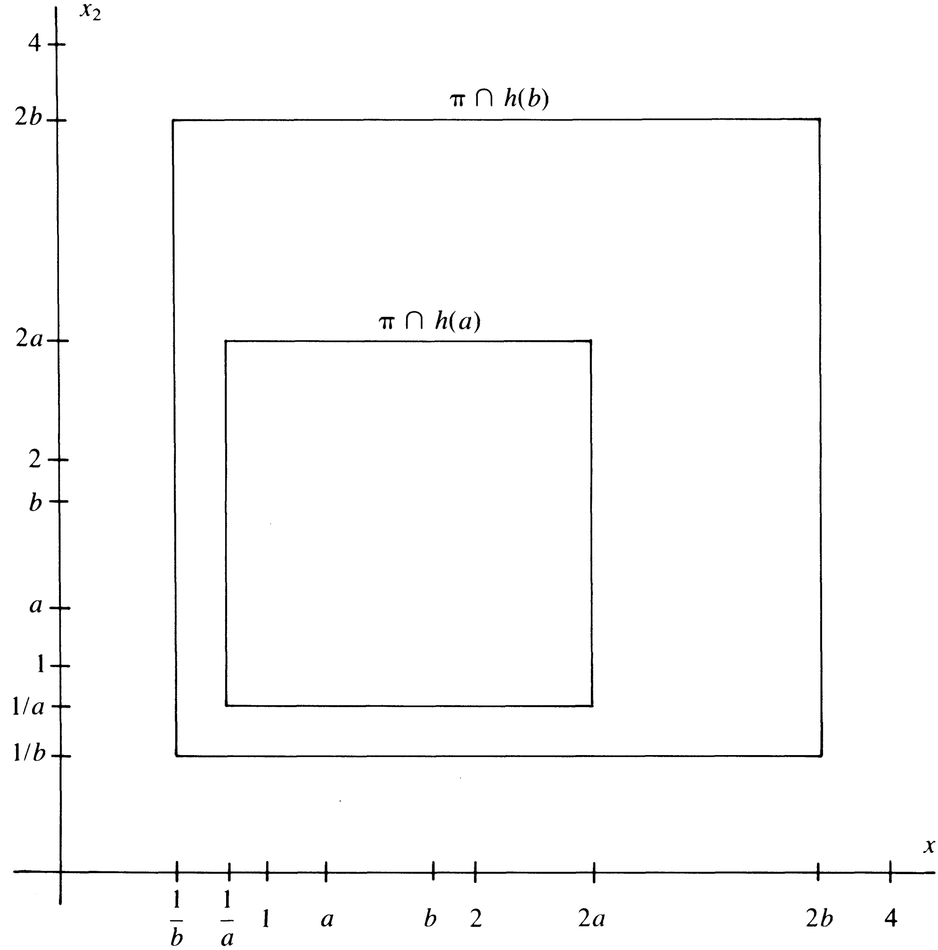
Obviously, if, in the statement of Theorem 2, the condition that g be an equivalent embedding of $S_{m,r}$ is omitted, the resulting statement is not true since there are continua in E^{n-1} which are homeomorphic to $S_{m,r}$ for $m < n-1$ and $r = n-m$. The following is an example of an uncountable collection G of disjoint continua in E^4 such that if g is an element of G, then g contains a continuum homeomorphic to $S_{2,2}$ but g is not embeddable in E^3.

In E^4, for each $\underline{a}$ in the open interval (1,2), let $C_1(\underline{a})$ denote the set of all points (x_1, x_2, x_3, x_4) satisfying the following: $1/\underline{a} \leqq x_1 \leqq 2\underline{a}$, $1/\underline{a} \leqq x_2 \leqq 2\underline{a}$, $x_3 = \underline{a}$, and $\underline{a} \leqq x_4 \leqq 2\underline{a}$. Let $C_2(\underline{a})$ denote the set of all points (x_1, x_2, x_3, x_4) satisfying the following:$1/\underline{a} \leqq x_1 \leqq 2\underline{a}$, $1/\underline{a} \leqq x_2 \leqq 2\underline{a}$, $1/\underline{a} \leqq x_3 \leqq \underline{a}$, and $x_4 = \underline{a}$. Let $B_1(\underline{a})$ and $B_2(\underline{a})$ denote the two-skeletons of $C_1(\underline{a})$ and $C_2(\underline{a})$ respectively. Then $B_1(\underline{a}) \cap B_2(\underline{a})$ equals $C_1(\underline{a}) \cap C_2(\underline{a})$ and is the convex hull of the four points $(1/\underline{a}, 1/\underline{a}, \underline{a}, \underline{a}), (2/\underline{a}, 1/\underline{a}, \underline{a}, \underline{a}), (1/\underline{a}, 2\underline{a}, \underline{a}, \underline{a})$, and $(2\underline{a}, 2\underline{a}, \underline{a}, \underline{a})$. Define $h(\underline{a})$ to be $B_1(\underline{a}) \cup B_2(\underline{a})$.

Let $P_0(\underline{a}) = (\underline{a}, \underline{a}, \underline{a}, \underline{a}), p_1(\underline{a}) = (\underline{a}, \underline{a}, \underline{a}, 2\underline{a}), p_2(\underline{a}) = (\underline{a}, \underline{a}, 1/\underline{a}, \underline{a}), p_3(\underline{a}) = (\underline{a}, \underline{a}, (\underline{a}+1)/2, (\underline{a}+2)/2)$, and $p_4(\underline{a}) = (\underline{a}, 2, \underline{a}, \underline{a})$. Denote $\langle p_0(\underline{a}), p_1(\underline{a}), p_3(\underline{a})\rangle$ by $t_1(\underline{a})$, $\langle p_0(\underline{a}), p_2(\underline{a}), p_3(\underline{a})\rangle$ by $t_2(\underline{a})$, $\langle p_0(\underline{a}), p_1(\underline{a}), p_4(\underline{a})\rangle$ by $t_3(\underline{a})$, and $\langle p_0(\underline{a}), p_2(\underline{a}), p_4(\underline{a})\rangle$ by $t_4(\underline{a})$. Then let $t(\underline{a}) = \bigcup_{i=1}^{4} t_i(\underline{a})$ and $g(\underline{a}) = t(\underline{a}) \cup h(\underline{a})$. It can be shown that $g(\underline{a})$ cannot be embedded in E^3 and that $g(\underline{a})$ contains a continuum homeomorphic to $S_{2,2}$. Define g to be the collection of all continua $G(\underline{a})$ for all numbers $\underline{a}$ in (1,2).

If $\underline{a}$ and $\underline{b}$ are numbers such that $1 < \underline{a} < \underline{b} < 2$, then $B_1(\underline{a})$ and $B_2(\underline{a})$ are subsets of the hyperplanes defined by $x_3 = \underline{a}$ and $x_4 = \underline{a}$ respectively, while $B_1(\underline{b})$ and $B_2(\underline{b})$ are subsets of the hyperplanes defined by $x_3 = \underline{b}$ and $x_4 = \underline{b}$ respectively. Consequently $B_1(\underline{a}) \cap B_1(\underline{b}) = \emptyset$, and $B_2(\underline{a}) \cap B_2(\underline{b}) = \emptyset$. Furthermore, since $B_2(\underline{a})$ lies on or between the two hyperplanes defined by $x_3 = 1/a$ and $x_3 = \underline{a}$, then $B_2(\underline{a}) \cap B_1(\underline{b}) = \emptyset$. If $B_1(\underline{a})$ and $B_2(\underline{b})$ intersect, then any points of intersection must lie in the 2-plane π defined by $x_3 = \underline{a}$ and $x_4 = \underline{b}$. The intersections of $B_1(\underline{a})$ and $B_2(\underline{b})$ with the plane π are two disjoint rectangles, as illustrated in Figure 1. Hence $h(\underline{a})$ and $h(\underline{b})$ do not intersect. In addition, $t(\underline{a}) \cap t(\underline{b}) = \emptyset$ since $t(\underline{a})$ and $t(\underline{b})$ are subsets of the hyperplanes defined by $x_1 = \underline{a}$ and $x_1 = \underline{b}$ respectively.

The triangular simplexes $t_1(\underline{a})$ and $t_2(\underline{a})$ are subsets of the 2-plane π' defined by $x_1 = \underline{a}$ and $x_2 = \underline{a}$. Let $\underline{c} \in (1,2)$ such that $\underline{c} \neq \underline{a}$. Then $\pi' \cap B_1(\underline{c})$ contains only $D_1 = (\underline{a}, \underline{a}, \underline{c}, \underline{c})$ and $D_2 = (\underline{a}, \underline{a}, \underline{c}, 2\underline{c})$. We note that D_1 is a point on the straight line interval q_1q_2, where $q_1 = (\underline{a}, \underline{a}, 1, 1)$ and $q_2 = (\underline{a}, \underline{a}, 2, 2)$, and D_2 is a point on the straight line interval q_3q_4, where $q_3 = (\underline{a}, \underline{a}, 1, 2)$ and $q_4 = (\underline{a}, \underline{a}, 2, 4)$. Likewise $\pi' \cap B_2(\underline{c})$ contains only D_1 and $D_3 = (\underline{a}, \underline{a}, 1/\underline{c}, \underline{c})$. Here we need only note that D_3 is a point on the curve M defined by $x_4 = 1/x_3$. Due to the fact that $[t_1(\underline{a}) \cup t_2(\underline{a})] \cap [q_1q_2 \cup q_3q_4 \cup M] = \{p_0(\underline{a}), p_1(\underline{a}), p_2(\underline{a})\}$ and the fact that these points are distinct from D_1, D_2 and D_3, it follows that $[t_1(\underline{a}) \cup t_2(\underline{a})] \cap h(\underline{c}) = \emptyset$. This is illustrated in Figure 2.



π: $x_3 = a$ and $x_4 = b$

FIGURE 1.

Lastly, the triangular simplex $t_3(\underline{a})$ is a subset of the 2-plane π'' defined by $x_1 = \underline{a}$ and $x_3 = \underline{a}$, and the triangular disk $t_4(\underline{a})$ is a subset of the 2-plane π''' defined by $x_1 = \underline{a}$ and $x_4 = \underline{a}$. Thus $t_3(\underline{a}) \cap B_1(\underline{c})$ and $t_4(\underline{a}) \cap B_2(\underline{c})$ are both empty. If $\underline{a} < \underline{c}$, then $\pi'' \cap B_2(\underline{c})$ contains only the points $(\underline{a}, 1/\underline{c}, \underline{a}, \underline{c})$ and $(\underline{a}, 2\underline{c}, \underline{a}, \underline{c})$ and is empty if $\underline{a} > \underline{c}$. Moreover, if $\underline{a} > \underline{c}$, $\pi''' \cap B_1(\underline{c})$ contains only the points $(\underline{a}, 1/\underline{c}, \underline{c}, \underline{a})$ and $(\underline{a}, 2\underline{c}, \underline{c}, \underline{a})$ and is empty if $\underline{a} < \underline{c}$. However, because $\underline{a} \leqq x_2 \leqq 2$ for both $t_3(\underline{a})$ and $t_4(\underline{a})$ and $1/\underline{c} < 1 < \underline{a} < 2 < \underline{c}$, it follows that $[t_3(\underline{a}) \cup t_4(\underline{a})] \cap [B_1(\underline{c}) \cup B_2(\underline{c})] = \emptyset$. We conclude that $t(\underline{a}) \cap h(\underline{c}) = \emptyset$, and the elements of G are disjoint.

We observe that in the construction of M_m^n, $n/2 \leqq m \leqq n-2$, if I^n is subdivided

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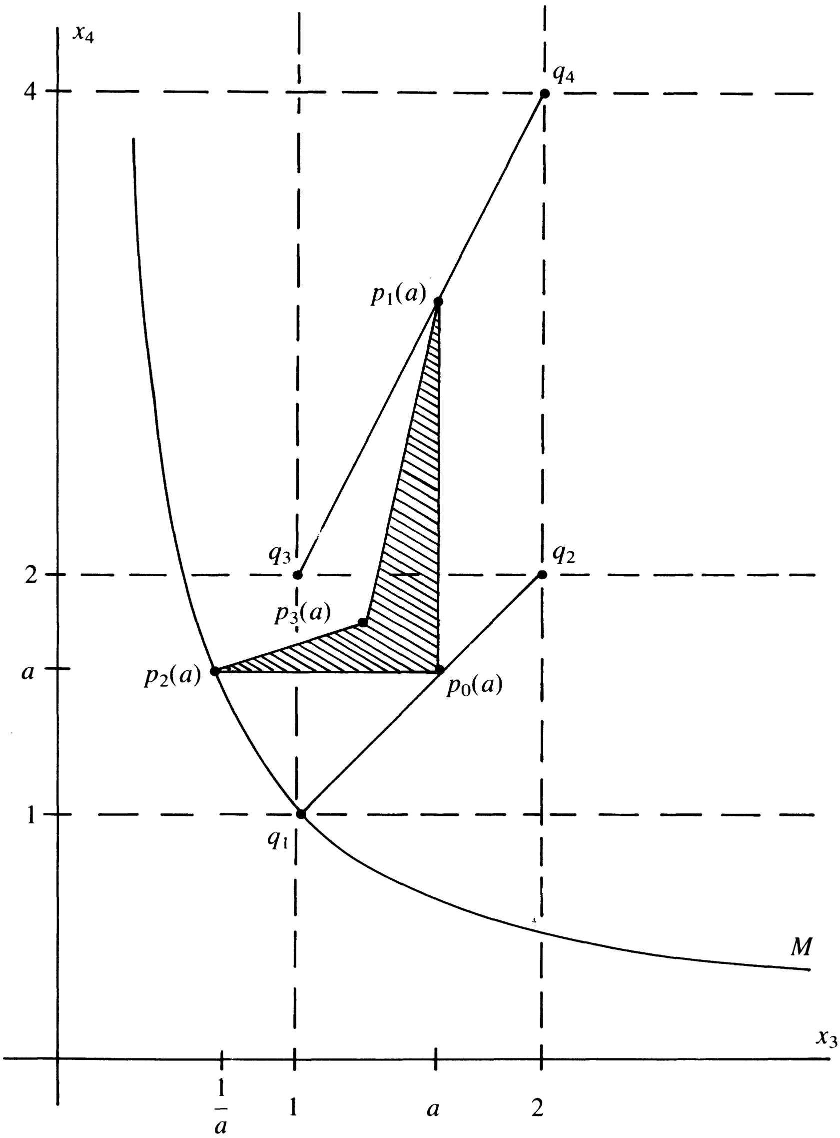

π': $x_1 = a$ and $x_2 = a$

FIGURE 2.

into k^{ni} congruent n-cubes at the ith step instead of 3^{ni} congruent n-cubes, the resulting set remains homeomorphic to M_m^n and contains an equivalent embedding of $S_{m,n-m}$.

QUESTIONS. Is there a set X in E^n which is homeomorphic to M_m^n, for $n/2 \leqq m \leqq n-2$, and which does not contain an equivalent embedding of $S_{m,n-m}$, and if there is, does E^n contain uncountable many disjoint such sets?

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QUESTIONS. Is there a set X in E^n which is homeomorphic to M_m^n, for $n/2 \leqq m \leqq n-2$, and which does not contain an equivalent embedding of $S_{m,n-m}$, and if there is, does E^n contain uncountable many disjoint such sets?

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REFERENCES

1. Mladen Bestvina, *Characterizing k-dimensional universal Menger compacta*, Bull. Amer. Math. Soc., **11**(1984), 369–370.

2. H. G. Bothe, *Ein eindimensionales Kompactum im E^3, das sich nicht lagetreau in die Mengersche Universalkurve einbetten läßt*, Fund. Math., **54** (1964), 251–258.

3. ———, *Universalmengen bezüglich der Lage im E^n*, Fund. Math., **56** (1964), 203–212.

4. K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York, 1966.

5. R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci., **14** (1928), 85–88.

6. M. A. Stan'ko, *Approximations of imbeddings of compacta in codimensions greater than two*, Soviet Math. Dokl., **12** (1971), 906–909.

7. G. S. Young, Jr., *A generalization of Moore's Theorem on simple triods,* Bull. Amer. Math. Soc., **50** (1944), 714.

Lamar University
Beaumont, Texas
77710

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