# ON THE IDEAL THEORY OF THE KRONECKER FUNCTION RING AND THE DOMAIN $D(X)$

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**1. Introduction.** Let $D$ be an integrally closed domain with identity having quotient field $L$. If $\{V_\alpha\}$ is the set of valuation overrings of $D$ and if $A$ is an ideal of $D$, then $\tilde{A} = \bigcap_\alpha A V_\alpha$ is an ideal of $D$ called the *completion* of $A$. If $X$ is an indeterminate over $D$ and $f \in D[X]$, then we denote by $A_f$ the ideal of $D$ generated by the coefficients of $f$. The Kronecker function ring $D^K$ of $D$ is defined by $D^K = \{f/g | f, g \in D[X], \tilde{A}_f \subseteq \tilde{A}_g\}$ (**4**, p. 558); and the domain $D(X)$ is defined by $D(X) = \{f/g | f, g \in D[X], A_g = D\}$ (**5**, p. 17). In this paper we wish to relate the ideal theory of $D$ to that of $D^K$ and $D(X)$ for the case in which $D$ is a Prüfer domain, a Dedekind domain, or an almost Dedekind domain.

**2. Preliminary results.** The Kronecker function ring was defined by Prüfer in (**6**) and was further investigated by Krull in (**4**). In (**4**) Krull showed that $D^K$ is an integral domain having quotient field $L(X)$, where $L$ is the quotient field of $D$, and that $D^K \cap L = D$. He further showed that $D^K$ is a Bezout domain, where a *Bezout domain* is defined to be a domain in which each finitely generated ideal is principal.

By a *valuation overring* of $D$ we shall mean a valuation ring $V$ such that $D \subseteq V \subseteq L$. Thus, let $V$ be a valuation overring of $D$ and suppose that $v$ is a valuation associated with $V$. If $f \in L[X] - \{0\}$, $f = f_0 + f_1X + \ldots + f_nX^n$, we define

$$v^*(f) = \min_{0 \leq i \leq n} \{v(f_i) | f_i \neq 0\}.$$

Then $v^*$ defines a valuation on $L(X)$ which is called the *trivial extension of $v$ to $L(X)$*, and if $V^*$ is the valuation ring of $L(X)$ associated with $v^*$, then $V^*$ is called the *trivial extension of $V$ to $L(X)$*. (We note that $v$ and $v^*$ have the same value group, thus $V$ and $V^*$ have the same rank.) With this notation and terminology, we state the following theorem which was proved by Krull (**4**, p. 560).

THEOREM 1. *Let $D$ be an integrally closed domain with identity, let $D^K$ be the Kronecker function ring of $D$, and let $\{W_\alpha\}$ be the collection of valuation overrings of $D^K$. If $V_\alpha = W_\alpha \cap L$, then $\{V_\alpha\}$ is the collection of valuation overrings of $D$, $W_\alpha = V_\alpha^*$ for each $\alpha$, and $D^K = \bigcap_\alpha V_\alpha^*$.*

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Received July 7, 1967. This paper is a portion of the author's doctoral dissertation which was written under the direction of Professor Robert W. Gilmer at Florida State University.

Let $R$ be a commutative ring with identity and let $S = \{f \in R[X] | A_f = R\}$. In (**5**, p. 17), Nagata showed that $S$ is a multiplicative system of $R[X]$ which contains only regular elements, and he then defined $R(X) = (R[X])_S$. It is easily seen that if $\{M_\beta\}$ is the collection of maximal ideals of $R$, and if $M_\beta[X]$ denotes the extension of $M_\beta$ to $R[X]$, then we also have that $S = R[X] - \bigcup_\beta M_\beta[X]$. We now state the following theorem which was proved by Nagata (**5**, p. 18).

THEOREM 2. *The set* $\{M_\beta R(X) | M_\beta$ *is a maximal ideal of* $R\}$ *is the collection of maximal ideals of* $R(X)$.

We now consider $D(X)$, where $D$ is a domain with identity having quotient field $L$. In particular, we relate the structure of $D(X)$ to that of quotient rings of $D$.

LEMMA 1. *If $S$ is a multiplicative system in $D$ and if $P$ is a prime ideal of $D$ such that $P \cap S = \emptyset$, then $D_P = (D_S)_{PD_S}$.*

*Proof.* Since $PD_S \cap D = P$ and $D \subseteq D_S$, it follows that $D_P \subseteq (D_S)_{PD_S}$. To show the reverse containment, suppose that $\delta = d/m \in (D_S)_{PD_S}$, where $d \in D_S$ and $m \in D_S - PD_S$. Then $d = d_1/s$, where $d_1 \in D$ and $s \in S$, and $m = m_1/t$, where $m_1 \in D - P$ and $t \in S$. Therefore, $\delta = d_1t/m_1s$, where $d_1t \in D$ and $m_1s \in D - P$; that is, $\delta \in D_P$. Thus, $(D_S)_{PD_S} \subseteq D_P$ and Lemma 1 follows.

LEMMA 2. *If $P$ is a prime ideal of $D$, then $(D[X])_{P[X]} = (D_P[X])_{PD_P[X]} = D_P(X)$.*

*Proof.* Let $S = D - P$. Then $S$ is a multiplicative system of $D[X]$ and $(D[X])_S = D_S[X] = D_P[X]$. Further, $P[X]$ is a prime ideal of $D[X]$ such that $P[X] \cap S = \emptyset$. Therefore, by Lemma 1, we have that $(D[X])_{P[X]} = (D[X])_{PD[X]} = (D[X]_S)_{PD[X]_S} = (D_P[X])_{PD_P[X]}$. That $(D_P[X])_{PD_P[X]} = D_P(X)$ follows from the fact that $PD_P$ is the unique maximal ideal of $D_P$.

THEOREM 3. *If $\{M_\beta\}$ is the collection of maximal ideals of $D$, then $D(X) = \bigcap_\beta D_{M_\beta}(X)$.*

*Proof.* We have seen that $\{M_\beta D(X)\}$ is the collection of maximal ideals of $D(X)$. Therefore, $D(X) = \bigcap_\beta (D(X))_{M_\beta D(X)}$. Let $S = D[X] - \bigcup_\beta M_\beta[X]$. From Lemmas 1 and 2 we have that $(D(X))_{M_\beta D(X)} = (D[X]_S)_{M_\beta[X]D[X]_S} = (D[X])_{M_\beta[X]} = (D_{M_\beta}[X])_{M_\beta D_{M_\beta}[X]} = D_{M_\beta}(X)$ for each $\beta$. Therefore, $D(X) = \bigcap_\beta D_{M_\beta}(X)$.

We now observe that if $D$ is an integrally closed domain, then $D(X) \subseteq D^K$. For, if $\delta \in D(X)$, then by definition of $D(X)$ there exists $f, g \in D[X]$ such that $A_g = D$ and such that $\delta = f/g$. Therefore, $\tilde{A}_f \subseteq \tilde{A}_g = D$, that is, $\delta \in D^K$. It then follows that $D \subseteq D(X) \subseteq D^K$ and $D = D(X) \cap L = D^K \cap L$.

https://doi.org/10.4153/CJM-1969-063-4 Published online by Cambridge University Press

**3. Related ideal theory of $D$, $D^K$, and $D(X)$.** An integral domain $D$ with identity is said to be a *Prüfer domain* provided that each finitely generated non-zero ideal of $D$ is invertible. Clearly then, a Bezout domain is a Prüfer domain; thus, it follows that $D^K$ is a Prüfer domain. Krull has shown (**4**, p. 554) that an integral domain $D$ is a Prüfer domain (*multiplikationsring* in the terminology of Krull) if and only if each valuation overring of $D$ is determined uniquely as the quotient ring of $D$ with respect to its centre on $D$. From this, it is easily seen that if $\{M_\beta\}$ is the collection of maximal ideals of $D$, then $D$ is a Prüfer domain if and only if the quotient ring $D_{M_\beta}$ is a valuation ring for each $\beta$. $D$ is said to be *almost Dedekind* if for each non-zero proper prime ideal $P$ of $D$, $D_P$ is a rank one discrete valuation ring (**3**, p. 813). We now relate the ideal theory of $D$, $D^K$, and $D(X)$ for the case in which $D$ is a Prüfer domain, a Dedekind domain, or an almost Dedekind domain. Since the ideal theory of such a domain is so closely related to the ideal theory of its valuation overrings, our results will depend on the relationship between the domains $V$, $V^K$, and $V(X)$, where $V$ is a valuation overring of $D$. Thus, we prove the following lemma.

LEMMA 3. *Let $V$ be a valuation ring having quotient field $L$. Then $V(X) = V^* = V^K$, where $V^K$ is the Kronecker function ring of $V$ and $V^*$ is the trivial extension of $V$ to $L(X)$.*

*Proof.* Let $\{V_\alpha\}$ be the collection of valuation overrings of $V$. Since $V \in \{V_\alpha\}$, if $A$ is an ideal of $V$, we have that $\tilde{A} = \bigcap_\alpha A V_\alpha = (AV) \cap (\bigcap_\alpha A V_\alpha) = A$.

Now, let $v$ be a valuation associated with $V$ and let $v^*$ be its trivial extension to $L(X)$. If $f, g \in V[X] - \{0\}$, $f = \sum_{i=0}^n f_i X^i$ and $g = \sum_{j=0}^m g_j X^j$, then $v^*(f) \geqq v^*(g)$ if and only if

$$\min_{0 \leqq i \leqq n} \{v(f_i) \mid f_i \neq 0\} \geqq \min_{0 \leqq j \leqq m} \{v(g_j) \mid g_j \neq 0\};$$

that is, if and only if $A_f \subseteq A_g$. We then have that $V^* = \{f/g \mid f, g \in V[X] - \{0\},\ v^*(f) \geqq v^*(g)\} \cup \{0\} = \{f/g \mid f, g \in V[X],\ g \neq 0,\ A_f \subseteq A_g\} = V^K$. Further, if

$$v(g_t) = \min_{0 \leqq j \leqq m} \{v(g_j) \mid g_j \neq 0\},$$

then it follows that $v(f_i) \geqq v(g_t)$ for each $f_i \neq 0$, $0 \leqq i \leqq n$, and $v(g_j) \geqq g_t$ for each $g_j \neq 0$, $0 \leqq j \leqq m$. Therefore, there exists $r_i \in V$ such that $f_i = r_i g_t$, $0 \leqq i \leqq n$, and there exists $s_j \in V$ such that $g_j = s_j g_t$, $0 \leqq j \leqq m$. If $f' = \sum_{i=0}^n r_i X^i$ and $g' = \sum_{j=0}^m s_j X^j$, then $f/g = g_t f'/g_t g' = f'/g'$, and $A_{g'} = V$ since $s_t = 1$. Consequently, $f/g \in V(X)$ and we have that $V^* = V^K \subseteq V(X)$. But we have already seen that $D(X) \subseteq D^K$ for an integrally closed domain $D$. Therefore, $V^* = V^K = V(X)$ as we wished to show.

THEOREM 4. *If $D$ is an integrally closed domain with identity, then the following conditions are equivalent:*

(1) *$D$ is a Prüfer domain*;
(2) $D(X) = D^K$;
(3) *$D(X)$ is a Prüfer domain*;
(4) *$D^K$ is a quotient ring of $D[X]$*;
(5) *Each prime ideal of $D(X)$ is the contraction of a prime ideal of $D^K$*;
(6) *Each prime ideal of $D(X)$ is the extension of a prime ideal of $D$.*

*Proof.* (1) $\Rightarrow$ (2). If $\{M_\beta\}$ is the collection of maximal ideals of $D$, then by Theorem 3, we have that $D(X) = \bigcap_\beta D_{M_\beta}(X)$. Since $D$ is a Prüfer domain, $D_{M_\beta}$ is a valuation ring for each $\beta$, and by the previous lemma, $D_{M_\beta}(X) = (D_{M_\beta})^*$, where $(D_{M_\beta})^*$ is the trivial extension of $D_{M_\beta}$ to $L(X)$. But

$$D^K \subseteq \bigcap_\beta (D_{M_\beta})^* = D(X).$$

Therefore, $D^K = D(X)$.

(2) $\Rightarrow$ (3). This is immediate since $D^K$ is a Prüfer domain.

(3) $\Rightarrow$ (1). Let $M$ be a maximal ideal of $D$. Then $MD(X)$ is a maximal ideal of $D(X)$ so that $D(X)_{MD(X)}$ is a valuation ring. But in the proof of Theorem 3 we showed that $D(X)_{MD(X)} = D_M(X)$. Therefore, since $D_M = D_M(X) \cap L$, $D_M$ is a valuation ring. It then follows that $D$ is a Prüfer domain.

Clearly, (2) implies both (4) and (5).

(4) $\Rightarrow$ (2). If $D^K$ is a quotient ring of $D[X]$, then $D^K = (D[X])_S$, where $S$ is the set of elements of $D[X]$ such that $1/f \in D^K$. But $1/f \in D^K$ if and only if $D \subseteq \tilde{A}_f$ and therefore, if and only if $D = A_f$. Consequently, $S = \{f \in D[X] | A_f = D\}$, and (2) follows.

To show that (5) implies (6), we need the following lemma.

LEMMA 4. *If $P'$ is a prime ideal of $D^K$ and if $P = P' \cap D$, then $P' \cap D[X] = P[X]$.*

*Proof.* It is clear that $P[X] \subseteq P' \cap D[X]$. To show the reverse containment, let $f \in P' \cap D[X]$, $f = f_0 + f_1X + \ldots + f_nX^n$. From the proof of (**4**, *Satz* 14, p. 559), we see that $(f_0, \ldots, f_n)D^K = (f_0 + f_1X + \ldots + f_nX^n)D^K \subseteq P'$. Thus, $f_i \in P' \cap D = P$ for each $i$, $0 \leqq i \leqq n$; that is, $f \in P[X]$.

(5) $\Rightarrow$ (6). Let $Q$ be a prime ideal of $D(X)$ and suppose that $Q = P' \cap D(X)$, $P'$ a prime ideal of $D^K$. From Lemma 4 we have that $P[X] = P' \cap D[X]$, where $P = P' \cap D$. It then follows that

$$Q \cap D[X] = (P' \cap D(X)) \cap D[X] = P' \cap D[X] = P[X].$$

Since $D(X)$ is a quotient ring of $D[X]$, $Q = P[X]D(X) = PD(X)$.

(6) $\Rightarrow$ (1). Let $P$ be a proper prime ideal of $D$. We show that $D_P$ is a valuation ring by showing that for $t \in L - \{0\}$, either $t \in D_P$ or $1/t \in D_P$.

Let $P'$ be a proper prime ideal of $D[X]$ such that $P' \subseteq \bigcup_\beta M_\beta[X]$. Then $P'D(X)$ is a proper prime ideal of $D(X)$; thus by assumption, there exists a prime ideal $M$ of $D$ such that $P'D(X) = MD(X) = M[X]D(X)$. Therefore, $P' = P'D(X) \cap D[X] = M[X]D(X) \cap D[X] = M[X]$, and it then follows that $P' \cap D = M[X] \cap D = M \neq (0)$.

Now let $t \in L$ and let $Q$ be the kernel of the canonical $D$-homomorphism $\phi$ from $D[X]$ onto $D[t]$ such that $\phi(X) = t$. Since $\phi(d) = d$ for $d \in D$, we have that $Q \cap D = 0$. Then from what we have just observed, $Q \not\subseteq \bigcup_\beta M_\beta[X]$. But $P[X] \subseteq \bigcup_\beta M_\beta[X]$; thus there exists $f(X) \in Q - P[X]$; that is, there exists $f(X) \in D[X]$ such that $f(t) = 0$ but $f(X) \notin P[X]$. It then follows that either $t$ or $1/t$ is in $D_P$ (**8**, p. 19). This completes the proof of Theorem 4.

THEOREM 5. *If $D$ is an integrally closed domain with identity, then the following statements are equivalent*:

(1) *$D$ is almost Dedekind*;

(2) *$D(X)$ is almost Dedekind*;

(3) *$D^K$ is almost Dedekind.*

*Proof.* (1) $\Rightarrow$ (2). Since an almost Dedekind domain is a Prüfer domain, it follows from Theorem 4 that $D(X)$ is a Prüfer domain. If $M_\beta D(X)$ is a maximal ideal of $D(X)$, $M_\beta$ a maximal ideal of $D$, then $D(X)_{M_\beta D(X)} = D_{M_\beta}(X) = (D_{M_\beta})^*$. But $D_{M_\beta}$ is a rank one discrete valuation ring; thus $(D_{M_\beta})^*$ is also a rank one discrete valuation ring. Therefore, $D(X)$ is one-dimensional and for each proper prime ideal $P$ of $D(X)$, $D(X)_P$ is a rank one discrete valuation ring. Thus, $D(X)$ is almost Dedekind.

(2) $\Rightarrow$ (3). If $D(X)$ is almost Dedekind, then it is also Prüfer; thus $D(X) = D^K$ by Theorem 4.

(3) $\Rightarrow$ (1). If $D^K$ is almost Dedekind, then it is one-dimensional; thus by (**1**, Corollary 2) each valuation overring of $D$ has dimension less than or equal to one. Since $D \subset L$, it follows from a theorem proved by Gilmer (**2**, p. 212) that $D$ is a one-dimensional Prüfer domain.

If $P$ is a proper prime ideal of $D$, then $D_P$ is a valuation overring of $D$; thus $(D_P)^*$ is a valuation overring of $D^K$. But since $D^K$ is a Prüfer domain, $(D_P)^* = (D^K)_{P'}$ for some prime ideal $P'$ of $D^K$. In particular, $(D_P)^*$ is a rank one discrete valuation ring. Therefore, $D_P$ is a rank one discrete valuation ring and it follows that $D$ is almost Dedekind.

If $D$ is an integral domain with identity, then $D$ is said to be a *Krull domain* provided there exists a collection $\{V_\alpha\}$ of valuation overrings of $D$ such that the following properties hold:

($E_1$) Each $V_\alpha$ has rank one and is discrete;

($E_2$) $D = \bigcap_\alpha V_\alpha$;

($E_3$) Each non-zero element of $D$ is a unit in all but a finite number of the $V_\alpha$;

($E_4$) For each $\alpha$, $V_\alpha$ is a quotient ring of $D$ with respect to its centre on $D$.

THEOREM 6. *If $D$ is an integrally closed domain, then the following statements are equivalent*:

(1) *$D$ is a Dedekind domain*;

(2) *$D(X)$ is a Dedekind domain*;

(3) *$D^K$ is a Dedekind domain*;

(4) *$D^K$ is Noetherian;*
(5) *$D^K$ is a Krull domain.*

*Proof.* (1) ⇒ (2). Since a Dedekind domain is almost Dedekind, it follows from Theorem 5 that $D(X)$ is almost Dedekind. But $D[X]$ is Noetherian since $D$ is, and $D(X)$ is a quotient ring of $D[X]$. Therefore, $D(X)$ is Noetherian, and consequently, $D(X)$ is a Dedekind domain (**7**, p. 275).

(2) ⇒ (3). Since $D(X)$ is a Dedekind domain, it is also a Prüfer domain. Hence, $D(X) = D^K$ and $D^K$ is a Dedekind domain.

That (3) implies (4) is immediate, since Dedekind domains are Noetherian, and that (4) implies (5) follows from the fact that integrally closed Noetherian domains are Krull domains (**8**, p. 82).

(5) ⇒ (3). If $D^K$ is a Krull domain, then any quotient ring of $D^K$ is also a Krull domain (**5**, p. 116). In particular, if $P$ is a proper prime ideal of $D^K$, then $(D^K)_P$ is a Krull domain. Therefore, $(D^K)_P$ is an intersection of rank one discrete valuation rings. But $(D^K)_P$ is itself a valuation ring; thus it is rank one and discrete. It then follows that each proper prime ideal of $D^K$ is minimal, and consequently, $D^K$ is a Dedekind domain (**8**, p. 84).

(3) ⇒ (1). If $D^K$ is a Dedekind domain, then it follows from Theorems 4 and 5 that $D^K = D(X)$ and $D$ is almost Dedekind.

Gilmer has shown that if $D$ is a domain with identity which is almost Dedekind, then $D$ is Dedekind if and only if each non-zero proper ideal of $D$ is contained in only finitely many maximal ideals (**3**, p. 815). Thus, let $A$ be a non-zero proper ideal of $D$ and let $\{M_\beta\}$ be the collection of maximal ideals of $D$. Then, by Theorem 2, $\{M_\beta D^K\}$ is the collection of maximal ideals of $D^K$. Further, since $D^K$ is a Dedekind domain, $AD^K$ is contained in only finitely many of the maximal ideals of $D^K$; thus it follows that $A \subseteq M_\beta$ for only finitely many $\beta$'s. Therefore, $D$ is a Dedekind domain.

## References


**1.** J. T. Arnold, *On the dimension theory of overrings of an integral domain* (submitted for publication).

**2.** R. W. Gilmer, *Domains in which valuation ideals are prime powers*, Arch. Math. *17* (1966), 210–215.

**3.** ——— *Integral domains which are almost Dedekind*, Proc. Amer. Math. Soc. *15* (1964), 813–818.

**4.** W. Krull, *Beiträge Zur Arithmetik kommutativer Integritätsbereiche*, Math. Z. *41* (1936), 545–577.

**5.** M. Nagata, *Local rings* (Interscience, New York, 1962).

**6.** H. Prüfer, *Untersuchungen über die Teilbarkeitseigenschaften in Körpern*, J. Reine Angew. Math. *168* (1932), 1–36.

**7.** O. Zariski and P. Samuel, *Commutative algebra*, Vol. 1 (Van Nostrand, Princeton, 1958).

**8.** ——— *Commutative algebra*, Vol. 2 (Van Nostrand, Princeton, 1960).



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