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On a sequence of Fourier coefficients

V. Venu Gopal Rao and B.N. Sahney

In this paper we establish $(c, 1)$ summability of the sequence $\{nB_n(x)\}$ and by using Tauber's Second Theorem, we deduce the convergence criterion of the conjugate series of a Fourier series.

1.

Let $f(t)$ be an integrable function periodic with period 2π and let its Fourier series and its conjugate series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) = \sum_{n=0}^{\infty} A_n(t) \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \left(b_n \cos nt - a_n \sin nt\right) = \sum_{n=1}^{\infty} B_n(t) , \tag{1.2}$$

respectively. We write

$$\psi(t) = f(x+t) - f(x-t) - l ,$$

$$\kappa(t) = f(x+t) - f(x-t) .$$

Fejér [3, pp. 55, 62] has shown that if $l = f(x+0) - f(x-0)$ exists and is finite, the sequence $\{nB_n(x)\}$ is summable (c, r) , $r > 1$, to the value l/π ; and if f is of bounded variation, the theorem holds true for $r > 0$. Obrechkoff [2] proved that if f is integrable (L) and if $t^{-1}|f(x+t)-f(x-t)-l|$ is integrable near $t = 0$, then $n^{-1} \sum_{r=1}^{n} rB_r(x) \to l/\pi$. Later Mohanty and Nanda [1] proved the following:

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THEOREM MN. *If*

$$\psi(t) = o\{(\log 1/t)^{-1}\} \text{, as } t \to 0 \text{,}$$

and

$$a_n = O\left(n^{-\delta}\right) \text{, } b_n = O\left(n^{-\delta}\right) \text{, } 0 < \delta < 1 \text{,}$$

then the sequence $\{nB_n(x)\}$ *is summable* $(c, 1)$ *to the value* l/π .

The object of this paper is to prove the following theorem:

THEOREM 1. *If*

$$\Psi(t) = \int_0^t \psi(u)du = o\left(t^{\Delta}\right) \text{, } \Delta > 1 \text{,} \tag{1.3}$$

and

$$\int_{t^{1/\Delta}}^{\delta} |d\theta(u)| = O\left(t^{-\eta}\right) \text{, } \pi > \delta > 0 \text{,} \tag{1.4}$$

where $\theta(u) = u^{-\eta}\psi(u)$ *and where* η *satisfies* $1 > \eta > 0$, *then* $\{nB_n(x)\}$ *is summable* $(c, 1)$ *to the value* l/π .

2.

Proof of Theorem 1. From Mohanty and Nanda [1], we write

$$\begin{aligned} n^{-1}\sum_{r=1}^{n} rB_r(x) - l/\pi &= \frac{1}{\pi}\int_0^{\pi} \{f(x+t)-f(x-t)-l\}g(n, t)dt + o(1) \\ &= \frac{1}{\pi}\int_0^{\pi} \psi(t)g(n, t)dt + o(1) \\ &= I + o(1) \text{,} \end{aligned} \tag{2.1}$$

say, where

$$\begin{aligned} g(n, t) &= -\frac{1}{n}\frac{d}{dt}\{\cos t + \cos 2t + \ldots + \cos nt\} \\ &= -\frac{1}{n}\frac{d}{dt}\left\{\frac{\sin nt}{\tan t/2} + \cos nt - 1\right\} \\ &= \left\{\frac{1}{4n}\,\frac{\sin nt}{\sin^2 t/2} - \frac{1}{2}\,\frac{\cos nt}{\tan t/2}\right\} + \frac{1}{2}\sin nt \text{.} \end{aligned}$$

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Then

$$\begin{aligned} I &= \int_0^{\pi} \psi(t) g(n, t) dt \\ &= \int_0^{\pi} \psi(t) \left\{ \frac{\sin nt}{4n \sin^2 t/2} - \frac{\cos nt}{t} \right\} dt + \frac{1}{2} \int_0^{\pi} \psi(t) \sin nt dt \\ &= \left\{ \int_0^{\pi/n} + \int_{\pi/n}^{(\pi/n)^{1/\Delta}} + \int_{(\pi/n)^{1/\Delta}}^{\delta} \right\} \psi(t) G(n, t) dt + o(1) \\ &= I_1 + I_2 + I_3 + o(1) , \end{aligned}$$

say, where $G(n, t) = \dfrac{\sin nt}{nt^2} - \dfrac{\cos nt}{t}$.

We have the following estimates:

$$G(n, t) = O(n^2 t) , \quad \pi/n \geq t \geq 0$$

$$G(n, t) = O(1/t) , \quad t > \pi/n$$

$$\frac{d}{dt} G(n, t) = G'(n, t) = O(n^2) , \quad \pi/n \geq t \geq 0$$

and

$$G'(n, t) = O(n/t) , \quad t > \pi/n .$$

Then

$$\begin{aligned} I_1 &= \int_0^{\pi/n} \psi(t) G(n, t) dt \\ &= \left[\Psi(t) G(n, t)\right]_0^{\pi/n} - \int_0^{\pi/n} \Psi(t) G'(n, t) dt \\ &= o\left\{ \left[t^{\Delta} n^2 t\right]_0^{\pi/n} - n^2 \int_0^{\pi/n} t^{\Delta} dt \right\} \\ &= o(1) ; \end{aligned} \tag{2.2}$$

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(2.3)
$$I_2 = \int_{\pi/n}^{(\pi/n)^{1/\Delta}} \psi(t)G(n, t)dt$$
$$= \int_{\pi/n}^{\alpha} \psi(t)G(n, t)dt \text{ , where } \alpha = (\pi/n)^{1/\Delta}$$
$$= [\Psi(t)G(n, t)]_{\pi/n}^{\alpha} - \int_{\pi/n}^{\alpha} \Psi(t)G'(n, t)dt$$
$$= o\left\{[t^{\Delta}.t^{-1}]_{\pi/n}^{\alpha} - n\int_{\pi/n}^{\alpha} t^{\Delta}.t^{-1}dt\right\}$$
$$= o\left\{[t^{\Delta-1}]_{\pi/n}^{\alpha} - n\int_{\pi/n}^{\alpha} t^{\Delta-1}dt\right\}$$
$$= o(1) .$$

Finally it remains to show that $I_3 = o(1)$. To evaluate I_3 we write

$$I_3 = \int_{\alpha}^{\delta} \psi(t)\frac{\sin nt}{nt^2}dt - \int_{\alpha}^{\delta} \psi(t)\frac{\cos nt}{t}dt$$
$$= I_{3.1} + I_{3.2} ,$$

say. Put

$$\theta(t) = t^{-\eta}\psi(t) , \quad \Theta(t) = \int_0^t |d\theta(u)| ,$$

then

$$\Theta(t) = O\left(t^{-\eta\Delta}\right) \text{ and } \theta(t) = O\left(t^{-\eta\Delta}\right) .$$

Then

$$I_{3.1} = \int_{\alpha}^{\delta} \psi(t)\frac{\sin nt}{nt^2}dt$$
$$= \int_{\alpha}^{\delta} \theta(t)\frac{\sin nt}{nt^{2-\eta}}dt = -\int_{\alpha}^{\delta} \theta(t)d\Lambda(t) ,$$

where

$$\Lambda(t) = \int_t^{\delta} \frac{\sin nt}{nt^{2-\eta}}dt = \frac{1}{nt^{2-\eta}}\int_t^{\xi} \sin nu\,du = O\left(n^{-2}t^{\eta-2}\right) .$$

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So

$$I_{3.1} = \int_\alpha^\delta \theta(t)d\Lambda(t) = [\theta(t)\Lambda(t)]_\alpha^\delta - \int_\alpha^\delta \Lambda(t)d\theta(t)$$

$$= P + Q ,$$

say.

$$P = O[t^{-\eta\Delta}n^{-2}t^{\eta-2}]_\alpha^\delta = O(n^{-2})[t^{\eta(1-\Delta)-2}]_\alpha^\delta = o(1) ;$$

and

$$|Q| \le \int_\alpha^\delta |\Lambda(t)||d\theta(t)| = O\left[n^{-2}\int_\alpha^\delta \frac{|d\theta(t)|}{t^{2-\eta}}\right]$$

$$= O(n^{-2})\left\{[t^{\eta-2}\Theta(t)]_\alpha^\delta - \int_\alpha^\delta t^{\eta-3}\Theta(t)dt\right\}$$

$$= O(n^{-2})\left\{[t^{\eta-2}.t^{-\eta\Delta}]_\alpha^\delta - \int_\alpha^\delta t^{\eta-3}.t^{-\eta\Delta}dt\right\}$$

$$= O(n^{-2})\left\{[t^{\eta(1-\Delta)-2}]_\alpha^\delta - \int_\alpha^\delta t^{\eta(1-\Delta)-3}dt\right\}$$

$$= o(1) .$$

Finally

$$I_{3.2} = \int_\alpha^\delta \psi(t)\frac{\cos nt}{t}dt = \int_\alpha^\delta \theta(t)\frac{\cos nt}{t^{1-\eta}}dt = -\int_\alpha^\delta \theta(t)d\chi(t)$$

where

$$\chi(t) = \int_t^\delta \frac{\cos nu}{u^{1-\eta}}du = \frac{1}{t^{1-\eta}}\int_t^\delta \cos nu\,du = O(n^{-1}t^{\eta-1}) .$$

Therefore

$$I_{3.2} = \int_\alpha^\delta \theta(t)d\chi(t)$$

$$= [\theta(t)\chi(t)]_\alpha^\delta - \int_\alpha^\delta \chi(t)d\theta(t)$$

$$= R - S ,$$

say;

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$$R = O\left\{\left[t^{-\eta\Delta}.n^{-1}t^{\eta-1}\right]_{\alpha}^{\delta}\right\}$$
$$= O\left\{n^{-1}\left[t^{\eta(1-\Delta)-1}\right]_{\alpha}^{\delta}\right\}$$
$$= o(1) ;$$

and

$$|S| \le \int_{\alpha}^{\delta} |\chi(t)||d\theta(t)|$$
$$= O\left(n^{-1}\int_{\alpha}^{\delta}\frac{|d\theta(t)|}{t^{1-\eta}}\right)$$
$$= O\left(n^{-1}\right)\left\{\left[t^{-\eta\Delta}.t^{\eta-1}\right]_{\alpha}^{\delta} - \int_{\alpha}^{\delta} t^{\eta(1-\Delta)-2}dt\right\}$$
$$= o(1) .$$

Finally

(2.4) $$I_3 = o(1) .$$

Hence from (2.1), (2.2), (2.3) and (2.4), we have

$$n^{-1}\sum_{r=1}^{n} rB_r(x) - l/\pi = o(1) \text{ , as } n \to \infty .$$

This completes the proof of Theorem 1.

3.

We have the following convergence criteria for the conjugate series:

THEOREM 2. *If*

(3.1) $$\int_{0}^{t} \kappa(u)du = o\left(t^{\Delta}\right) , \quad \Delta > 1 ,$$

and

(3.2) $$\int_{t^{1/\Delta}}^{\delta} \left|d\left(u^{-\eta}\kappa(u)\right)\right| = O\left(t^{-\eta}\right) , \quad 1 > \eta > 0 ,$$

then the allied series (1.2) *converges to the value*

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$$(3.3) \qquad \frac{1}{2\pi}\int_0^{\pi} \kappa(t)\cot t/2 dt ,$$

provided that the integral exists as a Cauchy integral at the origin.

Now we deduce Theorem 2 as a corollary of Theorem 1 employing the following:

TAUBER'S SECOND THEOREM. *If* $\sum U_n$ *is summable* (A) *, then a necessary and sufficient condition that it should be convergent is that the sequence* $\{nU_n\}$ *is summable* $(c, 1)$ *to the value zero.*

Proof of Theorem 2. The existence of the integral (3.3) as a Cauchy integral at the origin implies the summability (A) of the conjugate series (1.2) [3, p. 55].

Using Theorem 1, we find that the conditions (3.1) and (3.2) of Theorem 2 imply the summability $(c, 1)$ of the sequence $\{nB_n(x)\}$ to the value zero. The convergence of the series (1.2) follows from Tauber's Second Theorem.

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Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT;

Department of Mathematics,
Statistics and Computer Science,
The University of Calgary,
Alberta, Canada.

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