ON FINITE PLANE SETS CONTAINING FOR EVERY PAIR OF POINTS AN EQUIDISTANT POINT

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1. In [1] Melzak has posed the following problem: "A plane finite set X_n consists of $n \geq 3$ points and contains together with any two points a third one, equidistant from them. Does X_n exist for every n ? Must it consist of points lying on some two concentric circles (one of which may reduce to a point)? How many distinct (that is, not similar) X_n are there for a given n ? ..." We shall here provide a construction for uncountably many X_n for every $n > 4$, and a counterexample to the second question above.

2. An X_7 which does not lie on two concentric circles.

Consider the set shown in Figure 1, consisting of the points in E^2 whose circular polar coördinates are $(1, -\pi/3)$, $(1, 0)$, $(1, \pi/6)$, $(1, \pi/3)$, $(1, \pi/2)$, $(\sqrt{2}, \pi/4)$, and $(0, 0)$. If this set lay on the union of two circles then at least three of the first five points listed above would lie together on one of these circles, which must therefore be the unit circle about $(0, 0)$. Evidently no other circle with the same centre can contain both $(0, 0)$ and $(\sqrt{2}, \pi/4)$. An X_8 can also be constructed by adjoining to the above set the point $(2 \sin \pi/12, \pi/12)$.

3. A construction for uncountably many X_n $(n > 4)$.

We first observe without proof that X_3 and X_4 are unique up to similarity.

Let T_i be an equilateral triangle of unit side with vertices labelled a_i, b_i, c_i $(i=1, \ldots, k)$. Suppose these triangles to be so placed in E^2 that the vertices $a_1, \ldots, a_k$ coincide as a single point a , but without further restriction. Then a is an equidistant point from any two of the b's or c's ; and an equidistant point from a and b_i (respectively c_i) is c_i (respec-

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tively b_i). By allowing coincidences between the vertices of distinct triangles we can generate in this way uncountably many dissimilar X_n for each $n > 4$.

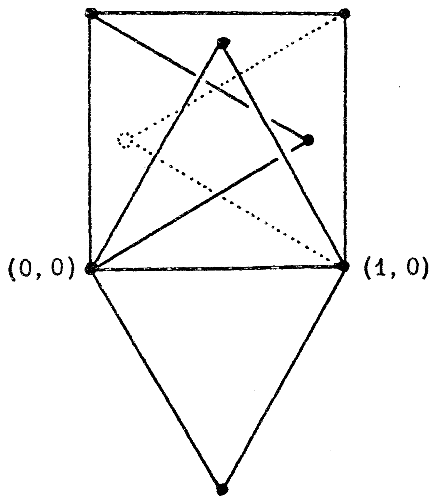

Figure 1

REFERENCE

1. Z.A. Melzak, Problems connected with convexity. Canad. Math. Bull. 8 (1965), pages 565-573: Problem (21) .

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