



# The Saddle-Point Method and the Li Coefficients

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*Abstract.* In this paper, we apply the saddle-point method in conjunction with the theory of the Nörlund–Rice integrals to derive precise asymptotic formula for the generalized Li coefficients established by Omar and Mazhouda. Actually, for any function  $F$  in the Selberg class  $\mathcal{S}$  and under the Generalized Riemann Hypothesis, we have

$$\lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

with

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j},$$

where  $\gamma$  is the Euler’s constant and the notation is as below.

## 1 Introduction

Let us consider the xi-function  $\xi(s) = s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s)$  and the Li coefficients  $(\lambda_n)_{n \geq 1}$  defined by

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ s^{n-1} \log \xi(s) \right]_{s=1}.$$

Then the Li criterion says that the Riemann Hypothesis holds if and only if the coefficients  $(\lambda_n)$  are positive numbers. Bombieri and Lagarias [2] obtained an arithmetic expression for the Li coefficients  $\lambda_n$  and gave an asymptotic formula as  $n \rightarrow \infty$ . More recently, Maslanka [10] computed  $\lambda_n$  for  $1 \leq n \leq 3300$  and empirically studied the growth behavior of the Li coefficients. Coffey [3, 4] studied the arithmetic formula and established a lower bound for the Archimedean prime contribution by means of series rearrangements using the Euler–Maclaurin summation. In [11], a generalization of the Li criterion for functions  $F$  in the Selberg class was given, and in [13] an explicit formula for the Li coefficients associated to  $F$  was established.

The object of this paper is to derive a precise asymptotic formula for the generalized Li coefficients using the saddle-point method.

The Selberg class  $\mathcal{S}$  consists of Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \quad \Re(s) > 1$$

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satisfying the following hypotheses.

- **Analytic continuation:** there exists a non-negative integer  $m$  such that  $(s-1)^m F(s)$  is an entire function of finite order. We denote by  $m_F$  the smallest integer  $m$  that satisfies this condition.
- **Functional equation:** for  $1 \leq j \leq r$ , there are positive real numbers  $Q_j$ ,  $\lambda_j$  and there are complex numbers  $\mu_j$ ,  $\omega$  with  $\Re(\mu_j) \geq 0$  and  $|\omega| = 1$ , such that

$$\phi_F(s) = \omega \overline{\phi_F(1 - \bar{s})}$$

where

$$\phi_F(s) = F(s) Q_F^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j).$$

- **Ramanujan hypothesis:**  $a(n) = O(n^\epsilon)$ .
- **Euler product:**  $F(s)$  satisfies

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients  $b(p^k)$  satisfying  $b(p^k) = O(p^{k\theta})$  for some  $\theta < \frac{1}{2}$ .

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, *i.e.*, that all non trivial (non-real) zeros lie on the critical line  $\Re(s) = \frac{1}{2}$ . The degree of  $F \in \mathcal{S}$  is defined by

$$d_F = 2 \sum_{j=1}^r \lambda_j.$$

The degree is well defined (although the functional equation is not unique by Legendre’s duplication formula). The logarithmic derivative of  $F(s)$  also has the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s}, \quad \Re(s) > 1,$$

where  $\Lambda_F(n) = b(n) \log n$  is an analogue of the Von Mangoldt function  $\Lambda(n)$  defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If  $N_F(T)$  counts the number of zeros of  $F(s) \in \mathcal{S}$  in the rectangle  $0 \leq \Re(s) \leq 1$ ,  $|\Im(s)| \leq T$  (according to multiplicities), one can show by standard contour integration the formula

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_F T + O(\log T),$$

in analogy to the Riemann–Von Mangoldt formula for Riemann’s zeta-function  $\zeta(s)$ , the prototype of an element in  $\mathcal{S}$ . For more details concerning the Selberg class we refer to the survey of Kaczorowski and Perelli [6].

## 2 The Li Criterion

Let  $F$  be a function in the Selberg class non-vanishing at  $s = 1$  and let us define the xi-function  $\xi_F(s)$  by  $\xi_F(s) = s^{m_F}(s - 1)^{m_F} \phi_F(s)$ . The function  $\xi_F(s)$  satisfies the functional equation  $\xi_F(s) = \omega \overline{\xi_F(1 - \bar{s})}$ . The function  $\xi_F$  is an entire function of order 1. Therefore by the Hadamard product, it can be written as

$$\xi_F(s) = \xi_F(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

where the product is over all zeros of  $\xi_F(s)$  in the order given by  $|\Im(\rho)| < T$  for  $T \rightarrow \infty$ . Let  $\lambda_F(n)$ ,  $n \in \mathbb{Z}$ , be a sequence of numbers defined by a sum over the non-trivial zeros of  $F(s)$  as

$$\lambda_F(n) = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right],$$

where the sum over  $\rho$  is

$$\sum_{\rho} = \lim_{T \rightarrow \infty} \sum_{|\Im \rho| \leq T}.$$

These coefficients are expressible in terms of power-series coefficients of functions constructed from the  $\xi_F$ -function. For  $n \leq -1$ , the Li coefficients  $\lambda_F(n)$  correspond to the following Taylor expansion at the point  $s = 1$

$$\frac{d}{dz} \log \xi_F \left(\frac{1}{1 - z}\right) = \sum_{n=0}^{+\infty} \lambda_F(-n - 1) z^n,$$

and for  $n \geq 1$ , they correspond to the Taylor expansion at  $s = 0$

$$\frac{d}{dz} \log \xi_F \left(\frac{-z}{1 - z}\right) = \sum_{n=0}^{+\infty} \lambda_F(n + 1) z^n.$$

Let  $Z$  be the multi-set of zeros of  $\xi_F(s)$  (counted with multiplicity). The multi-set  $Z$  is invariant under the map  $\rho \mapsto 1 - \bar{\rho}$ . We have

$$1 - \left(1 - \frac{1}{\rho}\right)^{-n} = 1 - \left(\frac{\rho - 1}{\rho}\right)^{-n} = 1 - \left(\frac{-\rho}{1 - \rho}\right)^n = 1 - \overline{\left(1 - \frac{1}{1 - \bar{\rho}}\right)^n}$$

and this gives the symmetry  $\lambda_F(-n) = \overline{\lambda_F(n)}$ . Using the corollary in [2, Theorem 1], we get the following generalization of the Li criterion for the Riemann hypothesis.

**Theorem 2.1** *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  non-vanishing at  $s = 1$ . All non-trivial zeros of  $F(s)$  lie in the line  $\Re(s) = 1/2$  if and only if  $\Re(\lambda_F(n)) > 0$  for  $n = 1, 2, \dots$*

Next, we recall the following explicit formula for the coefficients  $\lambda_F(n)$ . Let consider the following hypothesis:

$\mathcal{H}$ : there exists a constant  $c > 0$  such that  $F(s)$  is non-vanishing in the region:

$$\left\{ s = \sigma + it; \sigma \geq 1 - \frac{c}{\log(Q_F + 1 + |t|)} \right\}.$$

**Theorem 2.2** Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  satisfying  $\mathcal{H}$ . Then we have

$$\begin{aligned} (2.1) \quad \lambda_F(-n) &= m_F + n \left( \log Q_F - \frac{d_F}{2} \gamma \right) \\ &\quad - \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k} (\log k)^{l-1} - \frac{m_F}{l} (\log X)^l \right\} \\ &\quad + n \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(\lambda_j + \mu_j)} \right) \\ &\quad - \sum_{j=1}^r \sum_{k=2}^n \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \left( \frac{1}{l + \lambda_j + \mu_j} \right)^k, \end{aligned}$$

where  $\gamma$  is the Euler constant.

**Examples**

- In the case of the Riemann zeta function,  $m_\zeta = 1$ ,  $Q_\zeta = \pi^{-1/2}$ ,  $r = 1$ ,  $\lambda_1 = \frac{1}{2}$ , and  $\mu_1 = 0$ . With the equality

$$(-1)^k \sum_{l=0}^{+\infty} \left( \frac{1}{2l+1} \right)^k = (-1)^k \left( 1 - \frac{1}{2^k} \right) \zeta(k),$$

we find  $\lambda_\zeta$ , which was established by Bombieri and Lagarias [2, p. 281].

- For the Hecke  $L$ -functions,  $Q_F = \frac{\sqrt{N}}{2\pi}$ ,  $m_F = 0$ ,  $\lambda_1 = 1$ , and  $\mu_1 = \frac{1}{2}$ , we find  $\lambda_E(n)$ , which was established by X.-J. Li [9, p. 496].

### 3 Saddle-Point Method and the Nörlund–Rice Integrals

Given a complex integral with a contour traversing simple saddle-point, the saddle-point corresponds locally to a maximum of the integrand along the path. It is then natural to expect that a small neighborhood of the saddle-point might provide the dominant contribution to the integral. The saddle-point method is applicable precisely when this is the case and when this dominant contribution can be estimated by means of local expansions. The method then constitutes the complex-analytic counterpart of Laplace’s method for evaluating real integrals depending on a large parameter, and we can regard it as being

Saddle-point method = Choice of contour + Laplace’s method.

To estimate  $\int_A^B F(z) dz$ , it is convenient to set  $F(z) = e^{f(z)}$ , where  $f(z) \equiv f_n(z)$ , involves some large parameter  $n$ . We chose a contour  $\mathcal{C}$  through a saddle-point  $\eta$  such

that  $f'(\eta) = 0$ . Next, we split the contour as  $\mathcal{C} = \mathcal{C}^{(0)} \cup \mathcal{C}^{(1)}$ , and the following conditions are to be verified.

(i) On the contour  $\mathcal{C}^{(1)}$  the tails integral  $\int_{\mathcal{C}^{(1)}} F(z) dz$  is negligible

$$\int_{\mathcal{C}^{(1)}} F(z) dz = O\left(\int_{\mathcal{C}} F(z) dz\right).$$

(ii) Along  $\mathcal{C}^{(0)}$ , a quadratic expansion,

$$f(z) = f(\eta) + \frac{1}{2} f''(\eta)(z - \eta)^2 + O(\phi_n)$$

is valid, with  $\phi_n \rightarrow 0$  as  $n \rightarrow \infty$ , uniformly with respect to  $z \in \mathcal{C}^{(0)}$ .

(iii) The incomplete Gaussian integral taken over the central range is asymptotically equivalent to a complete Gaussian integral with ( $\epsilon = \pm 1$ ):

$$\int_{\mathcal{C}^{(0)}} e^{\frac{1}{2} f''(\eta)(z-\eta)^2} dz \sim \epsilon i \int_{-\infty}^{+\infty} e^{-|f''(\eta)| \frac{x^2}{2}} dx \equiv \epsilon i \sqrt{\frac{2\pi}{|f''(\eta)|}}.$$

Assuming (i), (ii), and (iii), one has, with  $\epsilon = \pm 1$

$$\frac{1}{2\pi} \int_A^B e^{f(z)} dz \sim \epsilon \frac{e^{f(\eta)}}{\sqrt{2\pi f''(\eta)}}.$$

This method is the main tool to prove our result. We finish this section by reviewing the definition of the Nörlund–Rice integral.

**Lemma 3.1** *Let  $f(s)$  be holomorphic in the half-plane  $\Re(s) \geq \eta_0 - \frac{1}{2}$ . Then the finite differences of the sequence  $(f(k))$  admit the integral representation*

$$\sum_{k=n_0}^n \binom{n}{k} (-1)^k f(k) = \frac{(-1)^n}{2i\pi} \int_{\mathcal{C}} f(s) \frac{n!}{s(s-1)\cdots(s-n)} ds,$$

where the contour of integration  $\mathcal{C}$  encircles the integers  $\{n_0, \dots, n\}$  in a positive direction and is contained in  $\Re(s) \geq \eta_0 - \frac{1}{2}$ .

**Proof** The integral on the right is the sum of its residues at  $s = n_0, \dots, n$ , which precisely equals the sum on the left. ■

### 4 Asymptotic Formula for the Li Coefficients

A natural problem is to determine the asymptotic behavior of the numbers  $\lambda_F(n)$ . Our main result in this paper is stated in the following theorem.

**Theorem 4.1** *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$ . Then, under the Generalized Riemann Hypothesis, we have*

$$\lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

where

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$$

and  $\gamma$  is the Euler constant.

**Remark 4.2** We conjecture that the asymptotic formula for the numbers  $\lambda_F(n)$  in Theorem 4.1 holds for any function in the Selberg class without any assumption.

For our purpose, it is sufficient to study sums of the form

$$(4.1) \quad H_n(m, k) = \sum_{l=2}^n (-1)^l \binom{n}{l} \frac{\zeta(l, \frac{m}{k})}{k^l},$$

where  $\zeta(s, q)$  is the Hurwitz zeta function given by

$$\zeta(s, q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}.$$

**Proposition 4.3**  $H_n(m, k)$ , defined by (4.1), satisfy the estimate

$$H_n(m, k) = \left(\frac{m}{k} - \frac{1}{2}\right) - \frac{n}{k} \left(\psi\left(\frac{m}{k}\right) + \log k + 1 - h_{n-1}\right) + a_n(m, k),$$

where the  $a_n(m, k)$  are exponentially small:

$$a_n(m, k) = \frac{1}{k} \left(\frac{2n}{\pi k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi n}{k}}\right) \cos\left(\sqrt{\frac{4\pi n}{k}} - \frac{5\pi}{8} - \frac{2\pi m}{k}\right) + O\left(n^{-1/4} e^{-2\sqrt{\frac{\pi n}{k}}}\right).$$

Here,  $h_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$  is a harmonic number, and  $\psi(x)$  is the logarithm derivative of the Gamma function.

**Proof** Convert the sum to the Nörlund–Rice integral, and extend the contour to the half-circle at positive infinity. The half-circle does not contribute to the integral. One obtains

$$H_n(m, k) = \frac{(-1)^n}{2i\pi} n! \int_{3/2-i\infty}^{3/2+i\infty} \frac{\zeta(s, \frac{m}{k})}{k^l s(s-1) \cdots (s-n)} ds.$$

Moving the integral to the left, one encounters a single pole at  $s = 0$  and a pole at  $s = 1$ . The residue of the pole at  $s = 0$  is

$$\text{Res}(s = 0) = \zeta\left(0, \frac{m}{k}\right) = -\frac{1}{k\pi} \sum_{l=1}^k \sin\left(\frac{2\pi lm}{k}\right) \psi\left(\frac{l}{k}\right) = -B_1\left(\frac{l}{k}\right) = \frac{1}{2} - \frac{m}{k},$$

where  $\psi$  is the digamma function,  $B_1$  is the Bernoulli polynomial of order 1, and

$$\text{Res}(s = 1) = \frac{n}{k} \left( \psi \left( \frac{m}{k} \right) + \log k + 1 - h_{n-1} \right).$$

Then we obtain

$$H_n(m, k) = \left( \frac{m}{k} - \frac{1}{2} \right) - \frac{n}{k} \left( \psi \left( \frac{m}{k} \right) + \log k + 1 - h_{n-1} \right) + a_n(m, k),$$

where

$$a_n(m, k) = O \left( e^{-\sqrt{kn}} \right)$$

for a constant  $K$  of order  $m/k$ . Indeed we have

$$a_n(m, k) = \frac{(-1)^n}{2i\pi} n! \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\zeta \left( s, \frac{m}{k} \right)}{k^l s(s-1) \cdots (s-n)}.$$

Recall that the Hurwitz zeta function satisfies the following functional equation

$$\zeta \left( 1-s, \frac{m}{k} \right) = \frac{2\Gamma(s)}{(k\pi k)^s} \sum_{l=1}^k \cos \left( \frac{\pi s}{2} - \frac{2\pi lm}{k} \right) \zeta \left( s, \frac{l}{k} \right).$$

Therefore,

(4.2)

$$\begin{aligned} a_n(m, k) &= -\frac{n!}{2ki\pi} \sum_{l=1}^k \int_{3/2-i\infty}^{3/2+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} \cos \left( \frac{\pi s}{2} - \frac{2\pi lm}{k} \right) \zeta \left( s, \frac{l}{k} \right) ds \\ &= -\frac{n!}{2ki\pi} \sum_{l=1}^k e^{i\frac{2\pi lm}{k}} \int_{3/2-i\infty}^{3/2+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} e^{-i\frac{\pi s}{2}} \zeta \left( s, \frac{l}{k} \right) ds \\ &\quad - \frac{n!}{2ki\pi} \sum_{l=1}^k e^{-i\frac{2\pi lm}{k}} \int_{3/2-i\infty}^{3/2+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} e^{i\frac{\pi s}{2}} \zeta \left( s, \frac{l}{k} \right) ds. \end{aligned}$$

For large values of  $n$ , those integrals will be evaluated by means of the saddle-point method. Note that the integrand in (4.2) has a minimum, on the real axis, near  $s = \sigma_0 = \sqrt{2ln/k}$ , and so the appropriate parameter is  $z = s/\sqrt{n}$ . Change  $s$  by  $z$ , and take  $z$  constant and  $n$  large. Then

$$(4.3) \quad a_n(m, k) = -\frac{1}{2i\pi} \sum_{l=1}^k k \left\{ e^{i\frac{2\pi lm}{k}} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{f(z)} dz + e^{-i\frac{2\pi lm}{k}} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\bar{f}(z)} dz \right\}.$$

We have

$$f(z) = \log n! + \frac{1}{2} \log n + \phi(z\sqrt{n}),$$

with

$$\phi(s) = -s \log\left(\frac{2\pi l}{k}\right) - i \frac{\pi s}{2} + \log\left(\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)}\right) + O\left(\left(\frac{l}{k+l}\right)^s\right),$$

using the approximation

$$\zeta(s, l/k) = (k/l)^s + O\left(\left(\frac{l}{k+l}\right)^s\right)$$

for large  $s$ . Furthermore,

$$\log \zeta(s) = \sum_{n=2}^{+\infty} \frac{\Lambda(n)}{n^s \log n},$$

where  $\Lambda(n)$  is the Von-Mangoldt function. The asymptotic expansion for the Gamma function is given by the Stirling expansion

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{j=1}^{+\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}},$$

where  $B_k$  are the Bernoulli numbers. Expanding to  $O(1/n)$  and collecting terms, we deduce

$$\begin{aligned} f(z) &= \frac{1}{2} \log n - z\sqrt{n} \left( \log\left(\frac{2\pi l}{k}\right) + i \frac{\pi}{2} + 2 - 2 \log z \right) \\ &\quad + \log(2\pi) - 2 \log z - \frac{z^2}{2} + \frac{1}{6z\sqrt{n}}(10 + z^2) \\ &\quad + \frac{1}{2n} \left( 1 - \frac{z^2}{2} - \frac{z^4}{6} + \frac{73}{72z^2} \right) + O(n^{-3/2}). \end{aligned}$$

The saddle-point is obtained by solving the equation  $f'(z) = 0$ , and we have

$$z_0 = (1 + i)\sqrt{\frac{\pi l}{k}}.$$

We need  $f''(z) = 2\sqrt{n}/z + O(1)$  to use the saddle-point formula. Substituting, we obtain

$$(4.4) \quad \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{f(z)} dz = \left(\frac{2\pi^3 l n}{k}\right)^{1/4} e^{\frac{i\pi}{8}} \exp\left(- (1 + i)\sqrt{\frac{4\pi l n}{k}}\right) + O\left(n^{-1/4} e^{-2\sqrt{\frac{\pi l n}{k}}}\right).$$



The integral for  $\bar{f}$  is the complex conjugate of (4.4) (having a saddle-point at the complex conjugate  $\bar{z}_0$ ). Finally, equations (4.3) and (4.4) together give

$$a_n(m, k) = \frac{1}{k} \left(\frac{2n}{\pi}\right)^{1/4} \sum_{l=1}^k \left(\frac{l}{k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi ln}{k}}\right) \cos\left(\sqrt{\frac{4\pi ln}{k}} - \frac{5\pi}{8} - \frac{2\pi lm}{k}\right) + O\left(n^{-1/4} e^{-2\sqrt{\frac{\pi n}{k}}}\right).$$

For large  $n$ , only the  $l = 1$  term contributes significantly, and so

$$a_n(m, k) = \frac{1}{k} \left(\frac{2n}{\pi k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi n}{k}}\right) \cos\left(\sqrt{\frac{4\pi n}{k}} - \frac{5\pi}{8} - \frac{2\pi m}{k}\right) + O\left(n^{-1/4} e^{-2\sqrt{\frac{\pi n}{k}}}\right),$$

which means that the terms  $a_n$  are exponentially small. ■

**Proof of Theorem 4.1** Without loss of generality, we assume that  $\mu_j$  is a real number. First, write the arithmetic formula of  $\lambda_F(-n)$  (equation (2.1)) as

$$(4.5) \quad \lambda_F(-n) = m_F + n\left(\log Q_F - \frac{d_F}{2}\gamma\right) - \sum_{l=1}^n \binom{n}{l} \eta_F(l-1) + n \sum_{j=1}^r \lambda_j \left(-\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(\lambda_j + \mu_j)}\right) - \sum_{j=1}^r I_j,$$

where

$$\eta_F(l) = \frac{(-1)^l}{l!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k} (\log k)^l - \frac{m_F}{l+1} (\log X)^{l+1} \right\}$$

are the generalized Stieltjes constants and

$$I_j = \sum_{k=2}^n \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \left(\frac{1}{l + \lambda_j + \mu_j}\right)^k.$$

Note that

$$I_j^{(1)} = \sum_{k=2}^n \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \frac{1}{(l + \lambda_j + \mu_j)^k} = \sum_{k=2}^n \binom{n}{k} (-1)^k \frac{\zeta(k, \lambda_j + \mu_j)}{(\lambda_j^{-1})^k},$$

which, with the above notation of  $H_n(m, k)$  (equation (4.1)), is equal to

$$I_j^{(1)} = H_n\left(1 + \frac{\mu_j}{\lambda_j}, \lambda_j^{-1}\right).$$

Applying Proposition 4.3 with  $m = 1 + \frac{\mu_j}{\lambda_j}$  and  $k = \lambda_j^{-1}$ , we deduce

$$(4.6) \quad I_j = \left( \lambda_j + \mu_j - \frac{1}{2} \right) - n\lambda_j \left( \psi(\lambda_j + \mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1} \right) + a_n \left( 1 + \frac{\mu_j}{\lambda_j}, \lambda_j^{-1} \right),$$

where

$$a_n \left( 1 + \frac{\mu_j}{\lambda_j}, \lambda_j^{-1} \right) = \lambda_j \left( \frac{2n}{\pi} \lambda_j \right)^{1/4} \exp(-\sqrt{4\pi n \lambda_j}) \cos \left( \sqrt{4\pi n \lambda_j} - \frac{5\pi}{8} - 2\pi(\lambda_j + \mu_j) \right) + O \left( n^{-1/4} e^{-2\sqrt{\pi n \lambda_j}} \right).$$

The  $a_n$  are exponentially small, then

$$(4.7) \quad a_n \left( 1 + \frac{\mu_j}{\lambda_j}, \lambda_j^{-1} \right) = O(1).$$

From (4.6) and (4.7), we obtain

$$(4.8) \quad I_j = \left( \lambda_j + \mu_j - \frac{1}{2} \right) - n\lambda_j \left\{ \psi(\lambda_j + \mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1} \right\} + O(n).$$

Summing (4.8) over  $j$ , we get

$$(4.9) \quad \sum_{j=1}^r I_j = \sum_{j=1}^r \left( \lambda_j + \mu_j - \frac{1}{2} \right) - n \sum_{j=1}^r \lambda_j \left\{ \psi(\lambda_j + \mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1} \right\} + O(n).$$

Using the expression

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{l=1}^{+\infty} \frac{z}{l(l+z)},$$

where  $\gamma$  is the Euler constant, and the estimate

$$h_n = \log n - \gamma + \frac{1}{2n} + O \left( \frac{1}{2n^2} \right),$$

we deduce from (4.5) and (4.9) that

$$\lambda_F(-n) = \left( \sum_{j=1}^r \lambda_j \right) n \log n + \left\{ \left( \sum_{j=1}^r \lambda_j \right) (\gamma - 1) + \log Q_F + \sum_{j=1}^r \lambda_j \log \lambda_j \right\} n - \sum_{l=1}^n \binom{n}{l} \eta_F(l-1) + O(n).$$

Recalling that  $d_F = \sum_{j=1}^r \lambda_j$  and noting that  $\lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$ , we have

$$(4.10) \quad \lambda_F(-n) = \frac{d_F}{2} n \log n + \left\{ \frac{d_F}{2}(\gamma-1) + \frac{1}{2} \log(\lambda Q_F^2) \right\} n - \sum_{l=1}^n \binom{n}{l} \eta_F(l-1) + O(n).$$

Now, we obtain a bound for  $S_F(n) = -\sum_{l=1}^n \binom{n}{l} \eta_F(l-1)$  in terms of

$$\lambda_F(-n, T) := \sum_{\rho: |\Im \rho| \leq T} 1 - \left(1 - \frac{1}{\rho}\right)^n,$$

where  $T$  is a parameter.

**Lemma 4.4** *If the Generalized Riemann Hypothesis holds for  $F \in \mathcal{S}$ , then*

$$S_F(n) = O(\sqrt{n} \log n).$$

**Proof** The proof is analogous to the argument used by Lagarias in [7]. We use a contour integral argument, and we introduce the kernel function

$$k_n := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{l=1}^n \binom{n}{l} \left(\frac{1}{s}\right)^l.$$

If  $C$  is a contour enclosing the point  $s = 0$  counterclockwise on a circle of small enough positive radius, the residue theorem gives

$$I(n) = \frac{1}{2i\pi} \int_C k_n(s) \left(-\frac{F'}{F}(s+1)\right) ds = \sum_{l=1}^n \binom{n}{l} \eta_{l-1} = S_F(n).$$

We deform the contour to the counterclockwise oriented rectangular contour  $C'$  consisting of vertical lines with real part  $\Re(s) = \sigma_0$  and  $\Re(s) = \sigma_1$ , where we will choose  $-3 < \sigma_0 < -2$ ,  $\sigma_1 = 2\sqrt{n}$  and the horizontal lines at  $\Im(s) = \pm T$ , where we will choose  $T = \sqrt{n} + \epsilon_n$  for some  $0 < \epsilon_n < 1$ . The residue theorem gives

$$\begin{aligned} I'(n) &= \frac{1}{2i\pi} \int_{C'} k_n(s) \left(-\frac{F'}{F}(s+1)\right) ds \\ &= S_F(n) + \sum_{\rho: |\Im \rho| \leq T} \left(1 + \frac{1}{\rho-1}\right)^n - 1 + O(1). \end{aligned}$$

The term  $O(1)$  evaluates the residues coming from the trivial zeros of  $F(s)$ . Using the symmetry  $\rho \mapsto 1 - \bar{\rho}$ , we can write

$$\left(\frac{1 - \bar{\rho}}{-\bar{\rho}}\right)^n - 1 = \left(\frac{\bar{\rho} - 1}{\bar{\rho}}\right)^n - 1.$$

Then

$$I'(n) = S_F(n) - \lambda_F(-n, T) + O(1).$$

We have

$$|\lambda_F(-n, \sqrt{n}) - \lambda_F(-n, T)| = O(\log n).$$

This follows from the observation that  $|T - \sqrt{n}| < 1$ , that there are  $O(\log n)$  zeros in an interval of length one at this height, and that for each zero  $\rho = \beta + i\gamma$  with  $\sqrt{n} \leq |\Im(\rho)| < \sqrt{n} + 1$  there holds

$$\left| \left( \frac{\rho - 1}{\rho} \right) \right| \leq \left| 1 + \frac{1}{n} \right|^{n/2} \leq 2.$$

We now choose the parameters  $\sigma_0$  and  $T$  appropriately to avoid the poles of the integrand. We may choose  $\sigma_0$  so that the contour avoids any trivial zero and  $T = \sqrt{n} + \epsilon_n$  with  $0 \leq \epsilon_n \leq 1$  so that the horizontal lines do not approach closer than  $O(\log n)$  to any zero of  $F(s)$ . Recall from [16] that for  $-2 < \Re(s) < 2$  there holds

$$\frac{F'}{F}(s) = \sum_{\{\rho; |\Im(\rho-s)| < 1\}} \frac{1}{s - \rho} + O(\log(Q_F(1 + |s|))).$$

Then on the horizontal line in the interval  $-2 \leq \Re(s) \leq 2$ , we have

$$\left| \frac{F'}{F}(s + 1) \right| = O(\log^2 T).$$

The Euler product for  $F(s)$  converges absolutely for  $\Re(s) > 1$ , hence the Dirichlet series for  $\frac{F'}{F}(s)$  converges absolutely for  $\Re(s) > 1$ . More precisely, for  $\sigma = \Re(s) > 1$

$$\left| \frac{F'}{F} \right|(\sigma) < \infty.$$

For  $\sigma = \Re(s) > 2$ , we obtain the bound

$$\left| \frac{F'}{F}(s) \right| \leq \left| \frac{F'}{F} \right|(\sigma) \leq 2^{-(\sigma-2)}.$$

Consider the integral  $I'(n)$  on the vertical segment ( $L_1$ ) having  $\sigma_1 = 2\sqrt{n}$ . We have

$$\left| \left( 1 - \frac{1}{s} \right)^n - 1 \right| \leq \left( 1 + \frac{1}{\sigma_1} \right)^n + 1 \leq \left( 1 + \frac{1}{2\sqrt{n}} \right)^n \leq \exp(\sqrt{n}/2) < 2^{\sqrt{n}}.$$

Then

$$\left| \frac{F'}{F}(s) \right| \leq C_0 2^{-2(\sqrt{n}+2)}.$$

Furthermore, the length of the contour is  $O(\frac{n}{\log n})$ , and we obtain  $|I'_{L_1}| = O(1)$ . Let  $s = \sigma + it$  be a point on one of the two horizontal segments. We have  $T \geq \sqrt{n}$ , so that

$$\left| 1 + \frac{1}{s} \right| \leq 1 + \frac{\sigma + 1}{\sigma^2 + T^2}.$$

By hypothesis  $T^2 \geq n$ , so for  $-2 \leq \sigma \leq 2$ , we have

$$|k_n(s)| \leq \left(1 + \frac{3}{4+n}\right)^n + 1 = O(1)$$

and

$$\left|\frac{F'}{F}(s)\right| = O(\log^2 T) = O(\log^2 n),$$

since we have chosen the ordinate  $T$  to stay away from zeros of  $F(s)$ . We step across the interval  $(L_2)$  toward the right, in segments of length 1, starting from  $\sigma = 2$ . Furthermore,

$$\left|\frac{k_n(s+1)+1}{k_n(s)+1}\right| \leq \left(1 + \frac{1}{T^2}\right)^n \leq e,$$

and we obtain an upper bound for  $|k_n(s)\frac{F'}{F}(s)|$  that decreases geometrically at each step. After  $O(\log n)$  steps it becomes  $O(1)$ , and the upper bound is

$$|I'_{L_2, L_4}(n)| = O(\log^2 n + \sqrt{n}) = O(\sqrt{n}).$$

For the vertical segment  $(L_3)$  with  $\Re(s) = \sigma_0$ , we have  $|k_n(s)| = O(1)$  and  $|\frac{F'}{F}(s)| = O[Q_F(\log(|s| + 1))]$ . Since the segment  $(L_3)$  has length  $O(\sqrt{n})$ , we obtain

$$|I'_{L_3}| = O(\sqrt{n} \log n).$$

Totalling the above bounds gives

$$S_F(n) = \lambda_F(-n, T) + O(\sqrt{n} \log n),$$

with  $T = \sqrt{n} + \epsilon_n$ . If the Generalized Riemann Hypothesis holds for  $F(s)$ , then we have  $|1 - \frac{1}{\rho-1}| = 1$ . Since each zero contributes a term of absolute value at most 2 to  $\lambda_F(-n, T)$ , we obtain using the zero density estimate ( $N_F(T) \sim T \log T$ )

$$\lambda_F(-n, T) = O(T \log T + 1).$$

Therefore  $\lambda_F(-n, \sqrt{n}) = O(\sqrt{n} \log n)$ , and Lemma 4.4 follows. ■

Using Lemma 4.4 and the expression (4.10) of  $\lambda_F(-n)$  and  $\lambda_F(-n) = \overline{\lambda_F(n)}$ , we obtain

$$\lambda_F(n) = \frac{d_F}{2} n \log n + \left\{ \frac{d_F}{2} (\gamma - 1) + \frac{1}{2} \log(\lambda Q_F^2) \right\} n + O(\sqrt{n} \log n),$$

which concludes the proof of Theorem 4.1. ■

**Examples**

- In the case of the Riemann zeta function, we have  $d_\zeta = 1$ ,  $Q_\zeta = \pi^{-1/2}$ , and  $\lambda = \frac{1}{2}$ . This proves again under the Riemann Hypothesis the asymptotic formula established by A. Voros in [17, equation (17), p. 59].

- Also, in the case of the principal  $L$ -function  $L(s, \pi)$  attached to an irreducible cuspidal unitary automorphic representation of  $GL(N)$ , as in Rudnick and Sarnak [14, §2], we have  $D_L = N$ ,  $Q_L = Q(\pi)\pi^{-N/2}$ , and  $\lambda = 2^{-n}$ . We find under the Generalized Riemann Hypothesis the asymptotic formula for  $\lambda_n(\pi)$  established by Lagarias in [7, equations (1.12) and (1.13), p. 4].

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