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On uniqueness of invariant measures for random walks on HOMEO$^+$($\mathbb{R}$)

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Abstract. We consider random walks on the group of orientation-preserving homeomorphisms of the real line $\mathbb{R}$. In particular, the fundamental question of uniqueness of an invariant measure of the generated process is raised. This problem was studied by Choquet and Deny [Sur l'équation de convolution $\mu = \mu * \sigma$. *C. R. Acad. Sci. Paris* **250** (1960), 799–801] in the context of random walks generated by translations of the line. Nowadays the answer is quite well understood in general settings of strongly contractive systems. Here we focus on a broader class of systems satisfying the conditions of recurrence, contraction and unbounded action. We prove that under these conditions the random process possesses a unique invariant Radon measure on $\mathbb{R}$. Our work can be viewed as following on from Babillot *et al* [The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Ann. Probab.* **25**(1) (1997), 478–493] and Deroin *et al* [Symmetric random walk on HOMEO$^+$($\mathbb{R}$). *Ann. Probab.* **41**(3B) (2013), 2066–2089].

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1. *Introduction*

Let Homeo$^+$($\mathbb{R}$) denote the group of orientation-preserving homeomorphisms of the real line $\mathbb{R}$. We shall consider the (left) random walk on Homeo$^+$($\mathbb{R}$), that is, the sequence of random homeomorphisms

$$\ell_n := \mathbf{g}_n \cdots \mathbf{g}_1$$

obtained by iterated composition products of a sequence $(\mathbf{g}_n)_{n\in\mathbb{N}}$ of independent and identically distributed (i.i.d.) $\mathrm{Homeo}^+(\mathbb{R})$-valued random variables. We denote by μ the common distribution of the $\mathbf{g}_n$. We shall always assume that μ is a *discrete* probability measure on $\mathrm{Homeo}^+(\mathbb{R})$. This sequence of random transformations induces a *stochastic dynamical system* (or *an iterated random function system*) on the real line, that is, the Markov chain $(X_n^x)_{n\in\mathbb{N}}$ defined recursively for any starting value $X_0^x = x \in \mathbb{R}$ by the formula

$$X_n^x := \mathbf{g}_n(X_{n-1}^x) = \ell_n(x) \quad \text{for } n \geq 1.$$

The associated Markov kernel is of the form

$$Pf(x) := \sum_{g\in\Gamma} f(g(x))\mu(g) \quad \text{for any bounded Borel-measurable function } f \text{ on } \mathbb{R}.$$

Here Γ denotes the discrete support of μ, that is, $\Gamma := \{g \in \mathrm{Homeo}^+(\mathbb{R}) : \mu(g) > 0\}$.

We are interested in the case when the Markov chain $(X_n^x)_{n\in\mathbb{N}}$ does not escape to infinity. Namely, we always suppose that the following hypothesis is satisfied.

($\mathfrak{R}$) The Markov chain is (uniformly topologically) *recurrent*, that is there exists a compact interval $\mathcal{I} \subset \mathbb{R}$ such that for every $x \in \mathbb{R}$ the sequence $(X_n^x)_{n\in\mathbb{N}}$ visits $\mathcal{I}$ infinitely often almost surely (a.s.).

Condition ($\mathfrak{R}$) entails immediately that there exists an *invariant Radon measure* ν for the system generated by μ, that is, a measure, finite on compact sets, satisfying

$$\int_{\mathbb{R}} f(x)\, d\nu(x) = \int_{\mathbb{R}} Pf(x)\, d\nu(x)$$

for any $f \in C_C(\mathbb{R})$, the space of continuous functions with compact support. This measure can be either finite or infinite.

The fundamental question of this paper is to decide whether an invariant measure is unique up to a multiplicative constant. This problem has been widely studied for different kind of systems: the now classical Choquet–Deny theorem [12] can been seen as one of the first results in this direction. It says that the Lebesgue measure on $\mathbb{R}$ is the unique Radon invariant measure for systems generated by translations that are recurrent and do not have discrete orbits. Among other interesting results we would like to mention fundamental works on strongly contractive systems initiated by Furstenberg [17]; see also [14, 22, 27]. In these works, under various contracting assumptions, it was proved that there exists a unique invariant probability measure.

A weaker contraction property (called local stability) has been proposed to deal with systems that have infinite Radon invariant measures. This property was first used by Babillot, Bougerol and Elie [5] in the case of systems generated by centred random affinities. Next it was studied in a much more general setting by Benda [6], Peigné and Woess [26] and Deroin *et al* [13]. The latter paper contains a detailed study of the uniqueness of an invariant measure for random walks on $\mathrm{Homeo}^+(\mathbb{R})$ under the hypothesis of the measure μ being symmetric. Using a conjugation of the reals to some open interval, say $(0, 1)$, we obtain some results for random walks on the group of orientation-preserving homeomorphisms of the interval $(0, 1)$. Initially such walks were considered by L. Alsedà and M. Misiurewicz who studied some function systems consisting of piecewise linear homeomorphisms and proved the existence of a unique probability measure (see [1]). More general function systems were investigated by Gharaei and Homburg in [19]. Recently D. Malicet obtained unique ergodicity as a consequence of the contraction principle for time-homogeneous random walks on the topological group of homeomorphisms defined on the circle and interval (see [24]). His proof, in turn, is based upon an invariance principle of Avila and Viana (see [4]). A simple proof of unique ergodicity on the open interval $(0, 1)$ for a wide class of iterated function systems is given in [10].

The main goal of this paper is to show that the uniqueness of an invariant measure can be obtained assuming, besides recurrence, the following two conditions that only involve the action of Γ (the support of μ) on $\mathbb{R}$.

($\mathfrak{C}$) *Contraction (or proximality) of the action.* There exists an interval $\mathcal{I} \subset \mathbb{R}$ such that for any compact set $K \subset \mathbb{R}$ there is some g belonging to the semigroup generated by Γ such that $g(K) \subset \mathcal{I}$.

($\mathfrak{U}$) *Unboundedness of the action.* For every $x \in \mathbb{R}$ we have $g_1(x) < x < g_2(x)$ for some $g_1, g_2 \in \Gamma$.

The first of this conditions says that it is possible to shrink any bounded set at finite distance. We will see that the second condition is equivalent to the question of whether one can reach $+\infty$ and $-\infty$ from any starting point x.

From now on, uniqueness will mean the existence of a unique, up to a scalar factor, invariant Radon measure. The main purpose of the paper is to prove that under the above conditions the invariant measure is unique.

THEOREM 1.1. *Assume that a stochastic dynamical system, generated by a discrete distribution μ on* Homeo$^+(\mathbb{R})$*, satisfies assumptions ($\mathfrak{R}$), ($\mathfrak{C}$) and ($\mathfrak{U}$). Then the system admits a unique invariant Radon measure ν.*

The study of invariant measures is strictly related to the issue of closed Γ-invariant sets, that is, closed sets $M \subseteq \mathbb{R}$ such that $gM \subseteq M$ for all $g \in \Gamma$. In fact, for any invariant measure ν its support supp ν is a closed Γ-invariant subset of $\mathbb{R}$. One of the crucial questions that the present paper explores is whether a closed Γ-invariant set can be contained in the support of different invariant ergodic measures. Recall that an invariant measure ν is *ergodic* if for any $A \subseteq \mathbb{R}$ such that ν_A, the restriction of ν to A, is invariant, we obtain that either $\nu_A = \nu$ or $\nu_A \equiv 0$. The following theorem gives a quite complete answer to this question under the recurrence and unboundedness hypotheses only.

THEOREM 1.2. *Assume that a stochastic dynamical system, generated by a discrete distribution μ on* Homeo$^+(\mathbb{R})$*, satisfies assumptions ($\mathfrak{R}$) and ($\mathfrak{U}$).*

(1) *Let ν_1 and ν_2 be two ergodic invariant Radon measures such that* supp $\nu_1 \subseteq$ supp ν_2 *and* supp ν_1 *is not discrete. Then $\nu_1 = C\nu_2$ for some constant $C > 0$.*

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(2) *The support of every ergodic invariant Radon measure ν either is minimal among the closed Γ-invariant sets or contains a Γ-invariant discrete set.*
(3) *For any minimal closed Γ-invariant set M there exists a unique ergodic invariant Radon measure ν such that $M = \operatorname{supp} \nu$.*

The proof of Theorem 1.2 will be given in §3. In §4 we will show that Theorem 1.1 is a consequence of this result together with the contraction hypothesis and the ergodic decomposition of invariant measures. We would like to point out that the results of these two theorems are quite optimal and that conditions ($\mathfrak{R}$), ($\mathfrak{C}$) and ($\mathfrak{U}$) are all needed to ensure uniqueness. In §5 we shall provide a number of examples and discuss our hypothesis.

In this paper we would also like to show how the general theorem (Theorem 1.1) can be applied to several specific but interesting situations. For instance, we will prove that an immediate consequence is the uniqueness of an invariant measure for recurrent affine recursions.

COROLLARY 1.3. *Let μ be a discrete measure on $\Gamma \subset \mathrm{Homeo}^+(\mathbb{R})$. Assume that every $g \in \Gamma$ is of the form $g(x) = A(g)x + B(g)$ for $x \in \mathbb{R}$. Moreover, assume that there exists $g_0 \in \Gamma$ such that $A(g_0) < 1$. Then, if conditions ($\mathfrak{R}$) and ($\mathfrak{U}$) hold, the corresponding stochastic dynamical system admits a unique invariant measure ν.*

This result is well known but we give here a new proof of it. In particular, it is not based on the Lipschitz property of affine transformations. The proof is valid both in the contractive case (when there exists a stationary probability measure [**17**]) and in the centred case (when the invariant measure has infinite mass [**5**]).

The recurrence ($\mathfrak{R}$) and contraction ($\mathfrak{C}$) conditions can be easily verified when homeomorphisms are repulsive at $\pm\infty$. In Lemma 5.1 we will present some general criteria for systems that are asymptotically linear, such as affine recursions. As a consequence, using a conjugation, one can obtain the following results for C^2-diffeomorphisms of the interval.

COROLLARY 1.4. *Let μ be a finitely supported measure on the group of increasing diffeomorphisms in $C^2([0, 1])$. Assume that:*
($\mathfrak{R}'$) $\sum_{h\in\operatorname{supp}\mu} \mu(h) \ln h'(0) \geq 0$ *and* $\sum_{h\in\operatorname{supp}\mu} \mu(h) \ln h'(1) \geq 0$;
($\mathfrak{C}'$) *there exists $h \in \operatorname{supp}\mu$ such that $h'(0) > 1$ and $h'(1) > 1$;*
($\mathfrak{U}'$) *for every $x \in (0, 1)$ there exist $h_1, h_2 \in \operatorname{supp}\mu$ such that $h_1(x) < x < h_2(x)$.*
Then there exists a unique invariant Radon measure on $(0, 1)$.

In §A we shall discuss some seminal results on ergodic invariant measures for Markov–Feller processes on locally compact metric spaces. In particular, we will give an explicit proof of the ergodic decomposition of a general invariant Radon measure as an integral over all ergodic Radon measures.

2. *Basic notions and preliminary results*

In this section we give the fundamental notions and basic facts about invariant Radon measures that will play an important role in the sequel.

2.1. *Random walks on* Homeo$^+(\mathbb{R})$ *and associated dynamical systems.* We denote by Homeo$^+(\mathbb{R})$ the set of orientation-preserving homeomorphisms of the real line. We consider the left random walk on Homeo$^+(\mathbb{R})$, that is, the Markov chain

$$\ell_n := \mathbf{g}_n \cdots \mathbf{g}_1$$

obtained by composition product of a sequence $(\mathbf{g}_n)_{n\in\mathbb{N}}$, which is a sequence of i.i.d. Homeo$^+(\mathbb{R})$-valued random variables whose distribution is a discrete measure μ. Let

$$\Gamma := \{g \in \text{Homeo}^+(\mathbb{R}) : \mu(g) > 0\} \subset \text{Homeo}^+(\mathbb{R})$$

be the discrete support of μ. The space of trajectories of the random walk is then the infinite product space $\Gamma^{\mathbb{N}}$. This space will be equipped with the product measure $(\Gamma^{\mathbb{N}}, \mu^{\otimes\mathbb{N}})$. The associated probability law will be denoted by $\mathbb{P}$. We denote by

$$\Gamma^* := \{g = g_1 \cdots g_n \in \text{Homeo}^+(\mathbb{R}) \text{ for some } g_i \in \Gamma\}$$

the semigroup generated by Γ. Observe that Γ^* is countable and may be equipped with the discrete topology.

We denote by $\mathcal{B}(\mathbb{R})$ the collection of all Borel subsets of $\mathbb{R}$, by $B(\mathbb{R})$ the family of all Borel-measurable bounded (real-valued) functions with the supremum norm $\|\cdot\|_\infty$ and by $C(\mathbb{R})$ the subspace of $B(\mathbb{R})$ consisting of all continuous functions. The subfamily of $C(\mathbb{R})$ consisting of all continuous functions with compact support is denoted by $C_C(\mathbb{R})$.

Since the semigroup Γ^* acts on $\mathbb{R}$, we can introduce the stochastic dynamical system on the real line $(X_n^x)_{n\in\mathbb{N}}$ corresponding to the left random walk on Homeo$^+(\mathbb{R})$, that is, for any $x \in \mathbb{R}$ we define the Markov chain

$$X_n^x := \mathbf{g}_n(X_{n-1}^x) = \ell_n(x) \quad \text{for } n \geq 1$$

and $X_0^x = x$. The transition probability for this Markov chain is given by the formula

$$P(x, A) = \sum_{g\in\Gamma} \mathbf{1}_A(g(x))\mu(g) \quad \text{for } x \in \mathbb{R} \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

It induces a positive contraction P on $B(\mathbb{R})$ defined by

$$Ph(x) := \sum_{g\in\Gamma} h(g(x))\mu(g) \quad \text{for } h \in B(\mathbb{R}). \tag{1}$$

For any Radon measure ν on $\mathbb{R}$, let $\mathbb{P}_\nu$ be the measure defined on the trajectories of the Markov chain $(X_n)_{n\in\mathbb{N}}$ where X_0 is distributed according to ν. More precisely, $\mathbb{P}_\nu$ is a measure on the space $\mathbb{R}^{\mathbb{N}}$ (endowed with the product σ-algebra) such that for any finite collection of compact intervals I_i, $i = 0, \ldots, n$, the measure of the cylinder $[I] = I_0 \times \cdots \times I_n \times \mathbb{R} \times \cdots$ is defined by

$$\begin{aligned}\mathbb{P}_\nu([I]) &= \mathbb{P}_\nu(X_0 \in I_0, \ldots, X_n \in I_n)\\ &:= \sum_{g_1,\ldots,g_n\in\Gamma} \mu(g_1)\cdots\mu(g_n) \int_{\mathbb{R}} \mathbf{1}_{I_0}(x)\mathbf{1}_{I_1}(g_1(x))\cdots\mathbf{1}_{I_n}(g_n\cdots g_1(x))\, d\nu(x).\end{aligned}$$

Observe that if ν is a Radon measure of infinite mass then $\mathbb{P}_\nu$ is not a probability measure but it is finite on the cylinders whose bases I_i are compact intervals.

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2.2. *Invariant measure and recurrence.* An *invariant Radon measure* for the system induced by a measure μ is a Borel measure ν that is finite on compact sets and satisfies

$$\nu(f) = \int_{\mathbb{R}} \mathbb{E} f(X_1^x)\nu(dx) = \sum_{g\in\Gamma} \int_{\mathbb{R}} f(g(x))\nu(dx)\mu(g) = \nu(Pf)$$

for any $f \in C_C(\mathbb{R})$. In short, we shall say the ν is *invariant* for μ, or that ν is a μ-invariant measure. It is easy to check that if ν is invariant for μ, then the measure $\mathbb{P}_\nu$ is invariant for the shift τ of $\mathbb{R}^{\mathbb{N}}$.

It is well known that recurrence hypothesis ($\mathfrak{R}$) immediately entails the existence of a μ-invariant Radon measure. Indeed, it is easy to see that the operator P is *topologically conservative*, that is, there exists a bounded set $K \subset \mathbb{R}$ such that

$$\sum_{k=0}^{\infty} P^k \mathbf{1}_K(x) = \infty \quad \text{for every } x \in \mathbb{R}.$$

Actually, ($\mathfrak{R}$) implies that this condition holds with $K = \mathcal{I}$. Then Lin's result [23, Theorem 5.1] ensures the existence of a μ-invariant Radon measure ν.

2.3. *Support of an invariant measure and closed Γ-invariant sets.* The analysis of μ-invariant measures is strictly related to the study of *closed* Γ*- invariant sets* of $\mathbb{R}$, that is, closed sets $M \subseteq \mathbb{R}$ such that $gM \subseteq M$ for all $g \in \Gamma$. In fact, for any μ-invariant measure ν its support

$$\text{supp } \nu := \{x \in \mathbb{R} : \nu(V_x) > 0 \text{ for every open neighbourhood } V_x \text{ of } x\}$$

is a closed Γ-invariant set of $\mathbb{R}$. To check Γ-invariance, take $x \in \text{supp } \nu$, $g_0 \in \Gamma$ and V an open neighbourhood of $g_0(x)$. Then

$$\nu(V) = \sum_{g\in\Gamma} \mu(g)\nu(g^{-1}V) \geq \mu(g_0)\nu(g_0^{-1}V) > 0,$$

since $g_0^{-1}V$ is an open-neighbourhood of x.

If ($\mathfrak{R}$) holds then, thanks to Lin's theorem, any closed Γ-invariant set contains the support of at least one μ-invariant Radon measure ν. In particular, if there exist two disjoint closed Γ-invariant sets, there are at least two different invariant measures.

To decide whether a Γ-invariant set can be (or contains) the support of different invariant measures it is essential to characterize *minimal closed* Γ*-invariant* sets, that is, closed Γ-invariant sets not containing other closed Γ-invariant sets except the empty set and itself.

2.4. *Unboundedness hypothesis.* The last of the fundamental hypotheses of our paper is as follows.

($\mathfrak{U}$) *Unboundedness of the action.* For every $x \in \mathbb{R}$ we have $g_1(x) < x < g_2(x)$ for some $g_1, g_2 \in \Gamma$.

This guarantees that any closed Γ-invariant set is unbounded. In fact, we have the following easy lemma.

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LEMMA 2.1. *Hypothesis* ($\mathfrak{U}$) *is satisfied if and only if for any* $x \in \mathbb{R}$,

$$\sup_{g\in\Gamma^*} g(x) = +\infty \quad \textit{and} \quad \inf_{g\in\Gamma^*} g(x) = -\infty.$$

In particular, if condition ($\mathfrak{U}$) *is satisfied then any non-empty* Γ*-invariant set is unbounded on both sides.*

Proof. Suppose first that ($\mathfrak{U}$) holds and $x_0 = \sup_{g\in\Gamma^*} g(x) < \infty$. Then for all $g_0 \in \Gamma$,

$$g_0(x_0) = \sup_{g\in\Gamma^*} (g_0 g)(x) \leq \sup_{g\in\Gamma^*} g(x) = x_0,$$

which contradicts ($\mathfrak{U}$).

Conversely, assume that there is an $x \in \mathbb{R}$ such that $g_1(x) \leq x$ for all $g_1 \in \Gamma$. Since all the homeomorphisms preserve the order, $g_2(g_1(x)) \leq g_2(x) \leq x$ for all $g_1, g_2 \in \Gamma$. Thus the induction argument yields $g(x) \leq x$ for all $g \in \Gamma^n$ and $n \in \mathbb{N}$. This finally implies that $\sup_{g\in\Gamma^*} g(x) \leq x$. □

In particular, under condition ($\mathfrak{U}$) the support of any invariant measure is unbounded in both directions. Note also that if ($\mathfrak{U}$) holds for Γ it also holds for Γ^{-1}.

2.5. *Ergodic measures and ratio ergodic theorem.* Among μ-invariant measures, ergodic measures play a special role. We present here the main facts and we refer to §A for a more detailed discussion.

For any measurable $A \subseteq \mathbb{R}$ denote by ν_A the restriction of ν to A. The restriction is called *trivial* if either $\nu(A) = 0$ or $\nu(\mathbb{R} \setminus A) = 0$. We say that a measure ν is *ergodic* if for any $A \in \mathcal{B}(\mathbb{R})$ such that the restriction ν_A is invariant, it must be also trivial. In our setting we can say that if an invariant measure is ergodic, then any closed Γ-invariant set M either is null or has full measure: $\nu(M) = 0$ or $\nu(\mathbb{R} \setminus M) = 0$. In §A.2 we give a more detailed discussion of other equivalent characterizations of ergodic measures.

Ergodic measures can be seen as atomic bricks that are used to construct any invariant measure. In fact, any invariant measure ν can be decomposed into ergodic components, in the sense that there exist a measurable set $\mathcal{E}_\nu$ of ergodic measures and a finite measure η_ν on $\mathcal{E}_\nu$ such that

$$\nu(f) = \int_{\mathcal{E}_\nu} \nu_e(f) d\eta_\nu(e) \quad \text{for all } f \in C_C(\mathbb{R}). \tag{2}$$

In Theorem A.6 we provide a proof of this decomposition for conservative Markov–Feller processes. Note that the above decomposition entails that if there are two different invariant measures, there must exist at least two different ergodic measures. Another consequence is that if ν is invariant, there exists an ergodic measure ν_e such that $\operatorname{supp} \nu_e \subseteq \operatorname{supp} \nu$. In fact, for η_ν-almost all $e \in \mathcal{E}_\nu$ we have $\nu_e(\mathbb{R} \setminus \operatorname{supp} \nu) = 0$. Hence we have $\operatorname{supp} \nu_e \subseteq \operatorname{supp} \nu$.

A fundamental property of ergodic μ-invariant Radon measures, which we will often use in the sequel, is the ratio ergodic theorem (or the Chacon–Ornstein theorem), which

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gives the asymptotic behaviour of the partial sum defined by

$$S_n\phi(x) := \phi(\mathbf{g}_n \cdots \mathbf{g}_1(x)) + \cdots + \phi(\mathbf{g}_1(x)) + \phi(x) = \sum_{k=0}^{n} \phi(X_n^x) \tag{3}$$

for any measurable function $\phi \in L^1(\mathbb{R}, \nu)$ and $x \in \mathbb{R}$. Observe that if ϕ is the indicator function of some set A, then $S_n\phi(x) = S_n\mathbf{1}_A(x)$ is the number of visits in A up to time n for the Markov chain $(X_n^x)_{n\in\mathbb{N}}$ starting at x.

Whenever recurrence condition $(\mathfrak{R})$ is satisfied, it follows that for any arbitrary function Φ whose support contains a recurrent interval $\mathcal{I}$ we have $S_n\Phi(x) \to +\infty$ for any $x \in \mathbb{R}$, as $n \to \infty$.

If ν is ergodic for any non-negative function $\Phi \in L^1(\mathbb{R}, \nu)$ we have $\nu(\Phi) > 0$ if and only if $S_n\Phi(x) \to +\infty$ for ν-almost all x and in this case the Chacon–Ornstein theorem [11] guarantees that for any $\phi \in L^1(\mathbb{R}, \nu)$ the limit

$$\lim_{n\to\infty} \frac{S_n\phi(x)}{S_n\Phi(x)} = \frac{\nu(\phi)}{\nu(\Phi)} \tag{4}$$

exists for $\mu^{\mathbb{N}}$-almost all sequences $(g_1, g_2, \ldots) \in \Gamma^{\mathbb{N}}$ and ν-almost all $x \in \mathbb{R}$. This is a consequence of the fact that the shift τ is a contraction on the space $L^1(\mathbb{R}^{\mathbb{N}}, \mathbb{P}_\nu)$ and that $\mathbb{P}_\nu$ is ergodic, if ν is ergodic (see §A and, in particular, Corollary A.5 for a more complete discussion of these results).

2.6. *Measures with atoms.* The following lemma is useful when we have to deal with some invariant measures ν that have *atoms*, that is, for which there exists $x \in \mathbb{R}$ such that $\nu(\{x\}) > 0$. It essentially says that one can have invariant measures with atoms only if the orbits of action of Γ^{-1} are somehow discrete.

LEMMA 2.2. *Assume that condition $(\mathfrak{R})$ is satisfied. Let ν be a μ-invariant Radon measure with atoms, and let K be a compact interval that contains the recurrence interval $\mathcal{I}$ and some atoms. Then there exists $x_0 \in K$ such that the orbit $(\Gamma^{-1})^* x_0 \cap K$ is finite.*

Proof. Let ν be a μ-invariant Radon measure with atoms. We shall abbreviate $\nu(\{x\})$ to $\nu(x)$. Analogously, we shall also write $\mathbf{1}_x$ for $\mathbf{1}_{\{x\}}$. Note first that because ν is a Radon measure, there are at most countably many atoms and the mass of all atoms in K is finite. Therefore $\sup_{x\in K} \nu(x)$ is finite, and there is $x_K \in K$ such that $\nu(x_K) = \sup_{x\in K} \nu(x)$ (note, however, that it could be not uniquely determined). We will prove that for any compact set K containing some atoms of ν,

$$\nu(y) = \nu(x_K) \quad \text{for any } y \in (\Gamma^{-1})^* x_K \cap K. \tag{5}$$

Since the total mass of K is finite, this will imply that $(\Gamma^{-1})^* x_K \cap K$ is finite.

Let $O = (\Gamma \cup \Gamma^{-1})^* x_K$ be the orbit of x_K under the action of the group generated by Γ endowed with the discrete topology. Note that O is a Γ- and Γ^{-1}-invariant countable set. We can define on O a countable Markov chain X_n (which is just the restriction to O of the

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Markov chain defined on $\mathbb{R}$) with the transition kernel

$$p(x, y) = \mathbb{P}(\mathbf{g}_1(x) = y) = \sum_{g\in\Gamma} \mathbf{1}_y(g(x))\mu(g) \quad \text{for any } x, y \in O.$$

Let $\overline{\nu}$ be the measure on O defined by $\overline{\nu}(x) := \nu(x)$. Observe that $\overline{\nu} = \nu|_O$ and $\overline{\nu}$ remains μ-invariant, that is,

$$\overline{\nu}(x) = \sum_{y\in O} p(y, x)\overline{\nu}(y) = \sum_{g\in\Gamma}\sum_{y\in O} \mathbf{1}_x(g(y))\mu(g)\overline{\nu}(y) = \sum_{g\in\Gamma} \mu(g)\overline{\nu}(g^{-1}(x)) \quad \text{for } x \in O. \tag{6}$$

In fact, we have

$$\begin{aligned}\sum_{g\in\Gamma}\sum_{y\in O} \mathbf{1}_x(g(y))\mu(g)\overline{\nu}(y) &= \sum_{g\in\Gamma}\int_{\mathbb{R}} \mathbf{1}_O(y)\mathbf{1}_x(g(y))\, d\nu(y)\mu(g)\\ &= \sum_{g\in\Gamma}\int_{\mathbb{R}} \mathbf{1}_x(g(y))\, d\nu(y)\mu(g) = \nu(x).\end{aligned}$$

Consider the induced Markov chain on $O_K = O \cap K$ defined by the kernel

$$p_K(x, y) := \mathbb{P}_x(X_T = y, T < \infty) = \sum_{n=1}^{\infty} \sum_{\underline{x}\in O_n(x,y)} p(x_1, x_2)\cdots p(x_{n-1}, x_n) \tag{7}$$

for $x, y \in O_K$, where $T := \inf\{n \geq 1 :\ X_n \in O_K\}$ is the first hitting time of O_K and

$$O_n(x, y) := \{\underline{x} \in O^{\mathbb{N}} :\ x_1 = x, x_n = y \text{ and } x_i \notin O_K \text{ for all } 1 < i < n\}.$$

Since K contains $\mathcal{I}$, the stopping time T is finite $\mu^{\otimes\mathbb{N}}$-a.s. for every $x \in O_K$, thus the kernel is stochastic, that is,

$$p_K(x, O_K) = 1 \quad \text{for } x \in O_K. \tag{8}$$

The restriction of $\overline{\nu}$ to O_K is an invariant measure for the Markov kernel p_K (see, for instance, [25, Proposition 10.4.6 and Theorem 10.4.7]), that is,

$$\overline{\nu}(y) = \sum_{x\in O_K} p_K(x, y)\overline{\nu}(x).$$

Consider now the reversed Markov chain $\widehat{X}_n$ on O defined by recursive action of the $\mathbf{g}_i^{-1}$ with the kernel

$$\widehat{p}(x, y) := \mathbb{P}(\mathbf{g}_1^{-1}(x) = y) = \mathbb{P}(x = \mathbf{g}_1(y)) = p(y, x) \quad \text{for any } x, y \in O, \tag{9}$$

and the induced kernel on K,

$$\widehat{p}_K(x, y) := \mathbb{P}_x(\widehat{X}_{\widehat{T}} = y, \widehat{T} < \infty),$$

where $\widehat{T} = \inf\{n \geq 1 : \widehat{X}_n \in O_K\}$. Observe also that, by (7) and (9), $\widehat{p}_K(x, y) = p_K(y, x)$. Thus for every $n \in \mathbb{N}$,

$$\overline{\nu}(x_K) = \sum_{y\in O_K} \overline{\nu}(y)p_K^n(y, x_K) = \sum_{y\in O_K} \overline{\nu}(y)\widehat{p}_K^n(x_K, y).$$

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Since the kernel $\widehat{p}_K(x, y)$ is sub-stochastic, $\sum_{y\in O_K} \widehat{p}_K(x, y) = \mathbb{P}_x(\widehat{T} < \infty) \le 1$ and $\overline{\nu}(y) \le \overline{\nu}(x_K)$ for $y \in O_K$, it follows that $\overline{\nu}(y) = \overline{\nu}(x_K)$ whenever there exists n such that $\widehat{p}^n_K(x_K, y) > 0$, that is, if $y \in (\Gamma^{-1})^* x_K$. This completes the proof. □

3. *Minimality of the supports. Proof of Theorem 1.2*

In this section we will prove Theorem 1.2. We need to consider the reverse random walk with step law given by the probability on $\mathrm{Homeo}^+(\mathbb{R})$ defined as

$$\widehat{\mu}(g) := \mu(g^{-1}),$$

and the associated Feller kernel

$$\widehat{P} f(x) := \sum_{g\in\Gamma} f(g^{-1}(x))\mu(g) = \sum_{g\in\Gamma^{-1}} f(g(x))\widehat{\mu}(g).$$

Theorem 1.2 will be a consequence of Propositions 3.1 and 3.2 which will be stated and proved below. We will see in the proof that these two propositions cover two complementary cases. The proof of the second proposition shares some arguments with the paper of Deroin *et al* [**13**] on symmetric random walks.

PROPOSITION 3.1. *Let ν_1 and ν_2 be two μ-invariant ergodic measures such that* $\mathrm{supp}\, \nu_1 \subseteq \mathrm{supp}\, \nu_2$. *Assume that there are a set M unbounded on both sides and an open interval J having at least two common points with* $\mathrm{supp}\, \nu_1$ *such that for any $u \in M$,*

$$N(\underline{g}, u) = \sup\{n : \ \ell_n^{-1}(u) \in J\} < \infty$$

for $\mu^{\otimes\mathbb{N}}$-almost all $\underline{g} = (g_1, g_2, \ldots)$. Then $\nu_1 = C\nu_2$ for some constant $C > 0$.

Proof. Note that to prove the result it is sufficient to ensure that for arbitrary $a, b \in M$ such that $\mathcal{I} \subset (a, b)$ and any $z \in (a, b)$,

$$\frac{\nu_1[a, z)}{\nu_1[a, b)} = \frac{\nu_2[a, z)}{\nu_2[a, b)}. \tag{10}$$

Indeed, taking the difference, we obtain that for all $a < z_1 < z_2 < b$,

$$\nu_1[z_1, z_2) = C(a, b)\nu_2[z_1, z_2)$$

with $C(a, b) = \nu_1[a, b)/\nu_2[a, b) \in (0, \infty)$. Thus ν_1 and ν_2 coincide up to a constant on $[a, b)$. Observe that $\nu_i[a, b) \ge \nu_i(\mathcal{I}) > 0$. To extend this equality to the whole line it is sufficient to appeal to unboundedness of M. Taking sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset M$ such that $a_n \to -\infty$ and $b_n \to +\infty$, we deduce the equality of both measures on $\mathbb{R}$.

We now turn to the proof of (10). Fix $a, b \in M$ such that the recurrent set $\mathcal{I}$ is contained in (a, b). The assumptions of the proposition assure that there exist two distinct $y_1, y_2 \in J \cap \mathrm{supp}\, \nu_1$; we can assume $y_1 < y_2$. Taking two sufficiently small neighbourhoods of these points, we can find two disjoint intervals J_1 and J_2 such that $\nu_1(J_i) > 0$ and $J_i \subset J$, $i = 1, 2$. Note that for any $x_i \in J_i$ $(i = 1, 2)$, $z > a$ and any $n \ge N(\underline{g}) := \max\{N(\underline{g}, a), N(\underline{g}, b)\} + 1$ we have

$$\mathbf{1}_{[a,z)}(\ell_n(x_1)) = \mathbf{1}_{[\ell_n^{-1}(a),\ell_n^{-1}(z))}(x_1) \ge \mathbf{1}_{[\ell_n^{-1}(a),\ell_n^{-1}(z))}(x_2) = \mathbf{1}_{[a,z)}(\ell_n(x_2)) \tag{11}$$

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since $x_1 < x_2$ and $\ell_n^{-1}(a) \notin J \supset [x_1, x_2]$. Similarly, one can check that

$$\mathbf{1}_{[a,b)}(\ell_n(x_1)) = \mathbf{1}_{[a,b)}(\ell_n(x_2)) \tag{12}$$

using that also $\ell_n^{-1}(b) \notin J$ for appropriately large n.

Observe that also $\nu_2(J_i) > 0$, because $y_i \in \operatorname{supp} \nu_1 \subseteq \operatorname{supp} \nu_2$. By the Chacon–Ornstein theorem (4) there exist $x_1 \in J_1$ and $x_2 \in J_2$ such that for $\mu^{\otimes\mathbb{N}}$-almost every (a.e.) $(g_i)_{n\in\mathbb{N}} \in \Gamma^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} = \frac{\nu_1[a,z)}{\nu_1[a,b)} \quad \text{and} \quad \lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_2)}{S_n\mathbf{1}_{[a,b)}(x_2)} = \frac{\nu_2[a,z)}{\nu_2[a,b)}. \tag{13}$$

Appealing to the definition of S_n given in (3), formulas (11) and (12) yield for any $n > N(\underline{g})$,

$$S_n\mathbf{1}_{[a,z)}(x_1) - S_{N(\underline{g})}\mathbf{1}_{[a,z)}(x_1) \geq S_n\mathbf{1}_{[a,z)}(x_2) - S_{N(\underline{g})}\mathbf{1}_{[a,z)}(x_2)$$

and

$$S_n\mathbf{1}_{[a,b)}(x_1) - S_{N(\underline{g})}\mathbf{1}_{[a,b)}(x_1) = S_n\mathbf{1}_{[a,b)}(x_2) - S_{N(\underline{g})}\mathbf{1}_{[a,b)}(x_2).$$

Recall that since the recurrent set $\mathcal{I}$ is a subset of (a, b), $S_n\mathbf{1}_{[a,b)}(x_i) \to \infty$ for $i = 1, 2$ $\mu^{\otimes\mathbb{N}}$-a.s. Hence on a set of probability 1 we have

$$\begin{aligned}
\frac{\nu_1[a,z)}{\nu_1[a,b)} &= \lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} \\
&= \lim_{n\to\infty} \frac{S_{N(\underline{g})}\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} + \frac{S_n\mathbf{1}_{[a,z)}(x_1) - S_{N(\underline{g})}\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1) - S_{N(\underline{g})}\mathbf{1}_{[a,b)}(x_1)} \\
&\quad \cdot \frac{S_n\mathbf{1}_{[a,b)}(x_1) - S_{N(\underline{g})}\mathbf{1}_{[a,b)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} \\
&\geq \lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_2) - S_{N(\underline{g})}\mathbf{1}_{[a,z)}(x_2)}{S_n\mathbf{1}_{[a,b)}(x_2) - S_{N(\underline{g})}\mathbf{1}_{[a,b)}(x_2)} = \lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_2)}{S_n\mathbf{1}_{[a,b)}(x_2)} = \frac{\nu_2[a,z)}{\nu_2[a,b)},
\end{aligned}$$

the penultimate equality by the fact that $S_n\mathbf{1}_{[a,b)}(x_2) \to \infty$ as $n \to \infty$. Interchanging in (13) the role of measures ν_1 and ν_2, that is, choosing $x_1 \in J_1$ and $x_2 \in J_2$ such that

$$\lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_2)}{S_n\mathbf{1}_{[a,b)}(x_2)} = \frac{\nu_1[a,z)}{\nu_1[a,b)} \quad \text{and} \quad \lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} = \frac{\nu_2[a,z)}{\nu_2[a,b)},$$

we arrive at the converse inequality

$$\frac{\nu_1[a,z)}{\nu_1[a,b)} \leq \frac{\nu_2[a,z)}{\nu_2[a,b)},$$

thus concluding (10). This completes the proof. □

PROPOSITION 3.2. *Let ν_1 and ν_2 be two ergodic invariant measures such that* $\operatorname{supp} \nu_1 \subseteq \operatorname{supp} \nu_2$ *and ν_1 has no atoms. Suppose that there exists a $\widehat{\mu}$-invariant Radon measure $\widehat{\nu}$ such that* $\operatorname{supp} \nu_1 \subseteq \operatorname{supp} \widehat{\nu}$*. Then $\nu_1 = C\nu_2$ for some positive constant C.*

The existence of the measure $\widehat{\nu}$ enables us to ensure that the number of visits to a given interval of processes $(X_n^x)_{n\in\mathbb{N}}$ and $(X_n^y)_{n\in\mathbb{N}}$ starting from two different points x and y does

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not differ too much if x and y are close enough. Our arguments are partially inspired by the techniques introduced in [13].

LEMMA 3.3. *Assume that* ($\mathfrak{R}$) *is satisfied. Let* ν *be an ergodic* μ*-invariant measure, and let* $\widehat{\nu}$ *be a* $\widehat{\mu}$*-invariant Radon measure. Let* a *and* b *be two points of the support of* $\widehat{\nu}$ *such that* $\nu[a, b) > 0$. *Fix two constants* $p, \varepsilon \in (0, 1)$, *and let* $\delta = \min\{\widehat{\nu}(I_{a,\varepsilon}), \widehat{\nu}(I_{b,\varepsilon})\} > 0$, *where* $I_{c,\varepsilon} := [c, c + \varepsilon)$ *for arbitrary* $c \in \mathbb{R}$. *Then for* ν*-a.e.* y *and any* $x < y$ *satisfying* $\widehat{\nu}[x, y) < (1 - p)\delta$,

$$\mu^{\otimes\mathbb{N}}\bigg(\bigg\{\underline{g} : \varlimsup_{n\to\infty}\bigg|\frac{S_n\mathbf{1}_{[a,b)}(x)}{S_n\mathbf{1}_{[a,b)}(y)} - 1\bigg| \le \frac{\nu(I_{a,\varepsilon}) + \nu(I_{b,\varepsilon})}{\nu[a, b)}\bigg\}\bigg) \ge p.$$

Proof. We start with an observation that if two points x and y are close with respect to the distance measured by $\widehat{\nu}$, that is, if $\widehat{\nu}[x, y) < (1 - p)\delta$, then with probability at least p the distance between two trajectories $(X_n^x)_{n\in\mathbb{N}}$ and $(X_n^y)_{n\in\mathbb{N}}$ remains small, that is,

$$\mathbb{P}(\lim_{n\to\infty} \widehat{\nu}[X_n^x, X_n^y) < \delta) = \mu^{\otimes\mathbb{N}}(\{\underline{g} : \lim_{n\to\infty} \widehat{\nu}[X_n^x, X_n^y) < \delta\}) \ge p. \tag{14}$$

This fact was proved in [13, Lemma 6.6], nevertheless for the reader's convenience we present here a complete argument. Note first that since the measure $\widehat{\nu}$ is $\widehat{\mu}$-invariant, the sequence $\widehat{\nu}[X_n^x, X_n^y)$ forms a positive martingale, thus by the martingale convergence theorem it converges to a non-negative random variable $v(x, y)$. Fatou's lemma entails that

$$\mathbb{E}v(x, y) \le \lim_{n\to\infty} \mathbb{E}[\widehat{\nu}[X_n^x, X_n^y)] = \widehat{\nu}[x, y)$$

and, finally, by the Markov inequality we obtain

$$\mathbb{P}(\{v(x, y) > \delta\}) \le \mathbb{P}\bigg(\bigg\{v(x, y) > \frac{\widehat{\nu}[x, y)}{1 - p}\bigg\}\bigg) \le \frac{(1 - p)\mathbb{E}v(x, y)}{\widehat{\nu}[x, y)} \le 1 - p,$$

thus completing the proof of (14).

To proceed further we need an additional auxiliary inequality. Namely, note that for any $x < y$ we have

$$\begin{aligned}
|\mathbf{1}_{[a,b)}(x) - \mathbf{1}_{[a,b)}(y)| &= \mathbf{1}_{[a,b)}(y)\mathbf{1}_{(-\infty,a)}(x) + \mathbf{1}_{[a,b)}(x)\mathbf{1}_{[b,+\infty)}(y)\\
&\le \mathbf{1}_{I_{a,\varepsilon}}(y) + \mathbf{1}_{\{[x,y)\supseteq I_{a,\varepsilon}\}}(y) + \mathbf{1}_{I_{b,\varepsilon}}(y) + \mathbf{1}_{\{[x,y)\supseteq I_{b,\varepsilon}\}}(y)\\
&\le \mathbf{1}_{I_{a,\varepsilon}}(y) + \mathbf{1}_{I_{b,\varepsilon}}(y) + \mathbf{1}_{\{\widehat{\nu}[x,y)\ge\widehat{\nu}(I_{a,\varepsilon})\}}(y) + \mathbf{1}_{\{\widehat{\nu}[x,y)\ge\widehat{\nu}(I_{b,\varepsilon})\}}(y)\\
&\le \mathbf{1}_{I_{a,\varepsilon}}(y) + \mathbf{1}_{I_{b,\varepsilon}}(y) + 2 \cdot \mathbf{1}_{\{\widehat{\nu}[x,y)\ge\delta\}}(y).
\end{aligned}$$

Replacing x, y by $\ell_k(x)$ and $\ell_k(y)$ respectively, and then summing over k, we obtain for any $x < y$ and $n \ge 0$.

$$\begin{aligned}
&|S_n\mathbf{1}_{[a,b)}(x) - S_n\mathbf{1}_{[a,b)}(y)|\\
&\qquad \le S_n\mathbf{1}_{I_{a,\varepsilon}}(y) + S_n\mathbf{1}_{I_{b,\varepsilon}}(y) + 2\mathrm{card}\{k \le n : \widehat{\nu}[X_k^x, X_k^y) \ge \delta\}.
\end{aligned} \tag{15}$$

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Since $\nu[a,b) > 0$, the Chacon–Ornstein theorem (4) entails that

$$\lim_{n\to\infty} \frac{S_n\mathbf{1}_{I_{a,\varepsilon}}(y) + S_n\mathbf{1}_{I_{b,\varepsilon}}(y)}{S_n\mathbf{1}_{[a,b)}(y)} = \frac{\nu(I_{a,\varepsilon}) + \nu(I_{b,\varepsilon})}{\nu[a,b)} \tag{16}$$

for ν-a.e. y. Furthermore, for ν-a.e. y, $S_n\mathbf{1}_{[a,b)}(y)$ converges to $+\infty$, since $\nu[a,b) > 0$. Now, fix a y for which the above limit exists and take arbitrary $x < y$ such that $\widehat{\nu}[x,y] < (1-p)\delta$. Then, in view of (14), on a set of probability at least p we have $\lim_{n\to\infty} \widehat{\nu}[X_n^x, X_n^y) < \delta$. Thus invoking (15) on the intersection of this set with the set of full measure for which (16) holds, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \left| \frac{S_n\mathbf{1}_{[a,b)}(y) - S_n\mathbf{1}_{[a,b)}(x)}{S_n\mathbf{1}_{[a,b)}(y)} \right| &\le \lim_{n\to\infty} \frac{S_n\mathbf{1}_{I_{a,\varepsilon}}(y) + S_n\mathbf{1}_{I_{b,\varepsilon}}(y)}{S_n\mathbf{1}_{[a,b)}(y)} \\ &= \frac{\nu(I_{a,\varepsilon}) + \nu(I_{b,\varepsilon})}{\nu[a,b)}\end{aligned}$$

and the proof of the lemma is complete. □

Proof of Proposition 3.2. We will now prove that for any $a, b \in \operatorname{supp} \nu_1 \subseteq \operatorname{supp} \widehat{\nu}$ such that $\nu_1[a,b) > 0$ and $\nu_2[a,b) > 0$, and for any $z \in (a,b]$,

$$\frac{\nu_1[a,z)}{\nu_1[a,b)} = \frac{\nu_2[a,z)}{\nu_2[a,b)}. \tag{17}$$

The desired result will be shown by the same argument as in the proof of Proposition 3.1, using the fact that $\operatorname{supp} \nu_1$ is unbounded.

Step 1. First we will prove that (17) holds for $z \in \operatorname{supp} \widehat{\nu}$ such that $\nu_1[a,z) > 0$. Fix $p \in (1/2, 1)$, choose $\varepsilon > 0$ such that the intervals $I_{a,\varepsilon}$, $I_{b,\varepsilon}$ and $I_{z,\varepsilon}$ are pairwise disjoint and put

$$\delta := \min\{\widehat{\nu}(I_{a,\varepsilon}), \widehat{\nu}(I_{b,\varepsilon}), \widehat{\nu}(I_{z,\varepsilon})\} > 0.$$

We claim that there exist two disjoint open intervals I_1, I_2 and an interval $I_0 \supset I_1 \cup I_2$ such that

$$\sup\{x \in I_2\} \le \inf\{x \in I_1\}, \quad \nu_1(I_1) > 0, \quad \nu_1(I_2) > 0 \quad \text{and} \quad \widehat{\nu}(I_0) < (1-p)\delta.$$

In fact, let $\mathcal{I}$ be an open interval such that $\operatorname{supp} \nu_1 \cap \mathcal{I} \neq \emptyset$. Since ν_1 has no atoms, $\operatorname{supp} \nu_1 \cap \mathcal{I}$ contains infinitely many points. Thus there exists a strictly monotone sequence $z_n \in \operatorname{supp} \nu_1 \cap \mathcal{I}$. Suppose that z_n is increasing (the decreasing case is similar). Consider the open neighbourhood of z_n defined by $J_n := ((z_n + z_{n-1})/2, (z_n + z_{n+1})/2)$. Intervals $J_n' = [(z_n + z_{n-1})/2, (z_n + z_{n+1})/2)$ for $n \in \mathbb{N}$ are disjoint and contained in the bounded interval $\mathcal{I}$, whence $\widehat{\nu}(J_n')$ converges to 0 and thus $\widehat{\nu}(J_n) < (1-p)\delta/2$ for any sufficiently large n. We take $I_2 = J_n$, $I_1 = J_{n+1}$ and $I_0 = J_n' \cup J_{n+1}'$.

Since $\nu_1(I_1) > 0$ and $\nu_1[a,b) > \nu_1[a,z) > 0$, by the Chacon–Ornstein theorem (4), and appealing twice to Lemma 3.3 (first for points a, b and then for a, z), we deduce that there exists $x_1 \in I_1$ such that $\mu^{\otimes\mathbb{N}}$-a.s.,

$$\lim_{n\to\infty} \frac{S_n\mathbf{1}_{[a,z)}(x_1)}{S_n\mathbf{1}_{[a,b)}(x_1)} = \frac{\nu_1[a,z)}{\nu_1[a,b)}, \tag{18}$$

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and for all $x_2 \in I_2$ with probability greater than $1 - 2(1-p) = 2p - 1 > 0$,

$$\limsup_{n\to\infty} \left| \frac{S_n \mathbf{1}_{[a,b)}(x_2)}{S_n \mathbf{1}_{[a,b)}(x_1)} - 1 \right| \leq \frac{\nu(I_{a,\varepsilon}) + \nu(I_{b,\varepsilon})}{\nu[a,b)},$$

$$\limsup_{n\to\infty} \left| \frac{S_n \mathbf{1}_{[a,z)}(x_2)}{S_n \mathbf{1}_{[a,z)}(x_1)} - 1 \right| \leq \frac{\nu(I_{a,\varepsilon}) + \nu(I_{b,\varepsilon})}{\nu[a,z)}.$$

Now since $\nu_2(I_2) > 0$ and $\nu_2[a,b) > 0$ we can chose $x_2 \in I_2$ such that $\mu^{\otimes\mathbb{N}}$-a.s.,

$$\lim_{n\to\infty} \frac{S_n \mathbf{1}_{(a,z)}(x_2)}{S_n \mathbf{1}_{(a,b)}(x_2)} = \frac{\nu_2[a,z)}{\nu_2[a,b)}. \tag{19}$$

Thus, with $\mu^{\otimes\mathbb{N}}$ probability at least $2p - 1 > 0$, we can write

$$\left| \frac{\nu_1[a,z)}{\nu_1[a,b)} \times \frac{\nu_2[a,b)}{\nu_2[a,z)} - 1 \right| = \lim_{n\to\infty} \left| \frac{S_n \mathbf{1}_{[a,b)}(x_2)}{S_n \mathbf{1}_{[a,b)}(x_1)} : \frac{S_n \mathbf{1}_{[a,z)}(x_2)}{S_n \mathbf{1}_{[a,z)}(x_1)} - 1 \right|$$

$$\leq \left(\frac{\nu_1(I_{a,\varepsilon}) + \nu_1(I_{b,\varepsilon})}{\nu_1[a,b)} + \frac{\nu_1(I_{a,\varepsilon}) + \nu_1(I_{z,\varepsilon})}{\nu_1[a,z)} \right) : \left(1 - \frac{\nu_1(I_{a,\varepsilon}) + \nu_1(I_{z,\varepsilon})}{\nu_1[a,z)} \right),$$

where for the last inequality we used the inequality $|(1+\epsilon_n)/(1+\eta_n) - 1| \leq |\epsilon_n| + |\eta_n|/(1 - |\eta_n|)$. Since the measure ν_1 is atomless, sending ε to 0 in the last estimates proves (17).

Step 2. We will now prove that (17) holds for any $a, b \in \operatorname{supp} \nu_1$ and any $z \in (a,b]$. Let

$$\overline{z} := \min\{x : x \in \operatorname{supp} \nu_1 \cap [z, +\infty)\} \in \operatorname{supp} \nu_1,$$

$$\underline{z} := \max\{x : x \in \operatorname{supp} \nu_1 \cap (-\infty, z]\} \in \operatorname{supp} \nu_1.$$

In particular, $\nu_1[\underline{z}, \overline{z}] = 0$ because ν_1 has no atoms. For all $c < \underline{z} \leq z$ we have

$$\nu_1[c, \overline{z}) = \nu_1[c,z) + \nu_1[z, \overline{z}) = \nu_1[c,z), \tag{20}$$

$$\nu_1[c, \underline{z}) = \nu_1[c, \underline{z}) + \nu_1(\underline{z}, z) = \nu_1[c,z). \tag{21}$$

Since $(\mathfrak{U})$ holds there exists $a_0 \in \operatorname{supp} \nu_1$ such that $\nu_1[a_0, a) > 0$ and $\nu_2[a_0, a) > 0$, by the fact that $\operatorname{supp} \nu_1 \subset \operatorname{supp} \nu_2$. Since $\operatorname{supp} \nu_1 \subseteq \operatorname{supp} \widehat{\nu}$, by step 1 for any $z \in (a,b] \cap \operatorname{supp} \nu_1$ we have

$$\nu_1[a_0, \underline{z}) = C\nu_2[a_0, \underline{z}) \quad \text{and} \quad \nu_1[a_0, \overline{z}) = C\nu_2[a_0, \overline{z}), \tag{22}$$

with $C := \nu_1[a_0, b)/\nu_2[a_0, b) \in (0, \infty)$. Observing that $a \leq \underline{z} \leq \overline{z} \leq b$ and applying (20), (21) and (22), one obtains

$$\nu_1[a,z) = \nu_1[a, \overline{z}) = \nu_1[a_0, \overline{z}) - \nu_1[a_0, a) = C\nu_2[a_0, \overline{z}) - C\nu_2[a_0, a)$$
$$= C\nu_2[a, \overline{z}) \geq C\nu_2[a, z)$$

and

$$\nu_1[a,z) = \nu_1[a, \underline{z}) = \nu_1[a_0, \underline{z}) - \nu_1[a_0, a) = C\nu_2[a_0, \underline{z}) - C\nu_2[a_0, a)$$
$$= C\nu_2[a, \underline{z}) \leq C\nu_2[a, z).$$

Thus $\nu_1[a,z) = C\nu_2[a,z)$ and (17) follows taking the quotient. The proof is complete. □

Proof of Theorem 1.2. To prove (1), let ν_1 and ν_2 be two ergodic invariant Radon measures such that supp $\nu_1 \subseteq$ supp ν_2 and supp ν_1 is not discrete in $\mathbb{R}$.

We will consider the following two complementary cases:

($\mathfrak{a}$) there exists an open interval J having at least two common points with supp ν_1 such that

$$C_J := \left\{x \in \mathbb{R} \,\middle|\, \sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(x) < \infty\right\}$$

is not empty;

($\mathfrak{b}$) for all open intervals $J \subset \mathbb{R}$ having at least two common points with supp ν_1 we have

$$\sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(x) = \infty \quad \text{for any } x \in \mathbb{R}.$$

The theorem is a consequence of Proposition 3.1 for case ($\mathfrak{a}$) and Proposition 3.2 for case ($\mathfrak{b}$).

Case ($\mathfrak{a}$). We claim that, in this case, the set C_J is unbounded on both sides and for any $u \in C_J$,

$$N(\underline{g}, x) := \sup\{n : g_1^{-1} \cdots g_n^{-1}(x) \in J\} < \infty \tag{23}$$

for $\mu^{\otimes\mathbb{N}}$-a.e. $\underline{g} = (g_1, g_2, \ldots)$. Then the fact that $\nu_1 = C\nu_2$ is a consequence of Proposition 3.1, with $M = C_J$.

To prove the claim observe that

$$\sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(x) = \mathbb{E}\bigg[\sum_{n=0}^{\infty} \mathbf{1}_J(\mathbf{g}_1^{-1} \cdots \mathbf{g}_n^{-1}(x))\bigg] = \mathbb{E}(\mathrm{card}\{n : \mathbf{g}_1^{-1} \cdots \mathbf{g}_n^{-1}(x) \in J\}).$$

In particular, for $x \in C_J$ the sequence $\mathbf{g}_1^{-1} \cdots \mathbf{g}_n^{-1}(x)$ visits J finitely many times with probability 1, that is, $N(\underline{g}, x) < \infty$ $\mu^{\otimes\mathbb{N}}$-a.s.

Observe also that the set C_J is Γ^{-1}-invariant. In fact, if $x \in C_J$ and $g_0 \in \Gamma$, then

$$\begin{aligned}\infty > \sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(x) \geq \sum_{k=1}^{\infty} \widehat{P}^k \mathbf{1}_J(x) &= \sum_{g \in \Gamma} \sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(g^{-1}x)\mu(g) \\ &\geq \sum_{k=0}^{\infty} \widehat{P}^k \mathbf{1}_J(g_0^{-1}x)\mu(g_0).\end{aligned}$$

Thus $g_0^{-1}x \in C_J$, since $\mu(g_0) > 0$. In particular, since ($\mathfrak{U}$) holds also for Γ^{-1}, Lemma 2.1 entails that C_J is unbounded.

Case ($\mathfrak{b}$). We will prove that under condition ($\mathfrak{b}$), ν_1 has no atoms and there exists a $\widehat{\mu}$-invariant Radon measure $\widehat{\nu}$ such that supp $\nu_1 \subseteq$ supp $\widehat{\nu}$. Then the fact that ν_1 is a multiple of ν_2 follows from Proposition 3.2.

We first prove that ν_1 cannot have atoms. Let K be a compact set that contains the recurrence interval and an accumulation point of supp ν_1. If ν_1 had an atom in K then, according to Lemma 2.2, there would exist an $x_0 \in K$ such that its Γ^{-1}-orbit M_0 $(= (\Gamma^{-1})^* x_0)$ has a finite number of points in K. But since K contains an accumulation

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point of supp ν_1, there exists an interval $J \subset M_0^c$ that contains at least two distinct points y_1 and y_2 of supp ν_1. By Γ^{-1}-invariance of M_0 we deduce that for any $x \in M_0$, $g^{-1}x \notin J$ for $g \in \Gamma^*$. Thus $M_0 \subset C_J \neq \emptyset$, which leads to a contradiction.

Since there exists a compact interval J such that $C_J = \emptyset$, the Feller kernel

$$\widehat{P} f(x) := \sum_{g\in\Gamma} f(g^{-1}(x))\mu(g) = \sum_{g\in\Gamma^{-1}} f(g(x))\widehat{\mu}(g)$$

is topologically conservative and therefore it has at least one invariant Radon measure $\widehat{\nu}$ (see Lin's theorem [23, Theorem 5.1]). The set $M_0 := \operatorname{supp}\widehat{\nu}$ is then closed and Γ^{-1}-invariant. Suppose now that there exists $y \in \operatorname{supp}\nu_1$ but $y \notin \operatorname{supp}\widehat{\nu}$. Since $\operatorname{supp}\widehat{\nu}$ is closed and ν_1 has no atoms, there exists $\tilde{J} \subseteq \mathbb{R} \setminus \operatorname{supp}\widehat{\nu}$ that contains at least two distinct points y_1 and y_2 of supp ν_1. By Γ^{-1}-invariance of $\operatorname{supp}\widehat{\nu}$ we conclude as above that for any $x \in \operatorname{supp}\widehat{\nu}$, $g^{-1}x \notin \tilde{J}$ for any $g \in \Gamma^*$, that is, $\operatorname{supp}\widehat{\nu} \subset C_{\tilde{J}} \neq \emptyset$. This leads to a contradiction.

To prove (2), take ν to be an ergodic invariant measure and suppose it does not contain a Γ-invariant discrete set. Let $M \subseteq \operatorname{supp}\nu$ be a non-empty closed Γ-invariant set. Then, by recurrence, there exists an ergodic invariant measure ν_1 such that $\operatorname{supp}\nu_1 \subseteq M$. If supp ν does not contain a discrete set, then we can apply the first part of the theorem, obtaining that $\operatorname{supp}\nu_1 = \operatorname{supp}\nu$. Hence $M = \operatorname{supp}\nu$. This proves that supp ν is minimal.

Conversely, to prove (3), take a minimal closed Γ-invariant set M. Then, by recurrence ($\mathfrak{R}$), there exists an ergodic invariant measure ν_1 such that $\operatorname{supp}\nu_1 \subseteq M$. By minimality of M we have $\operatorname{supp}\nu_1 = M$. Take now another ergodic measure ν_2 such that $\operatorname{supp}\nu_2 = M = \operatorname{supp}\nu_1$. If M is not discrete we can apply the first part of the theorem to conclude that ν_1 and ν_2 coincide up to a multiplicative constant. If M is discrete observe that any $x \in M$ is an atom for both ν_1 and ν_2, thus, invoking the Chacon–Ornstein theorem, we obtain that for any bounded function ϕ with compact support and for all $x \in M$,

$$\frac{\nu_1(\phi)}{\nu_1(\Phi)} = \lim_{n\to\infty} \frac{S_n\phi(x)}{S_n\Phi(x)} \quad\text{and}\quad \frac{\nu_2(\phi)}{\nu_2(\Phi)} = \lim_{n\to\infty} \frac{S_n\phi(x)}{S_n\Phi(x)}.$$

From this we finally obtain that $\nu_1 = C\nu_2$ with $C = \nu_1(\Phi)/\nu_2(\Phi)$. This completes the proof of (3). □

4. *Uniqueness of an invariant measure: proof of Theorem 1.1*

We start with the following lemma.

LEMMA 4.1. *Assume that hypotheses ($\mathfrak{C}$) and ($\mathfrak{U}$) hold. Then any two non-empty and closed Γ-invariant sets M_1 and M_2 have non-empty intersection, that is, $M_1 \cap M_2 \neq \emptyset$.*

Proof. Assume to the contrary that $M_1 \cap M_2 = \emptyset$ for some Γ-invariant sets M_1 and M_2. Consider the class of all open intervals such that

$$\mathcal{J} = \{J = (a, b) : a \in M_1, b \in M_2 \text{ and } (a, b) \subset (M_1 \cup M_2)^c\}.$$

Observe that all the intervals belonging to $\mathcal{J}$ are disjoint. Furthermore, note that if the sets M_1 and M_2 are disjoint, then for all pairs $m_1 \in M_1$, $m_2 \in M_2$ such that $m_1 < m_2$ there

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exists $J = (a, b) \in \mathcal{J}$ such that $J \subset (m_1, m_2)$. Indeed, one can just take $a = \sup\{m \in M_1 : m \leq m_2\}$ and $b = \inf\{m \in M_2 : m \geq a\}$.

Let $\mathcal{I}$ be the compact interval that appears in ($\mathfrak{C}$). We claim that there are only finitely many intervals $J \in \mathcal{J}$ which are subsets of $\mathcal{I}$. Indeed, suppose that there are infinitely many elements $J_i = (a_i, b_i)$ of $\mathcal{J}$ such that $J_i \subset \mathcal{I}$ for $i \in \mathbb{N}$. Since both sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ are contained in the compact interval $\mathcal{I}$, there exists a subsequence $\{i_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that sequences $\{a_{i_k}\}_{k\in\mathbb{N}}$ and $\{b_{i_k}\}_{k\in\mathbb{N}}$ are convergent. We denote by a_0 and b_0 their corresponding limits. Recalling that both sets M_i are closed, we deduce that $a_0 \in M_1$ and $b_0 \in M_2$. On the other hand, since all the intervals J_{i_k} are disjoint and contained in the compact set $\mathcal{I}$, their diameters $|b_{i_k} - a_{i_k}|$ converge to zero. Thus $a_0 = b_0 \in M_1 \cap M_2$ and we obtain a contradiction.

Denote by $J_1, \ldots J_N$ all the disjoint intervals, elements of $\mathcal{J}$, contained in $\mathcal{I}$. In view of Lemma 2.1, since the sets M_1 and M_2 are Γ-invariant, they are unbounded. Thus there exists an additional interval $J_{N+1} \in \mathcal{J}$ disjoint from $\mathcal{I}$ and all the remaining chosen intervals J_i for $i \leq N$. Condition ($\mathfrak{C}$) entails the existence of $g \in \Gamma^*$ such that $g(J_1 \cup \cdots \cup J_N \cup J_{N+1}) \subset \mathcal{I}$. Since g is a homeomorphism preserving the order, for every $i \leq N + 1$ it maps intervals $J_i = (a_i, b_i)$ onto open intervals $g(J_i) = (g(a_i), g(b_i))$. Observe also that $g(a_i) \in M_1 \cap \mathcal{I}$ and $g(b_i) \in M_2 \cap \mathcal{I}$, thus for every $i \in \{1, \ldots, N + 1\}$ there exists $j_i \in \{1, \ldots, N\}$ such that $g(J_i) \supseteq J_{j_i}$ and then the pigeonhole principle entails that $j_{i_1} = j_{i_2}$ for some $i_1 \neq i_2$. This means that both $g(J_{i_1})$ and $g(J_{i_2})$ contain J_{j_1} and therefore cannot be disjoint. Moreover, $J_{i_1} \cap J_{i_2} \supset g^{-1}(J_{j_1}) \neq \emptyset$, thus contradicting the choice of the intervals J_{i_1} and J_{i_2} as disjoint sets. So, we finally arrive at the conclusion that two closed and Γ-invariant sets M_1 and M_2 must have a non-empty intersection. This completes the proof. □

Proof of Theorem 1.1. Suppose that there exist two different invariant Radon measures. Without loss of generality, using ergodic decomposition, we may assume that there exist two different ergodic Radon measures $\tilde{\nu}_1$ and $\tilde{\nu}_2$. We claim then that there are two different invariant ergodic Radon measures ν_1 and ν_2 such that $\operatorname{supp} \nu_1 \subseteq \operatorname{supp} \nu_2$.

If $\operatorname{supp} \tilde{\nu}_1 = \operatorname{supp} \tilde{\nu}_2$ the result holds by taking $\nu_1 = \tilde{\nu}_1$ and $\nu_2 = \tilde{\nu}_2$.

Consider now the second case when $\operatorname{supp} \tilde{\nu}_1 \neq \operatorname{supp} \tilde{\nu}_2$. Both sets $\operatorname{supp} \tilde{\nu}_i$ are Γ-invariant, therefore in view of Lemma 4.1 they must have non-empty intersection, that is, $K = \operatorname{supp} \tilde{\nu}_1 \cap \operatorname{supp} \tilde{\nu}_2 \neq \emptyset$. Since K is Γ-invariant, by ($\mathfrak{R}$) there exists an invariant ergodic Radon measure, say ν_1, whose support is contained in K. Keeping in mind that both sets $\operatorname{supp} \tilde{\nu}_1$ and $\operatorname{supp} \tilde{\nu}_2$ are different, at least one of them, say $\operatorname{supp} \tilde{\nu}_2$, must be greater than K. Then the couple ν_1 and $\nu_2 := \tilde{\nu}_2$ satisfies the claim.

Observe that conditions ($\mathfrak{C}$) and ($\mathfrak{U}$) imply that $M_1 := \operatorname{supp} \nu_1$ is not discrete. Indeed, if $\mathcal{I}$ is the interval appearing in ($\mathfrak{C}$), then for all compact intervals J,

$$\operatorname{Card}(M_1 \cap J) = \operatorname{Card}(g(M_1 \cap J)) \leq \operatorname{Card}(M_1 \cap \mathcal{I}),$$

where $g \in \Gamma$ is such that $g(J) \subseteq \mathcal{I}$, by the fact that M_1 is Γ-invariant. Further, since M_1 is unbounded one can choose a sequence of compact intervals J_n such that $\operatorname{Card}(M_1 \cap J_n) \to \infty$. Hence $\operatorname{Card}(M_1 \cap \mathcal{I}) = \infty$.

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Point (1) of Theorem 1.2 yields $\nu_1 = C\nu_2$, which leads to a contradiction. The proof is complete. □

5. *Examples and applications*

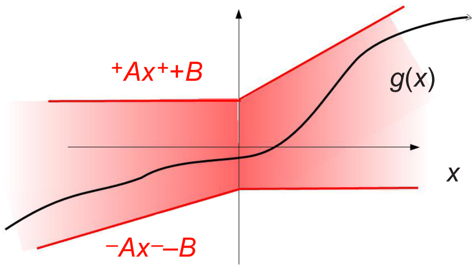
We will provide in this section some criteria to ensure recurrence and contraction of the system. In particular, we will focus on the study of systems induced by homeomorphisms whose behaviour we can control at the end points, such as asymptotically linear homeomorphisms and C^2-diffeomorphisms of the interval.

5.1. *Asymptotically linear systems.*

In this section we will focus on the study of systems induced by homeomorphisms that have a linear bound in the sense that for all $g \in \operatorname{supp} \mu$ there exist three positive numbers ${}^{+}A(g)$, ${}^{-}A(g)$ and $B(g)$ such that

$$-\, {}^{-}A(g)x^{-} - B(g) \le g(x) \le {}^{+}A(g)x^{+} + B(g) \quad \text{for all } x \in \mathbb{R}, \tag{24}$$

where $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$.



It can be easily shown that $g \in \text{Homeo}^+(\mathbb{R})$ satisfies (24) if the limits

$${}^{+}A(g) := \limsup_{x\to+\infty} \frac{g(x)}{x}, \quad {}^{-}A(g) := \limsup_{x\to-\infty} \frac{g(x)}{x}$$

are finite, and

$$\limsup_{x\to+\infty} [g(x) - {}^{+}A(g)x] < \infty \quad \text{and} \quad \liminf_{x\to-\infty} [g(x) - {}^{-}A(g)x] > -\infty.$$

These are sufficient conditions for (24) but examples of systems which do not satisfy these conditions, but for which (24) still holds, might be easily provided. Processes of this kind appear in many contexts of probability and related fields and have been investigated in several paper in the last years; see, for example, [**2**, **3**, **7**, **8**, **14**, **20**]. A fundamental example that has been widely studied is the affine recursion where $g(x) = A(g)x + B(g)$ (see [**9**] for a general overview). We refer to [**8**, §6] for a more detailed presentation of possible applications. In particular, condition (24) holds, after conjugation, for any increasing C^2-diffeomorphism h of the interval [0, 1], as we will see in the next section.

One expects that if ${}^{+}A$ and ${}^{-}A$ are sufficiently often smaller than 1, then the system will often be repelled away from infinity and thus will be recurrent or contracting. For instance one can prove the following sufficient criteria for hypothesis $(\mathfrak{C})$ and $(\mathfrak{R})$.

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LEMMA 5.1.

(1) *Suppose that there exists* $g \in \operatorname{supp} \mu$ *such that* (24) *holds with* ${}^+A(g)$ *and* ${}^-A(g)$ *smaller than 1. Then* ($\mathfrak{C}$) *holds.*
(2) *Suppose that* (24) *holds for every* $g \in \operatorname{supp} \mu$. *Then* ($\mathfrak{R}$) *holds in any of the following cases:*
   (a) $\log {}^{\pm}A(g)$ *and* $\log^+ B(g)$ *are* μ*-integrable and* $\int \log {}^{\pm}A(g)\, d\mu(g) < 0$;
   (b) *the support of* μ *is finite and* $\int \log {}^{\pm}A(g)\, d\mu(g) \le 0$;
   (c) ${}^-A = {}^+A = A$, $\log {}^{\pm}A(g)$ *and* $\log^+ B(g)$ *are* $(2+\varepsilon)$*-integrable for some* $\varepsilon > 0$, *and* $\int \log A(g)\, d\mu(g) = 0$.

Proof. We will use the linear bound assumed in (24) to compare the Markov chain $X_n^x = \ell_n(x) = \mathbf{g}_n \cdots \mathbf{g}_1(x)$ with the affine recursions

$$
\begin{aligned}
{}^+Y_n^x &:= {}^+A_n\, {}^+Y_{n-1} + B_n, \quad {}^+Y_0 = x^+,\\
{}^-Y_n^x &:= {}^-A_n\, {}^-Y_{n-1} + B_n, \quad {}^-Y_0 = x^-,
\end{aligned}
$$

where ${}^+A_n = {}^+A(\mathbf{g}_n)$, ${}^-A_n = {}^-A(\mathbf{g}_n)$ and $B_n = B(\mathbf{g}_n)$. It can then be verified by the inductive argument that

$$
-\,{}^-Y_n^x \le \mathbf{g}_n \cdots \mathbf{g}_1(x) \le {}^+Y_n^x. \tag{25}
$$

Proof of (1). Let $g \in \operatorname{supp} \mu$ be such that

$$
A := \max\{{}^+A(g), {}^-A(g)\} < 1,
$$

and set $B := B(g)$. It can be verified by induction (or applying (25) when $\mathbf{g}_i = g$ for all i) that

$$
-A^n x^- - \sum_{k=0}^{n-1} A^k B \le g^n(x) \le A^n x^+ + \sum_{k=0}^{n-1} A^k B. \tag{26}
$$

In particular, if $\beta := \sum_{k=0}^{\infty} A^k B$ then $|g^n(x)| \le A^n |x| + \beta$. Fix $I := [-2\beta, 2\beta]$ and take any interval $J = [a, b]$. Then for any sufficiently large n we obtain

$$
g^n(J) \subseteq [-A^n|a| - \beta, A^n|b| + \beta] \subseteq I.
$$

Proof of (2): (a) and (c). It is known that under hypothesis (a) or (c) the two-dimensional Markov chain $\{({}^+Y_n, {}^-Y_n)\}_{n\in\mathbb{N}}$ is recurrent in $\mathbb{R}^2$, that is, there exists a constant $K > 0$ such that for any starting point, with probability 1, $\max\{|{}^+Y_n|, |{}^-Y_n|\} < K$ for infinitely many n (see [**5**] and [**9**, §4.4.10]). From (25) it follows that X_n^x visits the interval $I = [-K, K]$ infinitely often.

Proof of (2): (b). Under hypothesis (b) one needs to be more careful. In fact, in this case each of the one-dimensional affine recursions ${}^+Y_n$ and ${}^-Y_n$ is recurrent, but the joint process $({}^+Y_n, {}^-Y_n)$ may not be.

Let $K > 0$ be such that for all $x \in \mathbb{R}$ we have

$$
\mathbb{P}(|{}^+Y_n^x| < K \text{ i.o.}) = 1 \quad \text{and} \quad \mathbb{P}(|{}^-Y_n^x| < K \text{ i.o.}) = 1. \tag{27}
$$

If at some moment n one has $X_n^x \le {}^+Y_n^x < K$, and at some later moment $n' > n$ one has $-K < {}^-Y_{n'}^x \le X_{n'}^x$, then

$$X_m^x \in [-K, K \vee \max_{g\in\Gamma} g(-K)],$$

where $m = \min\{n'' \in [n, n'] : X_{n''}^x > K\}$. Since this event holds $\mathbb{P}$-a.s., by (27), the proof is complete. □

5.2. *C^2-diffeomorphisms of the interval.* Our main theorems and the above-mentioned results concerning asymptotically linear systems can be applied to stochastic dynamical systems on the interval generated by an increasing C^2-diffeomorphism of [0, 1]. Similar iterated function systems have been extensively studied recently (see [**1**, **10**, **19**, **24**]). A sufficient criterion for the uniqueness of an invariant measure in this situation has been stated in Corollary 1.4 and is a direct consequence of Theorem 1.1, Lemma 5.1 and the following result.

LEMMA 5.2. *Take the diffeomorphism of* (0, 1) *onto* $\mathbb{R}$ *defined by* $r(u) := -(1/u) + 1/(1-u)$. *Then for any increasing* C^2*-diffeomorphism* h *of* [0, 1]*, the conjugated homeomorphism*

$$h_r := r \circ h \circ r^{-1} \in \mathrm{Homeo}^+(\mathbb{R})$$

satisfies (24) with

$${}^+A(h_r) = \frac{1}{h'(1)} \quad \text{and} \quad {}^-A(h_r) = \frac{1}{h'(0)}.$$

Furthermore, if μ is a finitely supported measure on the family of increasing diffeomorphisms in $C^2([0, 1])$ *and μ_r is the conjugated measure on* $\mathrm{Homeo}^+(\mathbb{R})$*, a Radon measure ν on* (0, 1) *is μ-invariant if and only if the Radon measure on $\mathbb{R}$ of the form*

$$\nu_r(f) = r_*\nu(f) = \int_{[0,1]} f(r(x))\, d\nu(x)$$

is μ_r-invariant.

Proof. We have

$$\begin{aligned}
{}^+A(h_r) &= \limsup_{x\to+\infty} \frac{h_r(x)}{x} = \limsup_{x\to+\infty} \frac{r(h(r^{-1}(x))}{r(r^{-1}(x))} \\
&= \limsup_{u\to 1^-} \frac{r(h(u))}{r(u)} \quad \text{change of variable } u := r^{-1}(x) \\
&= \limsup_{u\to 1^-} \frac{1/(1-h(u))}{1/(1-u)} \quad \text{since } r(u) \sim \frac{1}{1-u} \text{ for } u \sim 1^- \\
&= \limsup_{u\to 1^-} \frac{1-u}{h(1)-h(u)} = \frac{1}{h'(1)} \quad \text{since } h(1) = 1.
\end{aligned}$$

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Furthermore, since h is $C^2(0,1)$, we have $h(u) = 1 + h'(1)(1-u) + \mathcal{O}((1-u)^2)$. Thus finally

$$\limsup_{x\to+\infty}\left[h_r(x) - \frac{x}{h'(1)}\right] = \limsup_{u\to 1^-}\left[\frac{1}{1-h(u)} - \frac{1}{h'(1)(1-u)}\right]$$
$$= \limsup_{u\to 1^-}\frac{\mathcal{O}((1-u)^2)}{h'(1)^2(1-u)^2} < \infty.$$

Similar calculations can be done near $-\infty$ and 0.

The second part of the lemma is obvious. □

5.3. *Counterexamples to Theorems 1.1 and 1.2.* In this section we intend to provide some examples of stochastic dynamical systems that have more than one invariant measure to explain that neither condition ($\mathfrak{C}$) nor ($\mathfrak{U}$) is sufficient alone to guarantee uniqueness.

5.3.1. *Contraction but not unboundedness.* Consider the stochastic dynamical systems generated by a set Γ of homeomorphisms that fix two distinct points a and b of $\mathbb{R}$ and are all repulsive at the end point. For instance, take μ that gives mass 1/2 to $h(x) = x^{1/3}$ and to $k(x) = x^{1/5}$. Then μ is contracting because $h^n[-K, K] = [-K^{1/3^n}, K^{1/3^n}]$ is in $[-2, 2]$ for any large n. The interval $[-2, 2]$ is also recurrent for similar reasons. This system does not have a unique invariant measure since δ_0 and δ_1, the Dirac measures in 0 and 1, are both μ-invariant.

5.3.2. *Unboundedness but not contraction.* An example of a recurrent stochastic dynamical system that satisfies ($\mathfrak{U}$) but not ($\mathfrak{C}$) is just given by the simple random walk on $\mathbb{Z} \subset \mathbb{R}$. In fact, take μ that gives mass 1/3 to $h_0(x) = x$, $h_+(x) = x + 1$ and $h_-(x) = x - 1 \in \text{Homeo}^+(\mathbb{R})$. It defines a recurrent Markov chain and is obviously unbounded. It possesses infinitely many Radon ergodic invariant measures given by the counting measures on $\mathbb{Z} + a \subseteq \mathbb{R}$ for any $a \in [0, 1)$. The Lebesgue measure on $\mathbb{R}$ is also invariant, but it is not ergodic.

5.3.3. *Ergodic measures with non-minimal support.* We give here an example to prove that an ergodic measure may have support that is not minimal. The idea is to start with a stochastic dynamical system generated by a measure $\overline{\mu}$ on the set of increasing C^2-diffeomorphisms of $[0, 1]$ that has a unique Radon measure $\overline{\nu}$ whose support is the whole interval $(0, 1)$. It follows then that $\overline{\nu}$ is ergodic. Let $\overline{\Gamma} = \text{supp}\,\overline{\mu}$. For any $\overline{g} \in \overline{\Gamma}$ define three homeomorphisms of $\mathbb{R}$:

$$g_0(x) := \overline{g}(\{x\}) + \lfloor x \rfloor, \quad g_+(x) := \overline{g}(\{x\}) + \lfloor x \rfloor + 1, \quad g_-(x) := \overline{g}(\{x\}) + \lfloor x \rfloor - 1,$$

where $\{x\}$ is the fractional part of x and $\lfloor x \rfloor$ the floor function. Heuristically the function g_0 fixes each integer interval $[n, n+1]$ and acts on each one of them as $\overline{g}$, while $g_\pm$ do the same but are then composed with a translation by ± 1. Let μ be the measure charging

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g_0, $g_\pm$ with mass equal to $\overline{\mu}(\overline{g})/3$. Then it can be proved that the measure

$$\nu(f) := \sum_{k=-\infty}^{+\infty} \int_0^1 f(y+k)\, d\overline{\nu}(y)$$

is a μ-invariant Radon ergodic measure whose support is the whole of $\mathbb{R}$. On the other hand, $\mathbb{Z} \subset \mathbb{R}$ is a discrete closed invariant set for μ (and the counting measure on $\mathbb{Z}$ is another ergodic measure).

5.3.4. *Non-recurrent system.* A classical example that shows that for non-recurrent systems a closed minimal Γ-invariant set can be a support of several invariant measures is a non-centred random walk on $\mathbb{Z}$. Suppose that $g(x) = x + B(g)$ with $B(g) \in \mathbb{Z}$. Further, suppose also that $\mathbb{E}(B(\mathbf{g}_1)) \neq 0$ and that there exists $\alpha \neq 0$ such that $\mathbb{E}(e^{-\alpha B(\mathbf{g}_1)}) = 1$. Then both the counting measure on $\mathbb{Z}$ and the measure on $\mathbb{Z}$ such that $\nu(x) = e^{\alpha x}$ for any $x \in \mathbb{Z}$ are invariant.

5.3.5. *Non-Radon invariant measures.* The restriction to Radon measures in Theorem 1.1 is indispensable. In the family of Borel measures the uniqueness of the invariant measure can easily be broken; see, for example, [5, Remark 2].

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A. *Appendix: Some results on ergodic invariant measures for Markov–Feller processes*

This part of our paper is devoted to the description of ergodic measures and to the proof of an ergodic decomposition for Markov–Feller processes on locally compact metric spaces (2). Some of the results of this section seem to be classical and have often been used in a different context in several works in this fields. They are based on the classical theory of positive contractions of L^1-spaces that is a powerful and general tool. However, we could not find a comprehensive reference specifically adapted to Markov–Feller processes with an invariant Radon measure. So we give a quick survey of the results that we need in our paper and explain how they can be deduced from the general theory. In particular, we give an explicit proof of the ergodic decomposition of a general invariant Radon measure as an integral over the class of ergodic Radon measures.

For a complete overview on the ergodic theory and related infinite measures, L^1-contractions and Markov processes we refer to the books by Foguel [16], Garsia [18] and Revuz [29]. For a glimpse at the theory we also suggest the nice informal survey of Zweimüller [30].

A.1. *Markov–Feller processes and L^1-contractions.* Let (X, ρ) be a locally compact metric space, and let $(X, \mathcal{B}(X), \nu)$ be a σ-finite measure space with Radon measure ν. Let P be an operator on $L^1(X, \nu)$ and $L^\infty(X, \nu)$ such that:

- P is *positive*, that is, $Pf \geq 0$ for $f \geq 0$;

- $P\mathbf{1}_X(x) = \mathbf{1}_X(x)$ for ν-a.e. $x \in X$;
- P is a *contraction* of $L^1(X, \nu)$, that is, the operator norm $\|P\|_1$ on this space is less than 1. This last condition is equivalent to the property of the measure ν called an *excessive measure*, that is,

$$\int_X Pf(x)\, d\nu(x) \le \int_X f(x)\, d\nu(x) \quad \text{for } f \in L^1(X, \nu),\ f \ge 0.$$

We shall call the quadruple $(X, \mathcal{B}(X), \nu, P)$ a Markov process. This process is said to be *Feller* if additionally we assume that $Pf \in C(X)$ for $f \in C_c(X)$. Here $C(X)$ denotes the space of continuous functions and $C_c(X)$ the subspace of continuous functions with compact support. A Radon measure ν will be called *invariant* for a given Markov–Feller process if $\nu(Pf) = \nu(f)$ for $f \in L^1(X, \nu)$.

Using the duality between $L^1(X, \nu)$ and $L^\infty(X, \nu)$, we can define a dual operator P^* on both $L^1(X, \nu)$ and $L^\infty(X, \nu)$. More precisely, for any $f \in L^1(X, \nu)$ (respectively, $f \in L^\infty(X, \nu)$) P^*f is the unique function in $L^1(X, \nu)$ (respectively, in $L^\infty(X, \nu)$) such that

$$\int_X P^*f(x)g(x)\, d\nu(x) = \int_X f(x)Pg(x)\, d\nu(x) \quad \text{for all } g \in L^\infty(X, \nu) \text{ (respectively, } g \in L^1(X, \nu)).$$

It can also easily be checked that P^* is a positive contraction of $L^1(X, \nu)$.

We can associate to a Markov operator P with an invariant measure ν the *space of the trajectories* of the associated Markov chain $(X_n)_{n\in\mathbb{N}}$, that is, the product space $X^{\mathbb{N}}$ equipped with the measure $\mathbb{P}_\nu$ such that for any finite collection of compact sets $I_i \subset X$, $i = 0, \ldots, n$, the measure of the cylinder $[I] = I_0 \times \cdots \times I_n \times \mathbb{R} \times \cdots$ is given by the formula

$$\mathbb{P}_\nu([I]) = \mathbb{P}_\nu(X_0 \in I_0, \ldots, X_n \in I_n) := \int_{\mathbb{R}^{n+1}} \mathbf{1}_{I_n}(x_n) \cdots \mathbf{1}_{I_0}(x_0) P(x_{n-1}, dx_n) \cdots P(x_1, dx_2) P(x_0, dx_1) \nu(dx_0).$$

Recall that $P(\cdot,\cdot) : X \times \mathcal{B}(X) \to [0, 1]$ denotes the transition probability for the Markov chain $(X_n)_{n\in\mathbb{N}}$ given by the formula

$$P(x, A) = P^*\mathbf{1}_A(x) \quad \text{for } x \in X \text{ and } A \in \mathcal{B}(X).$$

The shift τ on $X^{\mathbb{N}}$, that is, the map $\underline{x} = (x_0, x_1, \ldots) \mapsto \tau\underline{x} = (x_1, x_2, \ldots)$, induces the operator on $L^1(X^{\mathbb{N}}, \mathbb{P}_\nu)$ (and on $L^\infty(X^{\mathbb{N}}, \mathbb{P}_\nu)$) that will also be denoted by τ and defined by the formula $\tau f(\underline{x}) = f(\tau\underline{x})$. If ν is P-invariant, then $\mathbb{P}_\nu$ is τ-invariant. Thus τ is a positive contraction of $L^1(X^{\mathbb{N}}, \mathbb{P}_\nu)$.

More generally, let (W, ω) be a σ-finite measure space. A linear operator T of $L^1(W, \omega)$ is a *positive contraction* if it is positive ($Tf \ge 0$ whenever $f \ge 0$) and $\|T\|_1 \le 1$. We may define the adjoint operator $T^* : L^\infty(\omega) \to L^\infty(\omega)$ by the formula

$$\int_W T^*gf\, d\nu = \int_X gTf\, d\nu \quad \text{for } g \in L^\infty(\nu) \text{ and } f \in L^1(\nu).$$

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As we have seen above, we can associate to a Markov kernel P with a P-invariant measure ν at least three different contractions: the contraction P on $L^1(X, \nu)$, the contraction P^* also on $L^1(X, \nu)$ and the shift τ on $L^1(X^{\mathbb{N}}, \mathbb{P}_\nu)$. All of them possess interesting properties, but this abundance also generates some confusion. We will be mainly interested in the contractions P and τ. Let us just notice that the contraction P^* is particularly adapted to and widely used in the study of Harris recurrent Markov chains (see, for instance, Revuz [29] and Foguel [16]), but this is not so in our case.

These three contractions are deeply related and the dynamical systems they engender share often the same ergodic properties, as will be shown below. For some results in this direction see also the recent paper of Pène and Thomine [28, §2].

A.2. *Ergodic measures.* A fundamental property of L^1-contractions is ergodicity, saying that the space cannot be decomposed into smaller invariant pieces. More precisely, a Borel set A is called T*-invariant* (or *invariant*) if $T^*\mathbf{1}_A = \mathbf{1}_A$ (or equivalently if ν_A, the restriction of ν to A, is a T-invariant measure). An invariant measure ν is called *ergodic* if either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$ for any T-invariant set $A \subset X$.

For the contractions induced by a Markov operator P, we have in principle at least three definitions of ergodicity (for P, P^* and τ), but all of them coincide. Observe that the σ-algebras of invariant sets defined by the contractions P and P^* on $L^1(X, \nu)$ coincide (see [29, Ch. 4, Proposition 3.4]). Thus ν is P-ergodic if and only if it is P^*-ergodic. Furthermore, we have the following lemma.

LEMMA A.1. *Let P be a Markov–Feller operator, and let ν be an invariant measure. If ν is P-ergodic, then $\mathbb{P}_\nu$ is ergodic for the shift τ.*

Proof. Let $\underline{A} \subset X^{\mathbb{N}}$ be τ-invariant, that is, $\tau^{-1}\underline{A} = \underline{A}$ up to a $\mathbb{P}_\nu$-null measure set. Let

$$u(x) = \mathbb{P}_x(\underline{A}) = \mathbb{P}_x((X_n)_{n\geq 0} \in \underline{A}).$$

Observe that for ν-a.e. x,

$$u(x) = \mathbb{P}_x(\tau^{-1}\underline{A}) = \mathbb{P}_x((X_{n+1})_{n\geq 0} \in \underline{A}) = \mathbb{E}_x(\mathbb{P}_{X_1}(\underline{A})) = Pu(x).$$

Thus u is P-invariant and, since ν is P^*-ergodic, $u(x) = u_0$ for some constant u_0.

Now take B_n in $\sigma(X_0, X_1, \ldots, X_n)$, the σ-algebra generated by the first $n + 1$ coordinates. Then

$$\mathbb{P}_\nu(B_n \cap \underline{A}) = \mathbb{P}_\nu(B_n \cap \tau^{-(n+1)}\underline{A}) = \mathbb{E}_\nu(\mathbf{1}_{B_n}\mathbb{P}_{X_{n+1}}(\underline{A})) = u_0\mathbb{P}_\nu(B_n).$$

Since the set of functions $\mathbf{1}_{B_n}$ spans a dense subset of $L^1(\mathbb{P}_\nu)$, we see that $\mathbf{1}_{\underline{A}} = u_0$ must be constant, that is, $\underline{A}$ is $\mathbb{P}_\nu$ trivial. This completes the proof. □

In the specific context of this paper where P is induced by the action of a discrete measure μ on the group of Homeo(X), we can characterize invariant sets (and prove directly that P- and P^*-invariant sets coincide).

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LEMMA A.2. *Let P be the Markov operator defined by the action of a discrete distribution μ on the group of homeomorphisms of X as in (1), and let ν be an invariant Radon measure. Then for any measurable set $A \subset X$ the following conditions are equivalent:*

(1) $\nu(A \triangle g^{-1}A) = 0$ *for each* $g \in \Gamma$;

(2) $P^*\mathbf{1}_A = \mathbf{1}_A$ *in* $L^\infty(X, \nu)$;

(3) $P\mathbf{1}_A = \mathbf{1}_A$ *in* $L^\infty(X, \nu)$.

In particular, if M is a closed Γ-invariant set, then M is P-invariant for the Markov operator P.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Since ν is invariant, for any $f \in L^1(\nu)$ we have

$$\begin{aligned}\int_X f(x)P^*1_A(x)\,d\nu(x) &= \int_\Gamma\int_X f(g(x))\mathbf{1}_A(x)\,d\nu(x)d\mu(g)\\ &= \int_\Gamma\int_X f(g(x))\mathbf{1}_A(g(x))\,d\nu(x)d\mu(g)\\ &= \int_X f(x)\mathbf{1}_A(x)\,d\nu(x).\end{aligned}$$

(2) $\Rightarrow$ (1). Let $K_n \nearrow X$ be a sequence of increasing compact sets. Let $A^c = X \setminus A$ and $B_n := K_n \cap A^c$. Then, since $\mathbf{1}_{B_n} \in L^1(\nu)$,

$$\begin{aligned}\sum_{g\in\Gamma}\mu(g)\nu(g^{-1}B_n \cap A) &= \int_X\int_\Gamma \mathbf{1}_{B_n}(g(x))\mathbf{1}_A(x)\,d\nu(x)d\mu(g)\\ &= \int_X P\mathbf{1}_{B_n}(x)\mathbf{1}_A(x)\,d\nu(x)\\ &= \int_X \mathbf{1}_{B_n}(x)P^*\mathbf{1}_A(x)\,d\nu(x)\\ &= \int_X \mathbf{1}_{B_n}(x)\mathbf{1}_A(x)\,d\nu(x) = \nu(B_n \cap A) = 0.\end{aligned}$$

Thus $\nu(g^{-1}B_n \cap A) = 0$ for all $g \in \Gamma$ and

$$\nu((g^{-1}A)^c \cap A) = \nu(g^{-1}(A^c) \cap A) = \lim_{n\to\infty}\nu(g^{-1}B_n \cap A) = 0.$$

Observe that, since ν is P-invariant, also $P^*\mathbf{1}_{A^c} = \mathbf{1}_{A^c}$. Similarly, $\nu(g^{-1}A \cap A^c) = 0$ and finally we can conclude that

$$\nu(A \triangle g^{-1}A) = \nu(g^{-1}A \cap A^c) + \nu((g^{-1}A)^c \cap A) = 0.$$

(1)$\Leftrightarrow$ (3). Observe that $P\mathbf{1}_A(x) = \sum_{g\in\Gamma}\mu(g)\mathbf{1}_{g^{-1}A}(x)$. Thus

$$P\mathbf{1}_A = \mathbf{1}_A \Longleftrightarrow \mathbf{1}_{g^{-1}A} = \mathbf{1}_A \quad \text{for all } g \in \Gamma$$

since $\mu(g) > 0$ for all $g \in \Gamma$.

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Let M be a closed Γ-invariant set. Since $M \subseteq g^{-1}M$, we then have

$$\sum_{g\in\Gamma} \mu(g)\nu(g^{-1}M \triangle M) = \sum_{g\in\Gamma} \mu(g)(\nu(g^{-1}M) - \nu(M)) = 0$$

by the fact that ν is invariant. Therefore, $\nu(g^{-1}M \triangle M) = 0$ for all $g \in \Gamma$. □

A.3. *Chacon–Ornstein ergodic theorem for L^1-contractions.* The Chacon–Ornstein ratio ergodic theorem is an extremely powerful and general theorem to study the asymptotic behaviour of the partial sums

$$S_n f := \sum_{k=0}^{n} T^k f \quad \text{with } f \in L^1(W, \omega).$$

THEOREM A.3. (Chacon–Ornstein ergodic theorem) *Let T be a positive contraction of $L^1(W, \omega)$. Assume that the operator T is* conservative, *that is, there exists a strictly positive function $\Phi \in L^1(W, \omega)$ such that* $\lim_{n\to\infty} S_n\Phi(w) = +\infty$ *for ω-almost all $w \in W$. Then for any $f \in L^1(\omega)$ the limit*

$$Lf(w) := \lim_{n\to\infty} \frac{S_n f(w)}{S_n \Phi(w)} \quad \textit{exists and is finite for } \omega\textit{-a.e. } w. \tag{A.1}$$

Furthermore, the function Lf is invariant (that is, measurable with respect to $\mathcal{I}$, the σ-algebra of all T-invariant sets) and

$$\int Lf(x)\Phi(x)\, d\omega(x) = \int f(x)\, d\omega(x). \tag{A.2}$$

For a complete proof see, for instance, [18, Theorem 2.6.1] (see also [11], where the proof appeared for the first time). The ratio ergodic theorem enables us to give another characterization of ergodic measures.

LEMMA A.4. *Let $\mathcal{F}$ be a dense family in $L^1(\omega)$. An invariant measure ω is ergodic if and only if Lf is constant for all $f \in \mathcal{F}$.*

Proof. If ω is ergodic then the invariant σ-algebra is trivial and thus Lf is constant. In consequence, by (A.2), it is equal to $(\omega(f)/\omega(\Phi))$.

Suppose now that $Lf = (\omega(f)/\omega(\Phi))$ is ω-almost everywhere constant for any $f \in \mathcal{F}$. Let A be an invariant set. Since $T(\mathbf{1}_A f) = \mathbf{1}_A Tf$ (see, for instance, [18, Proposition 2.5.6]), it follows that $L(\mathbf{1}_A f) = \mathbf{1}_A Lf$ ω-almost everywhere and

$$\omega(\mathbf{1}_A f) = \omega(\Phi)L(\mathbf{1}_A f)(x) = \omega(\Phi)\mathbf{1}_A(x)Lf(x) = \mathbf{1}_A(x)\omega(f).$$

Since $f \in \mathcal{F}$ is arbitrary and $\mathcal{F}$ is a dense family in $L^1(\omega)$, the set A must be trivial, and we are done. □

A direct consequence of the previous lemma and of Lemma A.1, in the Markov–Feller operator case, is the following corollary that summarizes some of the fundamental results needed in our paper.

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COROLLARY A.5. *Let ν be an ergodic invariant Radon measure for the Markov–Feller operator P. Suppose that the Markov chain is recurrent, that is, there exists a compact set K such that $\mathbf{1}_K(x_n) + \cdots + \mathbf{1}_K(x_0) \to +\infty$ $\mathbb{P}_\nu$-almost everywhere. Then for any non-negative function $\phi \in L^1(X, \nu)$ we have $\nu(\phi) > 0$ if and only if $\phi(x_n) + \cdots + \phi(x_0) \to +\infty$ $\mathbb{P}_\nu$-almost everywhere, and in this case for all $f \in L^1(X, \nu)$:,*

$$\lim_{n\to\infty} \frac{f(x_n) + \cdots + f(x_0)}{\phi(x_n) + \cdots + \phi(x_0)} = \frac{\nu(f)}{\nu(\phi)} \quad \mathbb{P}_\nu\text{-almost everywhere}$$

Proof. Since $\mathbb{P}_\nu$ is τ-ergodic by Lemma A.1, applying Lemma A.4 to $\underline{f}(\underline{x}) := f(x_0)$ and $\underline{\Phi}(\underline{x}) := \Phi(x_0)$ with $\Phi > \mathbf{1}_K$, we obtain

$$\lim_{n\to\infty} \frac{f(x_n) + \cdots + f(x_0)}{\Phi(x_n) + \cdots + \Phi(x_0)} = \lim_{n\to\infty} \frac{\underline{f}(\tau^n \underline{x}) + \cdots + \underline{f}(\underline{x})}{\underline{\Phi}(\tau^n \underline{x}) + \cdots + \underline{\Phi}(\underline{x})}$$
$$= \frac{\mathbb{P}_\nu(\underline{f})}{\mathbb{P}_\nu(\underline{\Phi})} = \frac{\nu(f)}{\nu(\Phi)} \quad \mathbb{P}_\nu\text{-almost everywhere}$$

Take a non-negative function $\phi \in L^1(X, \nu)$ such that $\nu(\phi) > 0$. Then

$$S_n\phi(x) \sim \frac{\nu(\phi)}{\nu(\Phi)} S_n\Phi(x) \to \infty.$$

Conversely, if $\phi(x_n) + \cdots + \phi(x_0)$ tends to ∞, ν-almost everywhere, then, since the limit of $S_n\Phi/S_n\phi$ is finite (see, for instance, [16, Ch. III, Theorem D]) and equal to $\nu(\Phi)/\nu(\phi)$, by the previous step we obtain that $\nu(\phi) > 0$. □

A.4. *Ergodic decomposition of invariant measure.* This part of the paper is devoted to a complete proof of an ergodic decomposition for Markov–Feller processes on locally compact metric spaces. From this decomposition formula (2) will follow.

THEOREM A.6. *Let $(X, \mathcal{B}(X), \nu, P)$ be a Markov–Feller process. Assume that there exists a function $\Phi \in C(X) \cap L^1(\nu)$, $\Phi > 0$, such that $\sum_{n=1}^{\infty} P^n\Phi(x) = +\infty$ for all $x \in X$. Then there exists a measurable set $X_0 \subset X$ with $\nu(X \setminus X_0) = 0$ such that:*

(1) *for every $x \in X_0$ there exists a Radon measure ν_x such that*

$$\nu_x(f) = \lim_{n\to\infty} \frac{S_n f(x)}{S_n\Phi(x)} \quad \text{for all } f \in C_c(X); \tag{A.3}$$

(2) *for every non-negative $f \in L^1(\nu)$,*

$$\nu_x(f) = \lim_{n\to\infty} \frac{S_n f(x)}{S_n\Phi(x)} \quad \text{for } \nu\text{-a.e. } x \in X, \tag{A.4}$$

thus the function $x \mapsto \nu_x(f)$ is measurable and

$$\nu(f) = \int_X \nu_x(f)\Phi(x)\nu(dx); \tag{A.5}$$

(3) *ν_x is invariant and ergodic for any $x \in X_0$.*

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Although the above result has been used by several authors and is part of the folklore of the field, we are not aware of any explicit reference in the literature. In our understanding, in the specific case of Radon measures invariant under the action of a countable group, the ergodic decomposition could be deduced (with some work) from the paper of Greschonig and Schmidt [21, Theorem 1.4]. However, since their approach does not seem to apply to more general Markov–Feller processes, we present here an independent proof. This may be of interest in view of the future development of stochastic dynamical systems induced by transformations g_i that are not invertible or not countably generated.

We would also like to mention that in the ergodic decomposition obtained in the previous theorem, the set of ergodic measures ν_x *depends* on the measure ν. In this sense our result is weaker than that proved in [21], where the authors obtain the existence of the set of quasi-invariant ergodic measures that depends only on the group action.

Proof. First observe that since X is a locally compact metric space, there exist a countable increasing family of compact sets $(K_i)_{i\in\mathbb{N}}$ such that $K_i \nearrow X$ and a countable family of continuous functions $\mathcal{F} \subset C_C(X)$ dense in the space $C_c(X)$ (with the supremum norm) and such that if the support of f is contained in K_i, then for every $\varepsilon > 0$ there exists $h \in \mathcal{F}$ such that

$$\|f - h\|_\infty < \varepsilon \quad \text{and} \quad \operatorname{supp} h \subset K_{i+1}. \tag{A.6}$$

Thus, for every $f \in C_c(X)$ and $\delta > 0$, there exists $h \in \mathcal{F}$ such that

$$|f(x) - h(x)| < \delta\, \Phi(x) \quad \text{for all } x \in X.$$

Indeed, since $C_{i+1} = \inf_{x\in K_{i+1}} \Phi(x) > 0$, it suffices to take $h \in \mathcal{F}$ such that (A.6) holds for $\varepsilon = \delta/C_{i+1}$.

We will split the proof into four steps.

Step I. We define measures ν_x for ν-almost all $x \in X$ and prove (1). Let X_1 be the set of all $x \in X$ such that

$$Lh(x) := \lim_{n\to\infty} \frac{S_n h(x)}{S_n \Phi(x)} \quad \text{exists for all } h \in \mathcal{F}.$$

Since $\mathcal{F} \subset L^1(\nu)$ is countable, by the Chacon–Ornstein theorem, $\nu(X \setminus X_1) = 0$.

We shall prove that if $x \in X_1$, then the above limit exists for an arbitrary $f \in C_c(X)$. For this purpose we check that the sequence $((S_n f(x)/S_n\Phi(x))_{n\in\mathbb{N}})$ for $f \in C_c(X)$ and $x \in X_1$ satisfies the Cauchy condition. Fix $f \in C_c(X)$ and $\varepsilon > 0$. Let $h \in \mathcal{F}$ be such that $|f - h| < (\varepsilon/3)\Phi$. Then we have

$$\begin{aligned}\left|\frac{S_n f(x)}{S_n\Phi(x)} - \frac{S_m f(x)}{S_m\Phi(x)}\right| &\le \frac{S_n|h-f|(x)}{S_n\Phi(x)} + \frac{S_m|h-f|(x)}{S_m\Phi(x)} + \left|\frac{S_n h(x)}{S_n\Phi(x)} - \frac{S_m h(x)}{S_m\Phi(x)}\right| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for all $m, n \in \mathbb{N}$ sufficiently large. Since $\varepsilon > 0$ was arbitrary, the Cauchy condition is verified.

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Define now, for any $x \in X_1$, the functional on $C_c(X)$ by the formula

$$f \mapsto Lf(x) = \lim_{n\to\infty} \frac{S_n f(x)}{S_n \Phi(x)} \quad \text{for } f \in C_c(X).$$

Since this is a positive linear functional, it is represented by some regular measure ν_x, that is, $Lf(x) = \nu_x(f)$ for $f \in C_c(X)$, by the Riesz–Markov–Kakutani representation theorem. Obviously, ν_x is a Radon measure. This proves (A.3) for all $x \in X_0 \subseteq X_1$ of full measure.

Step II. We shall check that for any $f \in L^1(\nu)$ we have

$$\nu_x(f) = Lf(x) \quad \text{for } \nu\text{-almost all } x \in X \tag{A.7}$$

and prove (2). By (A.3), we already know that the last equality is true for all $f \in C_c(X)$. We will prove that by a continuity argument it can be extended to all functions $f \in L^1(\nu)$. However, observe that if the function is not continuous, the set of x where (A.7) holds may depend on f.

Let g_n, $n \in \mathbb{N}$, be a non-increasing family of non-negative measurable functions such that $g_n \searrow 0$ in $L^1(\nu)$. Then $Lg_n(x) \searrow 0$ for ν-a.e $x \in X$. In fact, since the operator L is positive and Lg_n is a non-increasing sequence of measurable functions, the limit $\overline{g}(x) := \lim_{n\to\infty} Lg_n(x)$ exists for ν-a.e. x and is non-negative. Furthermore,

$$\begin{aligned}
\int_X \overline{g}(x)\Phi(x)\nu(dx) &= \int_X \lim_{n\to\infty} Lg_n(x)\Phi(x)\nu(dx) \\
&\le \liminf_{n\to\infty} \int_X Lg_n(x)\Phi(x)\nu(dx) \quad \text{by Fatou's lemma} \\
&= \liminf_{n\to\infty} \int_X g_n(x)\nu(dx) \quad \text{by the Chacon–Orstein theorem} \\
&= 0.
\end{aligned}$$

Thus $0 = \overline{g}(x) := \lim_{n\to\infty} Lg_n(x)$ for ν-a.e x.

Let consider the class of functions

$$\mathcal{H} := \{f \text{ bounded mesurable function on } X : \ \nu_x(f\Phi) = L(f\Phi)(x) \ \nu\text{-a.s.}\}.$$

Consider the following assertions.

- If $f_n \in \mathcal{H}$ is a family of non-negative and increasing bounded functions converging to f, then $f \in \mathcal{H}$. In fact, since $f_n\Phi \nearrow f\Phi$ pointwise and in $L^1(\nu)$,

$$\begin{aligned}
\nu_x(f\Phi) &= \lim_{n\to\infty} \nu_x(f_n\Phi) \quad \text{by the monotone convergence theorem} \\
&= \lim_{n\to\infty} L(f_n\Phi)(x) \quad \text{since } f_n \in \mathcal{H} \\
&= \lim_{n\to\infty} [L(f\Phi)(x) - L((f - f_n)\Phi)(x)] \quad \text{by linearity of } L \\
v &= L(f\Phi)(x) - \lim_{n\to\infty} L((f - f_n)\Phi)(x) = L(f\Phi)(x)
\end{aligned}$$

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since $g_n = (f - f_n)\Phi \searrow 0$ pointwise and in $L^1(\nu)$, by the dominated convergence theorem.

- If U is an open subset of X, then $\mathbf{1}_U \in \mathcal{H}$. In fact, there exists a non-decreasing sequence of non-negative functions $f_n \in C_c(X)$ such that $f_n \nearrow \mathbf{1}_U$ pointwise. Since $f_n\Phi \in C_c(x)$, step I yields $f_n \in \mathcal{H}$, and consequently $\mathbf{1}_U \in \mathcal{H}$.
- If $f, g \in \mathcal{H}$, then $f + g$ and cf are in $\mathcal{H}$ for any real number c. This is a direct consequence of linearity of ν_x and L.

Applying the monotone class theorem for functions (see, for instance, [15, Theorem 5.2.2]), $\mathcal{H}$ contains all measurable bounded functions.

Now take a non-negative $f \in L^1(\nu)$ and an increasing sequence of compact sets $K_n \nearrow X$. Observe that

$$f_n(x) := \frac{f(x) \wedge n}{\Phi(x)} \quad \text{for } x \in K_n,$$

and $f_n(x) = 0$ otherwise. It is easy to check that f_n are bounded and $f_n\Phi \nearrow f$, both pointwise and in $L^1(\nu)$, thus, following the same reasoning as above, we obtain

$$\begin{aligned}
\nu_x(f) &= \lim_{n\to\infty} \nu_x(f_n\Phi) \quad \text{by the monotone convergence theorem}\\
&= \lim_{n\to\infty} L(f_n\Phi)(x) \quad \text{since } f_n \in \mathcal{H}\\
&= \lim_{n\to\infty} L(f)(x) - L(f - f_n\Phi)(x) \quad \text{by linearity of } L\\
&= Lf(x),
\end{aligned}$$

since $g_n = f - f_n\Phi \searrow 0$ pointwise and in $L^1(\nu)$. Invoking (A.2), this completes the proof of (2).

Step III. We will prove that there exists a set of full measure X_2 such that ν_x is P-invariant for all $x \in X_2$. Let X_2 be the set of all $x \in X_1$ such that:

(1) $\nu_x(f) = Lf(x)$ for all $f \in \mathcal{F}$;
(2) $\nu_x(Pf) = L(Pf)(x)$ for all $f \in \mathcal{F}$;
(3) $\nu_x(\Phi) = L\Phi(x)$ and $\nu_x(P\Phi) = L(P\Phi)(x)$.

Since $\mathcal{F}$ is countable and the desired equalities hold ν-almost everywhere, $\nu(X \setminus X_2) = 0$.

Observe now that for every $f \in C_c(X)$ and $x \in X_2$,

$$\begin{aligned}
Lf(x) &= \lim_{n\to\infty} \frac{S_nPf(x)}{S_n\Phi(x)} = \lim_{n\to\infty} \frac{S_nf(x) + P^{n+1}f(x) - f(x)}{S_n\Phi(x)}\\
&= \lim_{n\to\infty} \frac{S_nf(x)}{S_n\Phi(x)} = L(Pf)(x),
\end{aligned}$$

since f and Pf are bounded and $S_n\Phi \to \infty$. Thus, if $x \in X_2$ and $f \in \mathcal{F}$, we have

$$\nu_x(f) = Lf(x) = LPf(x) = \nu_x(Pf).$$

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Fix $f \in C_c(X)$ and $\varepsilon > 0$. Let $h \in \mathcal{F}$ be such that $|f - h| \le \varepsilon\Phi$. Thus $P|f - h| \le \varepsilon P\Phi$. Then it follows that

$$\begin{aligned}|\nu_x(Pf) - \nu_x(f)| &\le |\nu_x(Pf) - \nu_x(Ph)| + |\nu_x(Ph) - \nu_x(h)| + |\nu_x(f) - \nu_x(h)| \\ &= \varepsilon\nu_x(P\Phi) + 0 + \varepsilon\nu_x(\Phi) \\ &= \varepsilon(L(P\Phi)(x) + L\Phi(x)) \\ &= 2\varepsilon \quad \text{since } L(P\Phi)(x) = L\Phi(x) = 1.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain that $\nu_x(Pf) = \nu_x(f)$ for all $f \in C_c(X)$. Thus ν_x is P-invariant.

Step IV. We will prove that there exists a set of full measure $X_3 \subset X_2$ such that ν_x is ergodic for all $x \in X_3$. Take $f \in C_c(X)$ and observe that Lf is bounded (since $f \le C \cdot \Phi$ for some constant C, thus $Lf \le CL\Phi = C$) and, by the Chacon–Ornstein theorem, invariant. By [18, Proposition 2.5.6], $P(gLf) = (Pg)\,(Lf)$ for any $g \in L^1(\nu)$ and thus $L(gLf) = (Lg)\,(Lf)$. In particular, for ν-a.e. x,

$$\nu_x(fLg) \overset{(A.4)}{=} L(fLg)(x) = Lf(x)Lg(x) \overset{(A.4)}{=} \nu_x(f)\nu_x(g).$$

Let $X_3 \subseteq X_2$ be the set of all x such the latter equality holds for all $f, g \in \mathcal{F}$. Since $\mathcal{F}$ is countable. $\nu(X \setminus X_3) = 0$. Take $x \in X_3$ and fix $g \in \mathcal{F}$. Then

$$\nu_x(fLg) = \nu_x(f)\nu_x(g) \quad \text{for all } f \in \mathcal{F}.$$

Since $\mathcal{F}$ is dense in $L^1(\nu)$, it follows that $Lg(y) = \nu_x(g)$ for ν_x-almost all $y \in X$. Thus ν_x is an invariant measure such that Lg is ν_x-almost everywhere constant and ν_x is then ergodic, by Lemma A.4 applied to $\nu = \nu_x$. The proof is complete. □

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