A nonlinear complementarity problem
for monotone functions

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In this note we prove that for a monotone function that fixes
the origin, the complementarity problem for C^n always admits a
solution. If, moreover, the function is strictly monotone, then
zero is the unique solution. These results are stronger than
known results in this direction for two reasons: firstly, there
is no condition on the nature of the cone and secondly, no
feasibility assumptions are made.

1.   Preliminaries

Let C^n denote the n-dimensional complex space with hermitian norm
and the usual inner product. Let K denote a closed convex cone in C^n
and K^* its polar; that is,

$$K^* = \{y \in C^n : \mathrm{re}(x, y) \geq 0 \text{ for all } x \in K\} .$$

Given a continuous function $g : C^n \to C^n$, the nonlinear complementarity
problem in C^n consists of finding a point z such that

(1.1) $z \in K$, $g(z) \in K^*$, and $\mathrm{re}(g(z), z) = 0$.

In this note we shall be concerned with the solution of the nonlinear
complementarity problem for monotone functions. A mapping $f : C^n \to C^n$ is
said to be monotone on K if $\mathrm{re}(f(x)-f(y), x-y) \geq 0$ for every
$(x, y) \in K \times K$, and strictly monotone if strict inequality holds whenever

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$x \neq y$. The function f is said to be strongly monotone if there is a constant $c > 0$ such that, for each $(x, y) \in K \times K$,

$$re\big(f(x) - f(y), \ x - y\big) \geq c\|x - y\|^2 \ .$$

For each real number $r \geq 0$, we denote by K_r the following set

$$K_r = \{x \in K : \|x\| \leq r\} \ .$$

Several authors (see references [1], [2], [3], [5], [6], and [7]) have discussed the complementarity problem for R^n and C^n . The result we present here is an improvement of the results contained in the references above for two reasons. Firstly, no assumptions are made on the nature of the cone (such as polyhedral, pointed, and so on). Secondly, we do not make any feasibility assumption.

2.   The main theorem

We start by mentioning a modified version of a result of Hartman and Stampacchia [4].

LEMMA   2.1. *Let* $g : K \to C^n$ *be a continuous map. Then there is a point* $z_r \in K_r$ *such that*

(2.1) $$re\big(g(z_r), \ z - z_r\big) \geq 0$$

for all $z \in K_r$.

Proof. K_r is a closed convex and bounded set. Therefore Proposition 2.1 of [6] applies, and the desired result follows.

LEMMA   2.2. *Let* $g : K \to C^n$ *be a continuous monotone function on* K *that fixes the origin, and let* $x \in K$. *Then the continuous function* $\theta : R^+ \to R$ *defined by*

$$\theta(r) = re\big(g(rx), \ rx\big)$$

is monotone increasing.

Proof. Let $s \in R$. Since g is monotone, $re\big(g(sx) - g(0), \ sx\big) \geq 0$, implying that $re\big(g(sx), \ sx\big) \geq 0$. For any $r, s \in R$ with $r < s$, we have

$$re\big(g(rx) - g(sx), \ (r - s)x\big) \geq 0 \ .$$

Since $r - s < 0$, we have

$$re\big(g(rx)-g(sx),\ x\big) \le 0\ .$$

Thus

$$re\big(g(rx),\ rx\big) \le re\big(g(sx),\ rx\big) = (r/s)re\big(g(sx),\ sx\big) \le re\big(g(sx),\ sx\big)\ ,$$

which is the required result.

THEOREM 2.3. *Let* $g : K \to C^n$ *be a continuous function that fixes the origin and which, moreover, is monotone on* K . *Let* $z_r \in K_r$ *be the point as obtained in Lemma* 2.1. *Then* z_r *is a solution of the complementarity problem* (1.1).

Proof. Note first of all that since

$$re\big(g(z_r),\ z-z_r\big) \ge 0\quad \text{for all}\quad z \in K_r\ ,$$

it follows by taking $z = 0$ that

(2.2) $$re\big(g(z_r),\ z_r\big) \le 0\ .$$

Let $t \in [0,\ 1] = I$. We then have from Lemma 2.2 that

$$re\big(g(tz_r),\ tz_r\big) \le re\big(g(z_r),\ z_r\big) \le re\big(g(z_r),\ z\big)$$

for all $z \in K_r$. It is also evident that the second inequality above holds for all $z \in K$. Thus we have

$$-re\big(g(z_r),\ z\big) \le -re\big(g(z_r),\ z_r\big) \le -re\big(g(tz_r),\ tz_r\big)\ .$$

Since $-re\big(g(z_r),\ z_r\big) \ge 0$ by virtue of (2.2), it follows that

$$-re\big(g(tz_r),\ tz_r\big) \ge 0\ ,$$

so that we can apply Cauchy-Schwartz inequality to get

(2.3) $$-re\big(g(z_r),\ z\big) \le -re\big(g(tz_r),\ tz_r\big) \le \|g(tz_r)\|\|tz_r\|$$

for all $z \in K$ and $t \in I$.

Observe that the set $\{tz_r : t \in I\}$ is compact; thus g attains a maximum on this set, say M . Thus from (2.3) we have the inequality

$$-re\big(g(z_r),\ z\big) \le M\|z_r\|\ ,$$

which holds for all $z \in K$ and $t \in I$. Since t can be chosen to be arbitrarily small, it follows that

$$(2.4) \qquad\qquad -re\bigl(g(z_r),\ z\bigr) \leq 0$$

for all $z \in K$. This shows that $g(z_r) \in K^*$, and, moreover, (2.4) together with the inequality (2.2) implies that $re\bigl(g(z_r),\ z_r\bigr) = 0$. Thus z_r is a solution to the complementarity problem.

THEOREM 2.4. *Let* $f : K \to C^n$ *be a continuous function that fixes the origin and which, moreover, is strictly monotone on* K *;* *then zero is the unique solution to the complementarity problem.*

Proof. That a solution exists is clear from the theorem above. Moreover, due to strict monotonicity, such a solution is unique. Suppose that z_0 is the solution. Assume to the contrary that $z_0 \neq 0$. Let $\|z_0\| = k > 0$, and let $r = \tfrac{1}{2}k$. By applying Lemma 2.1 we get a $z_r \in K_r$ which is the solution of the complementarity problem. Since the solution is unique, $z_r = z_0$. This is impossible since $\|z_r\| \leq \tfrac{1}{2}k$. Thus we arrive at a contradiction and z_0 must be zero.

REMARK 1. A function $g : C^n \to C^n$ is said to be α-monotone on K if for some function $\alpha : R^+ \to R^+$ which is monotone increasing such that $\alpha(r) \to \infty$ as $r \to \infty$, the following inequality is satisfied:

$$re\bigl(g(x)-g(y),\ x-y\bigr) \geq \alpha(\|x-y\|)\|x-y\| \ .$$

In case $\alpha(r) = cr$ for some $c > 0$, g is said to be strongly monotone. It is clear that α-monotone functions are strictly monotone. Therefore Theorem 2.4 applies and zero is the unique solution to the complementarity problem under the same conditions.

REMARK 2. The arguments of Theorem 2.3 and Theorem 2.4 go through in case we have R^n instead of C^n .

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