AN OPERATOR SATISFYING DUNFORD'S CONDITION (C) BUT WITHOUT BISHOP'S PROPERTY (β)

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1. Introduction. For X a complex Banach space and U an open subset of the complex plane $\mathbb{C}$, let $\mathcal{O}(U, X)$ denote the space of analytic X-valued functions defined on U. This is a Fréchet space when endowed with the topology of uniform convergence on compact subsets, and the space X may be viewed as simply the constants in $\mathcal{O}(U, X)$. Every bounded operator T on X induces a continuous mapping T_U on $\mathcal{O}(U, X)$ given by $(T_U f)(\lambda) = (\lambda - T)f(\lambda)$ for every $f \in \mathcal{O}(U, X)$ and $\lambda \in U$. Corresponding to each closed $F \subset \mathbb{C}$ there is also an associated analytic subspace $X_T(F) = X \cap \operatorname{ran}(T_{\mathbb{C}/F})$. For an arbitrary $T \in \mathcal{L}(X)$, the spaces $X_T(F)$ are T-invariant, generally non-closed linear manifolds in X.

An operator $T \in \mathcal{L}(X)$ has the *decomposition property* (δ) provided that the space X decomposes as $X = X_T(\bar{U}_1) + X_T(\bar{U}_2)$ whenever $\{U_1, U_2\}$ is an open cover of the complex plane. $T \in \mathcal{L}(X)$ is *decomposable in the sense of Foias* provided that T has property (δ) and that the analytic subspaces $X_T(F)$ are closed whenever F is a closed subset of the plane; see [**1**], [**3**].

The *local resolvent of* T at a vector $x \in X$ is the set $\rho_T(x)$ consisting of all $\lambda \in C$ for which there is a open neighborhood U such that $x \in T_U\mathcal{O}(U, X)$. The *local spectrum of* T at x is $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$. If T is such that for every closed $F \subset \mathbb{C}$ the linear manifold $\{x \in X : \sigma_T(x) \subset F\}$ is closed in X, then T *satisfies Dunford's condition* (C); see [**4**, XVI.1]. We say that T has the *single-valued extension property* provided that T_U is injective for every open $U \subset \mathbb{C}$; equivalently, if $X_T(F) = \{x \in X : \sigma_T(x) \subset F\}$ for every closed $F \subset \mathbb{C}$; see [**6**, Proposition 1.1].

An operator T has *Bishop's property* (β) provided that for every open $U \subset \mathbb{C}$ the mapping T_U is injective and has closed range. Albrecht and Eschmeier [**2**] showed that property (β) completely characterizes the restrictions of decomposable operators to invariant subspaces and their analytic functional model shows that every Banach space operator is similar to the quotient of an operator with property (β); see [**5**]. Moreover, Albrecht and Eschmeier prove properties (β) and (δ) to be completely dual; an operator T has one of these precisely when T^* has the other.

Property (β) implies (C), and (C) in turn implies the single-valued extension property, [**6**, Proposition 1.2]. Therefore $T \in \mathcal{L}(X)$ has property (C) if and only if the analytic subspaces $X_T(F)$ are closed whenever F is a closed subset of the plane, and T is decomposable if and only if T has both properties (C) and (δ); equivalently, if and only if T has both properties (β) and (δ). Thus it is a natural question whether property (C) is strictly weaker than property (β). Laursen and Neumann mention it explicitly in [**6**], but this question has circulated informally for some time.

Example. Weighted shifts have proven to be a rich source of examples and are a favorite testing ground for operator theorists. The basic facts regarding weighted shifts are collected in Allen Shields' excellent survey [**11**].

Let $\alpha = (\alpha_n)_{n\geq 0}$ be a sequence of strictly positive real numbers with $\sup_{n\leq 0}\alpha_{n+1}/\alpha_n < \infty$, and consider the Hilbert space $H^2(\alpha)$, consisting of formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ satisfying

$$\|f\|_\alpha^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \alpha_n^2 < \infty.$$

An injective unilateral weighted shift can be realized as multiplication by the independent variable on a space $H^2(\alpha)$ for some α: let $T : H^2(\alpha) \to H^2(\alpha)$ be given by $(Tf)(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^{n+1}$. Shields describes the spectrum of T in terms of quantities r, r_1 and r_2, where

$$r(\alpha) = \lim_{n\to\infty}\left(\sup_{k\geq 0}\frac{\alpha_{n+k}}{\alpha_k}\right)^{1/n}, r_1(\alpha) = \lim_{n\to\infty}\left(\inf_{k\geq 0}\frac{\alpha_{n+k}}{\alpha_k}\right)^{1/n} \text{ and } r_2(\alpha) = \liminf_{n\to\infty}\alpha_n^{1/n}.$$

Clearly for any α we have $r_1 \leq r_2 \leq r$. By [**11**, Theorems 4 and 6], the shift T has norm $\sup_{n\geq 0}\alpha_{n+1}/\alpha_n$, spectrum $\sigma(T) = \{z : |z| \leq r\}$ and approximate point spectrum given by $\sigma_{ap}(T) = \{z : r_1 \leq |z| \leq r\}$. The point spectrum, $\sigma_p(T)$, is empty, and by [**11**, Theorem 8]

$$\{0\} \cup \{z : |z| < r_2\} \subset \sigma_p(T^*) \subset \{z : |z| \leq r_2\}.$$

Moreover by [**11**, Theorem 10], if $|\lambda| < r_2$, then the vector $k_\lambda(z) = \sum_{n=0}^{\infty} \bar{\lambda}^n \alpha_n^{-2} z^n$ is in $\ker(\lambda - T)^*$ and is the reproducing kernel for evaluation at λ:

$$\langle f, k_\lambda\rangle_{H^2(\alpha)} = \sum_{n=0}^{\infty}\hat{f}(n)\lambda^n.$$

In particular, if $r_2 > 0$ then each $f \in H^2(\alpha)$ is analytic on the disk $\{z : |z| < r_2\}$.

PROPOSITION. *With the notation above, we have the following results.* (1) *If* $r_2 = r$, *then* T *satisfies Dunford's condition* (C); *in fact,* $\sigma_T(f) = \sigma(T)$ *for every nonzero* f *in* $H^2(\alpha)$. (2) *If* $r_1 < r_2$, *then* T *does not have Bishop's property* (β).

Proof. If $r = 0$, then T is quasiinilpotent and therefore decomposable. Thus we may assume that $r > 0$. In this case, the first statement follows from [**8**, Proposition 1], but this is not hard to show directly. Indeed, if $f \in H^2(\alpha)$ is such that $\sigma_T(f) \neq \sigma(T)$ then there is an open subset U of $\{z : |z| < r_2\}$ and a function $\varphi \in \mathcal{O}(U, H^2(\alpha))$ such that $f = T_U\varphi$. For every $\lambda, \omega \in U$ we have that

$$\begin{aligned} f(\omega) &= \langle(\lambda - T)\varphi(\lambda), k_\omega\rangle = \langle\varphi(\lambda), (\lambda - T)^* k_\omega\rangle \\ &= (\lambda - \omega)\langle\varphi(\lambda), k_\omega\rangle \end{aligned}$$

In particular $f(\lambda) = 0$, for every $\lambda \in U$, and since f is analytic it follows that $f = 0$.

To show the second assertion, suppose that $r_1 < r_2$ and let D be the disk $D = \{z : |z| < r_2\}$. If $\varphi \in \mathcal{O}(D, H^2(\alpha))$ and $\lambda \in D$ let us write the power series for $\varphi(\lambda)$ as $\varphi(\lambda, z) = \sum_{n=0}^{\infty}\widehat{\varphi(\lambda)}(n)z^n$.

For every $\lambda \in D$, let $H_\lambda = k_\lambda^\perp = \{f \in H^2(\alpha) : f(\lambda) = 0\}$. Notice that the polynomials in H_λ are dense in H_λ; in fact,

$$\{p : p \text{ a polynomial}, p(\lambda) = 0\} \subset (\lambda - T)H^2(\alpha) \subset H_\lambda$$

and ran $(\lambda - T) = H_\lambda$ only if $|\lambda| < r_1$. Choose λ_0 in D with $|\lambda_0| \geq r_1$ and $f \in H_{\lambda_0} \backslash \operatorname{ran}(\lambda_0 - T)$. Let $(p_n)_{n\geq 1}$ be a sequence of polynomials in H_{λ_0} converging to f in $H^2(\alpha)$, and define φ and $(\varphi_n)_{n\geq 1}$ in $\mathcal{O}(D, H^2(\alpha))$ by $\varphi(\lambda) = \langle f, k_\lambda \rangle 1 - f$, and for each n, $\varphi_n(\lambda) = \langle p_n, k_\lambda \rangle 1 - p_n$; that is, $\varphi(\lambda, z) = f(\lambda) - f(z)$, and $\varphi_n(\lambda, z) = p_n(\lambda) - p_n(z)$ for every $\lambda, z \in D$. Notice that $\varphi \notin \operatorname{ran}(T_D)$ since $f = -\varphi(\lambda_0)$ is not in $\operatorname{ran}(\lambda_0 - T)$. Also, for every $\lambda \in D$ and for each n, there is a function q_n such that $q_n(\lambda, z)$ is a polynomial in z, analytic in λ and $\varphi_n(\lambda, z) = (\lambda - z)q_n(\lambda, z)$. In particular, each $\varphi_n \in \operatorname{ran}(T_D)$.

If K is a compact subset of D, there is a positive constant M such that $\|k_\lambda\|_\alpha \leq M$, for each $\lambda \in K$, and therefore

$$\sup_{\lambda \in K} \|\varphi_n(\lambda) - \varphi(\lambda)\|_\alpha \leq \sup_{\lambda \in K}(|\langle p_n - f, k_\lambda \rangle| + \|p_n - f\|_\alpha) \leq (M+1)\|p_n - f\|_\alpha.$$

It follows that $\varphi_n \to \varphi$ in $\mathcal{O}(D, H^2(\alpha))$, and thus T_D fails to have closed range.

To obtain the desired example, it remains only to construct an appropriate sequence α. Let $(\nu_j)_{j=0}^\infty$ be a sequence of nonnegative integers with $\nu_0 = 0$, satisfying $\nu_j > \nu_{j-1} + j - 1$ and

$$\prod_{\ell=1}^{j} (1+\ell)^{-\ell/\nu_j} > \frac{j}{1+j}$$

for each $j \geq 1$. Define the sequence α by $\alpha_n = 1$, for $0 \leq n \leq \nu_1$, and $\alpha_n = \prod_{\ell=1}^{j}(1+\ell)^{-\ell}$ if $\nu_i + j \leq n \leq \nu_{j+1} + j$.

Corollary. *Let α be the sequence defined above and let T be the injective unilateral weighted shift $Tf(z) = zf(z)$, for $f \in H^2(\alpha)$. Then $r_1(\alpha) = 0$ and $1 = r_2(\alpha) = r(\alpha) = \|T\|$. In particular, T has Dunford's property (C) but not Bishop's property (β).*

Proof. Clearly $\|T\| = \sup_n \alpha_{n+1}/\alpha_n = 1$; in fact, for each $n \geq 1$, if $\ell > n$, then $\alpha_{n+\nu_\ell}/\alpha_{\nu_\ell} = 1$. Thus $1 \geq \|T^n\| = \sup_{k\geq 0} \alpha_{n+k}/\alpha_k \geq 1$, and $r(\alpha) = \|T\| = 1$.

For every $j \geq 1$,

$$(\alpha_{j+\nu_j}/\alpha_{\nu_j})^{1/j} = \left(\frac{\prod_{\ell=1}^{j}(1+\ell)^{-\ell}}{\prod_{\ell=1}^{j-1}(1+\ell^{-\ell})} \right)^{1/j} = \frac{1}{j+1};$$

it follows that $r_1(\alpha) = 0$.

Finally, if $n \geq 1$ and if $\nu_j + j \leq n < \nu_{j+1} + j + 1$, then

$$\alpha_n^{1/n} = \prod_{\ell=1}^{j}(1+\ell)^{-\ell/n} \geq \prod_{\ell=1}^{j}(1+\ell)^{-\ell/\nu_j} > \frac{j}{j+1},$$

and therefore $\liminf_n \alpha_n^{1/n} \geq 1$.

Remarks. It follows, for example, from [**7**, Theorem 2] that an injective unilateral weighted shift T has the decomposition property (δ) if and only if $\sigma(T) = \{0\}$, and in this case T is decomposable. If the sequence $(\alpha_{n+1}/\alpha_n)_{n\geq 0}$ is increasing [**11**, Section 7], then the corresponding shift T is hyponormal, and therefore has property (β) by [**9**]. By [**10**, Corollary 2], if

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the sequence $(\alpha_{n+1}/\alpha_n)_{n\geq 0}$ is convergent with limit r, then the corresponding shift has property (β) if $\left(r - \alpha_{n+1}/\alpha_n\right)_{n\geq 0} \in \ell^p$ for some p, with $1 \leq p < \infty$. Other than these cases, we do not know which shifts have Bishop's property (β).

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