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ON THE REGULARITY OF SLE TRACE

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Abstract

We revisit regularity of SLE trace, for all $\kappa \neq 8$, and establish Besov regularity under the usual half-space capacity parametrization. With an embedding theorem of Garsia–Rodemich–Rumsey type, we obtain finite moments (and hence almost surely) optimal variation regularity with index $\min(1 + \kappa/8, 2)$, improving on previous works of Werness, and also (optimal) Hölder regularity à la Johansson Viklund and Lawler.

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1. Introduction

It is classical that SLE_κ has a.s. continuous trace γ, any $\kappa \in (0, \infty)$. (The trivial case $\kappa = 0$ will be disregarded throughout.) With the exception of $\kappa = 8$, the (classical) proof has two steps: (1) estimates of moments of $|\hat{f}'_t(iy)|$ the derivative of the shifted inverse (Loewner) flow and (2) partition of (t, y)-space into Whitney-type boxes, together with a Borel–Cantelli argument. This strategy of proof is very standard (see [**RS05**, **Law05**] or also [**JVRW14**]) and was subsequently refined in [**JVL11**] and [**Lin08**] to show that SLE_κ (always in half-space parametrization) is Hölder continuous on any compact set in $(0, \infty)$ with any Hölder exponent less than

$$\alpha_*(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{\kappa + 8}};$$

on compact sets in $[0, \infty)$, the critical exponent has to be modified to

$$\alpha_0(\kappa) = \min(\alpha_*(\kappa), 1/2).$$

On the other hand, based on Aizenman–Burchard techniques, it was shown in [**Wer12**], under the (technical) condition $\kappa \leqslant 4$, that SLE enjoys p-variation regularity for any

$$p < p_* = 1 + \frac{\kappa}{8}.$$

Loosely stated, the main result of this paper is a Besov regularity for SLE of the form.

THEOREM 1.1. *Assume $\kappa > 0$, and fix $T > 0$. Then, for suitable $\delta \in (0, 1), q > 1$ we have*

$$\mathbb{E}\|\gamma\|^q_{W^{\delta,q};[0,T]} < \infty.$$

In particular, for a.e. realization of SLE trace, we have $\gamma(\omega)|_{[0,T]} \in W^{\delta,q}$.

For the (classical) definition of the Besov space $W^{\delta,q}$, see (4.1) below; we also postpone the important precise description of possible values δ, q. In the spirit of step (1), cf. the discussion at the very beginning of the introduction, moment estimates on $|\hat{f}'_t(iy)|$, taken in the sharp form of [**JVL11**], are an important ingredient in establishing Theorem 1.1. In turn, this theorem unifies and extends previous works [**JVL11**, **Wer12**] by exhibiting Besov regularity as common source of both (optimal) Hölder *and* p-variation regularity for SLE trace. More specifically, we have the following first corollary which recovers the optimal Hölder exponent of SLE trace, as previously established in [**JVL11**] and in fact improves their almost-sure statement to finite moments.

COROLLARY 1.2. *Assume $\kappa \neq 8$. On compact sets in $[0, \infty)$ (respectively $(0, \infty)$), the α-Hölder norm of SLE curve with $\alpha < \alpha_0$ (respectively $\alpha < \alpha_*$) has finite q-moment, for some $q > 1$.*

The following extends [**Wer12**] (from $\kappa \leqslant 4$) to all $\kappa \neq 8$, and again improves from a.s.-finiteness to existence of moments. As this paper neared completion, we learned that Lawler and Werness [**LW**] found an independent proof of a.s. finite p-variation of SLE trace (for $\kappa < 8$).

COROLLARY 1.3. *Assume $\kappa \neq 8$. On compact sets in $[0, \infty)$ the p-variation norm of SLE curve with $p > p_*$ has finite q-moment, for some $q > 1$.*

As a further corollary, precise statement left to Corollary 5.3, we note that our p-variation result implies an upper bound on the Hausdorff dimension of the SLE trace, along the lines of the original Rohde–Schramm paper. Our approach should also allow to bypass step (2), cf. the discussion in the very beginning, in proving continuity of SLE trace, but we do not focus on this aspect here.

We give some discussion about the basic ideas. We note that every α-Hölder continuous curve is automatically of finite p-variation, with $p = 1/\alpha$. In some prominent cases, this yields the correct (optimal) p-variation regularity: for instance, Brownian motion is $(1/2 - \varepsilon)$-Hölder and then of finite $(2 + \varepsilon)$-variation, $\varepsilon > 0$. However, already for elements in the Cameron–Martin space $W^{1,2}[0, 1]$, that is, absolutely continuous curves h with $\int_0^1 |\dot{h}|^2\, dt < \infty$, this fails: in general, such an h is $1/2$-Hölder (hence automatically of 2-variation) but in fact has finite 1-variation. The same phenomena is seen for SLE,

$$p_* < \frac{1}{\alpha_*} \leqslant \frac{1}{\alpha_0}.$$

Let us also note that, from the stochastic point of view, the half-space parametrization is not fully satisfactory as it induces an artificial, directed view on SLE. A decisive advantage of p-variation is its invariance with respect to reparametrization, related at least in spirit to the natural parametrization introduced in [**LR15**, **LS11**]. The above Cameron–Martin example in fact holds a key message: Sobolev regularity is ideally suited to guarantee both α-Hölder and p-variation regularity with $p < 1/\alpha$. More quantitatively, the (elementary) embedding

$$W^{1,2} \subset C^{\text{1-var}} \cap C^{1/2\text{-Höl}}$$

(always on compacts sets in $[0, \infty)$) has been generalized in [**FV06**], by a delicate application of the Garsia–Rodemich–Rumsey inequality, to Besov spaces as follows:

THEOREM 1.4 (Besov-variation embedding). *Assume $\delta \in (0, 1)$, $q \in (1, \infty)$ such that $\delta - 1/q > 0$. Set $p := 1/\delta$, $\alpha := \delta - 1/q$. Then there exists a constant C, such that for all $0 \leqslant s < t < \infty$,*

$$\|x\|_{p\text{-var};[s,t]} \leqslant C|t - s|^{\alpha} \|x\|_{W^{\delta,q};[s,t]}. \tag{1.1}$$

The estimates holds for arbitrary continuous paths x, even with values in general metric spaces; for us, of course, x takes values in $\mathbb{C}$. Note that the left-hand side dominates the increment $x_{s,t}$, so that the following (classical) Besov–Hölder embedding appears as immediate consequence,

$$\|x\|_{\alpha\text{-Höl};[s,t]} \leqslant C \|x\|_{W^{\delta,q};[s,t]}. \tag{1.2}$$

The point of Theorem 1.4 is the 'gain' $p < 1/\alpha$, which can be substantial for integrability parameter $q \ll \infty$. For instance, $W^{\delta,2}$ ($q = 2$) is closely related to the Cameron–Martin space of fractional Brownian motion (fBm) in the rough regime with Hurst parameter $H \in (0, 1/2)$; in this case $\delta = H + 1/2$ and the above implies that all such paths fall into the reign of Young integration (which requires p-variation with $p < 2$), which is certainly not implied by $(H - \varepsilon)$-Hölder regularity of such Cameron–Martin paths. (This was a crucial ingredient in the development of Malliavin calculus for rough differential equations driven by fBm, see e.g. [**CF10**, **CHLT15**].)

On the other hand, the gain $1/\alpha - p = O(1/q)$ vanishes when $q \uparrow \infty$, which is exactly the reason why Brownian motion, which has q-moments for all $q < \infty$, has p-variation regularity no better than what is implied by its α-Hölder regularity. This, however, is not the case for SLE and our starting point is precisely to estimate q-moments for increments of SLE curves which in turn leads to a.s. $W^{\delta,q}$-regularity (and actually some finite moments of these Besov norms). By careful booking-keeping, and optimizing over, the possible choices of δ, q (for given κ) we then obtain the desired variation and Hölder regularity of SLE. Surprisingly perhaps, the entire regularity proof is channelled through knowledge of moments of the increments of the curve (very much in the spirit of Kolmogorovs criterion – which in fact is little more than the embedding $W^{\delta,q} \subset C^{\alpha\text{-Höl}}$). At last, our work suggests a viable new route towards (an analytic proof) for existence of SLE trace when $\kappa = 8$, for the Garsia–Rodemich–Rumsey based proof in [**FV06**] offers the flexibility to go beyond the Besov scale and for example, allows to deal with logarithmic modulus, to be pursued elsewhere.

2. Moments of the derivative of the inverse flow

Fix $\kappa \in (0, \infty)$. Let $U_t = \sqrt{\kappa} B_t$, where B is a standard Brownian motion. Let (g_t) be the downward flow SLE, that is, solutions to

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z \quad \text{for } z \in \mathbb{H},$$

and let $f_t = g_t^{-1}$ and $\hat{f}_t(z) = f_t(z + U_t)$. Let γ be the SLE$_\kappa$ curve. It follows from [**RS05**] and [**LSW04**] that a.s. for all $t \geqslant 0$,

$$\gamma(t) = \lim_{u \to 0^+} \hat{f}_t(iu).$$

Suppose

$$-\infty < r < r_c := \frac{1}{2} + \frac{4}{\kappa}$$

$$q := q(r) = r\left(1 + \frac{\kappa}{4}\right) - \frac{\kappa r^2}{8}$$

$$\zeta := \zeta(r) = r - \frac{\kappa r^2}{8}.$$

Through out this note, we always assume $r < r_c$. Note that q is strictly increasing with r on an interval which contains $(-\infty, r_c)$. The following moment estimate will be important.

LEMMA 2.1. *For fixed $\kappa \in (0, \infty)$, there exists a constant $c < \infty$ depending on r such that for all $s, y \in (0, 1]$,*

$$\mathbb{E}(|\hat{f}_s'(iy)|^q) \leqslant \begin{cases} cs^{-\zeta/2}y^\zeta & \text{when } s \geqslant y^2, \\ cA_s y^\zeta & \text{in general,} \end{cases}$$

where $A_s = \max(s^{-\zeta/2}, 1)$.

This is just a corollary of [**JVL11**, Lemma 4.1] in which they prove that for all $t \geqslant 1$,

$$\mathbb{E}[|\hat{f}_{t^2}'(i)|^q] \leqslant ct^{-\zeta}.$$

When $t \in (0, 1]$, the Koebe distortion theorem implies there is a constant c such that

$$|\hat{f}_{t^2}'(i)| \leqslant c.$$

By the scaling property of SLE

$$\hat{f}_s'(iy) \overset{(d)}{=} \hat{f}_{s/y^2}'(i),$$

hence

$$\mathbb{E}(|\hat{f}_s'(iy)|^q) = \mathbb{E}(|\hat{f}_{s/y^2}'(i)|^q) \leqslant \begin{cases} cs^{-\zeta/2}y^\zeta & \text{when } s \geqslant y^2, \\ c \leqslant cs^{-\zeta/2}y^\zeta & \text{when } s \leqslant y^2 \text{ and } \zeta > 0, \\ c \leqslant cy^\zeta & \text{when } s \leqslant y^2 \text{ and } \zeta \leqslant 0. \end{cases}$$

We also make use of the following two lemmas.

LEMMA 2.2 [**JVL11**, Lemma 3.5]. *There exists a constant C depending only on κ such that if $0 \leqslant t - s \leqslant y^2$ where $y = \text{Im}(z)$, then*

$$|f_t(z) - f_s(z)| \lesssim y|f_s'(z)|.$$

LEMMA 2.3 [**Law12**, Exercise 4]. *There exist $C > 0$ and $l > 1$ such that if $h : \mathbb{H} \to \mathbb{C}$ is a conformal transformation, then for all $x \in \mathbb{R}$, $y > 0$,*

$$|h'(xy + iy)| \leqslant C(x^2 + 1)^l |h'(iy)|.$$

3. Moment estimates for SLE increments

We prepare our statement with defining some set of 'suitable' r's. Unless otherwise stated, we always assume $\kappa \in (0, \infty)$.

$$I_0 := I_0(\kappa) := \{r \in \mathbb{R} : r < r_c\} \quad \text{with } r_c \equiv \frac{1}{2} + \frac{4}{\kappa},$$

$$I_1 := I_1(\kappa) := \{r \in \mathbb{R} : q > 1\} \quad \text{with } q = q(r) = \left(1 + \frac{\kappa}{4}\right)r - \frac{\kappa r^2}{8},$$

$$I_2 := I_2(\kappa) := \{r \in \mathbb{R} : q + \zeta > 0\} \quad \text{with } \zeta = \zeta(r) = r - \frac{\kappa r^2}{8}.$$

LEMMA 3.1. *One has $I_1 = (r_{1-}, r_{1+})$ with $r_{1\pm} = ((4 + \kappa) \pm \sqrt{\kappa^2 + 16})/\kappa$ and*

$$0 < r_{1-} < r_c < r_{1+}. \tag{3.1}$$

Moreover, $I_2 = (0, 2r_c)$, so that (cf. Figure 1 below)

$$I := I(\kappa) := I_0 \cap I_1 = I_0 \cap I_1 \cap I_2 = (r_{1-}, r_c). \tag{3.2}$$

Proof. Solving a quadratic equation, we see that $I_1 = \{r : q > 1\}$ is of the given form (r_{1-}, r_{1+}). Direct inspection of $q(r_c) > 1$ implies (3.1). At last, I_2 is given by those r for which $(1 + \kappa/4)r - \kappa r^2/8 + r - \kappa r^2/8 = r(2 + \kappa/4 - (\kappa/4)r) > 0$. It follows that $I_2 = (0, 4/\kappa(2 + \kappa/4)) = (0, 2r_c)$. $\qquad \square$

LEMMA 3.2. *Let $r \in I = I(\kappa)$, as defined in (3.2). Then, for any $0 < s \leqslant t \leqslant 1$,*

$$\mathbb{E}[|\gamma(t) - \gamma(s)|^q] \leqslant C(t - s)^{(q+\zeta)/2}(A_s + t^{-\zeta/2}) + C(t - s)^{(1/2)(q+\tilde{\zeta}/\theta)}t^{-\tilde{\zeta}/(2\theta)},$$

where $\tilde{r} \in (r, r_c)$ arbitrarily, $\tilde{q} = q(\tilde{r})$, $\tilde{\zeta} = \zeta(\tilde{r})$ and $\theta := \tilde{q}/q > 1$, and C is a positive constant depending on r and $\tilde{r}$ only.

Proof. Let $y = (t - s)^{1/2}$. Fix b such that $(q - 1)/q > b > (-\zeta - 1)/q$. The triangle inequality gives

$$\begin{aligned}
|\gamma(t) - \gamma(s)|^q &\lesssim |\gamma(t) - \hat{f}_t(iy)|^q + |\gamma(s) - \hat{f}_s(iy)|^q + |\hat{f}_t(iy) - \hat{f}_s(iy)|^q \\
&\lesssim |\gamma(t) - \hat{f}_t(iy)|^q + |\gamma(s) - \hat{f}_s(iy)|^q + |f_t(iy + U_s) \\
&\quad - f_s(iy + U_s)|^q + |f_t(iy + U_t) - f_t(iy + U_s)|^q. \tag{3.3}
\end{aligned}$$

We show that

$$\mathbb{E}(|\gamma(t) - \hat{f}_t(iy)|^q) \lesssim t^{-\zeta/2}(t - s)^{(\zeta+q)/2}.$$

Indeed, note that $\gamma(t) = \lim_{u \to 0^+} \hat{f}_t(iu)$

$$|\gamma(t) - \hat{f}_t(iy)|^q \leqslant \left(\int_0^y |\hat{f}_t'(iu)| \, du \right)^q$$

$$= \left(\int_0^y |\hat{f}_t'(iu)| u^b \cdot u^{-b} \, du \right)^q$$

$$\leqslant \left(\int_0^y |\hat{f}_t'(iu)|^q u^{bq} \, du \right) \left(\int_0^y u^{-bq/(q-1)} \right)^{q-1}$$

$$\text{by Hölder's inequality and } q > 1$$

$$\lesssim \left(\int_0^y |\hat{f}_t'(iu)|^q u^{bq} \, du \right) y^{q-1-bq} \quad \text{since } q - 1 - bq > 0.$$

So

$$\mathbb{E}(|\gamma(t) - \hat{f}_t(iy)|^q) \lesssim \left(\int_0^y \mathbb{E}(|\hat{f}_t'(iu)|^q) u^{bq} \, du \right) y^{q-1-bq}$$

$$\lesssim \left(\int_0^y t^{-\zeta/2} u^\zeta u^{bq} \, du \right) y^{q-1-bq}$$

$$\text{by Lemma 2.1 and since } t \geqslant t - s \geqslant u^2$$

$$\lesssim t^{-\zeta/2} y^{\zeta+bq+1} y^{q-1-bq} \quad \text{since } \zeta + bq > -1$$

$$= t^{-\zeta/2} (t-s)^{(\zeta+q)/2}.$$

In a similar way, we attain

$$\mathbb{E}(|\gamma(s) - \hat{f}_s(iy)|^q) \lesssim A_s (t-s)^{(\zeta+q)/2}. \tag{3.4}$$

Next

$$|f_t(iy + U_s) - f_s(iy + U_s)|^q \lesssim (y|\hat{f}_s'(iy)|)^q \quad \text{by Lemma 2.2.}$$

Therefore by Lemma 2.1

$$\mathbb{E}(|f_t(iy + U_s) - f_s(iy + U_s)|^q) \lesssim y^q \mathbb{E}(|\hat{f}_s'(iy)|^q) \lesssim A_s (t-s)^{(\zeta+q)/2}. \tag{3.5}$$

Now for the last term in (3.3)

$$|f_t(iy + U_t) - f_t(iy + U_s)|^q \leqslant \left(|U_t - U_s| \sup_{w \in [iy, iy + U_s - U_t]} |\hat{f}_t'(w)| \right)^q$$

$$\lesssim (|U_t - U_s| |\hat{f}_t'(iy)| ((|U_t - U_s|/y)^2 + 1)^l)^q$$

$$\text{by Lemma 2.3}$$

$$\lesssim |\hat{f}_t'(iy)|^q |U_t - U_s|^q ((|U_t - U_s|/y)^{2lq} + 1).$$

So let $X = |U_t - U_s|^q ((|U_t - U_s|/y)^{2lq} + 1)$, and with $\theta = \tilde{q}/q > 1$ for some $\tilde{r} \in (r, r_c)$ one has

$$\mathbb{E}(|f_t(iy + U_t) - f_t(iy + U_s)|^q) \lesssim \mathbb{E}(|\hat{f}'_t(iy)|^q X)$$
$$\lesssim (\mathbb{E}|\hat{f}'_t(iy)|^{q\theta})^{1/\theta} (\mathbb{E}X^{\theta^*})^{1/\theta^*},$$

where $\theta^* = \theta/(\theta - 1)$. Now since $U_t - U_s$ is a normal distribution,

$$\mathbb{E}(X^{\theta^*}) \lesssim y^{q\theta^*}.$$

So

$$\mathbb{E}(|f_t(iy + U_t) - f_t(iy + U_s)|^q) \lesssim y^q (\mathbb{E}|\hat{f}'_t(iy)|^{q\theta})^{1/\theta}$$
$$\lesssim \frac{y^{q+\zeta(\tilde{r})/\theta}}{t^{\tilde{\zeta}/(2\theta)}} = \frac{(t-s)^{(1/2)(q+\zeta(\tilde{r})/\theta)}}{t^{\tilde{\zeta}/(2\theta)}}. \qquad \square$$

4. Besov regularity of SLE

For each $\delta > 0$, $q \geqslant 1$ and for each measurable $\phi : [a, b] \to \mathbb{C}$, define its Besov (or fractional Sobolev) seminorm as

$$\|\phi\|_{W^{\delta,q};[a,b]} = \left(\int_a^b \int_a^b \frac{|\phi(t) - \phi(s)|^q}{|t - s|^{1+\delta q}} \, ds \, dt \right)^{1/q}. \tag{4.1}$$

The space of ϕ with $\|\phi\|_{W^{\delta,q};[a,b]} < \infty$ is a Banach space, denoted by $W^{\delta,q} = W^{\delta,q}([a, b])$.

Lemma 3.2 applies provided $r \in I$ and $\tilde{r} \in (r, r_c)$. We can then obtain $W^{\delta,q}$-regularity for SLE trace, restricted to some interval $[\varepsilon, 1]$, provided we find δ, q such that

$$\mathbb{E}\|\gamma\|^q_{W^{\delta,q};[\varepsilon,1]} = \int_\varepsilon^1 \int_\varepsilon^1 \frac{\mathbb{E}[|\gamma_t - \gamma_s|^q]}{|t - s|^{1+\delta q}} \, ds \, dt < \infty.$$

Though our focus is $\varepsilon = 0$, the case $\varepsilon > 0$, say $\varepsilon \in (0, 1)$, has noteworthy features and relates to a phase transition at $\kappa = 1$. Observe also that

$$\tilde{\zeta} \to \zeta, \tilde{q} \to q, \theta \to 1 \quad \text{as } \tilde{r} \downarrow r.$$

(Loosely speaking, by choosing $\tilde{r}$ sufficiently close to r, we can work with the limiting values whenever it comes to power-counting arguments.) Note that, as a consequence of Lemma 3.1, the condition $r \in I$ implies $\zeta + q > 0$ and then, with q positive (in fact, $q = q(r) > 1$ by definition of I_1),

$$\exists \delta \in \left(0, \frac{\zeta + q}{2q} \right). \tag{4.2}$$

To deal with the behaviour at s, t near 0^+, we further introduce

$$J_1 = J_1(\kappa) := \{r \in \mathbb{R} : \zeta(r) < 2\}.$$

LEMMA 4.1. $I \cap J_1 = I$ for $\kappa > 1$.

Proof. Obvious from $\zeta(r) \equiv r - \kappa r^2/8 = -\kappa/8(r - 4/\kappa)^2 + 2/\kappa \leqslant 2/\kappa$. $\quad\square$

THEOREM 4.2. *Assume $\kappa > 0$. If $r \in I \cap J_1$, $q = q(r)$ and δ is picked according to (4.2), then*

$$\mathbb{E}\|\gamma\|^q_{W^{\delta,q};[0,1]} < \infty.$$

For $\kappa \in (0, 1]$, under the weaker assumption $r \in I$ one still has

$$\mathbb{E}\|\gamma\|^q_{W^{\delta,q};[\varepsilon,1]} < \infty \quad \text{for any } \varepsilon \in (0, 1].$$

REMARK 4.3. The first part of this proposition applies to $\kappa = 8$.

REMARK 4.4. The conditions on r can be fully spelled out. For instance, when $\kappa > 1$, then $r \in I = I \cap J_1$ iff $r \in (r_{1-}, r_c) = (\kappa^{-1}(4+\kappa-\sqrt{\kappa^2 + 16}), 1/2+4/\kappa)$, cf. Lemma 3.1 above. When $\kappa \leqslant 1$, one takes additionally into account $r \notin [j_{1-}, j_{1+}]$ with

$$j_{1\pm} = 4\kappa^{-1}(1 \pm \sqrt{1 - \kappa}), \tag{4.3}$$

obtained from solving the quadratic inequality $\zeta < 2$.

Proof of Theorem 4.2. For $r \in I$ and $\tilde{r} \in (r, r_c)$, Lemma 3.2 estimates $\mathbb{E}|\gamma_t - \gamma_s|^q$ in terms of factors which are singular at $s = t = 0$, these will be controlled thanks to J_1, and factors which are singular at the diagonal $s = t$, to be controlled via the bound (4.2). More specifically, $r \in J_1$ guarantees the integrability of $\max(1, s^{-\zeta/2}, t^{-\zeta/2}, t^{-\tilde{\zeta}/(2\theta)})$. Indeed this is obvious for $s^{-\zeta/2}, t^{-\zeta/2}$ since the integrability-at-0^+-condition $(-\zeta/2 > -1)$ is precisely guaranteed by $\zeta(r) < 2$. The same is true for the exponent $-\tilde{\zeta}/(2\theta) = -\tilde{\zeta}/(2\tilde{q}/q)$, upon choosing $\tilde{r}$ close enough to r. A similar power-counting argument applies to $|t - s|^{(1/2)(q+\zeta)}$ respectively $|t - s|^{(1/2)(q+\tilde{\zeta}/\theta)}$. Taking into account factor $|t - s|^{1+\delta q}$ which appears in the definition of the $W^{\delta,q}$-norm, the integrability-at-diagonal condition becomes

$$\tfrac{1}{2}(q + \zeta) - (1 + \delta q) > -1$$

which is precisely what is guaranteed by (4.2). $\quad\square$

5. Optimal *p*-variation regularity of SLE

For each $p \geqslant 1$, and each continuous function ϕ defined on an interval $[a, b]$, we define its p-variation as

$$\|\phi\|_{p\text{-var};[a,b]} = \left(\sup_{\mathcal{P}} \sum_{i=1}^{\#\mathcal{P}} |\phi(t_i) - \phi(t_{i-1})|^p \right)^{1/p}$$

where the supremum is taken over partitions $\mathcal{P} = \{t_0, \ldots, t_n\}$ of $[a, b]$.

We would like to apply the embedding in (1.1). To do that, define (J_1 repeated for the reader's convenience)

$$J_1 := J_1(\kappa) := \{r \in \mathbb{R} : \zeta(r) < 2\},$$
$$J_2 := J_2(\kappa) = \{r \in \mathbb{R} : \zeta(r) + q(r) > 2\}.$$

Observe that $r \in J_2$ is precisely equivalent to

$$\exists \delta \in \left(\frac{1}{q(r)}, \frac{\zeta(r) + q(r)}{2q(r)} \right). \tag{5.1}$$

Write (a, b) for an open interval (of $\mathbb{R}$), and agree further that $(a, b) = \emptyset$ when $a = b$.

LEMMA 5.1. (i) $r \in J_2$ iff it is an element in the open interval with endpoints $1, 8/\kappa$ (and empty for $\kappa = 8$).

(ii) $r \in I \cap J_2$ iff $r \in (1, r_c) \equiv (1, 1/2 + 4/\kappa)$, in case $\kappa < 8$, and $r \in (8/\kappa, r_c)$ for $\kappa > 8$.

(iii) With $j_{1\pm}$ as introduced in (4.3),

$$I \cap J_1 \cap J_2 = \begin{cases} (1, j_{1-}) \cup (\min(j_{1+}, r_c), r_c) & \text{when } \kappa \in (0, 1], \\ (1, r_c) & \text{when } \kappa \in (1, 8), \\ \emptyset & \text{when } \kappa = 8, \\ (8/\kappa, r_c) & \text{when } \kappa \in (8, \infty). \end{cases}$$

(See Figure 3(b), and also Figure 2(b) for a zoom, just below $\kappa = 1$, where $(\min(j_{1+}, r_c), r_c) \neq \emptyset$.)

Proof. Part (i) and (ii) come from the fact that

$$\zeta(r) + q(r) - 2 = -\frac{\kappa}{4}(r - 1)\left(r - \frac{8}{\kappa} \right).$$
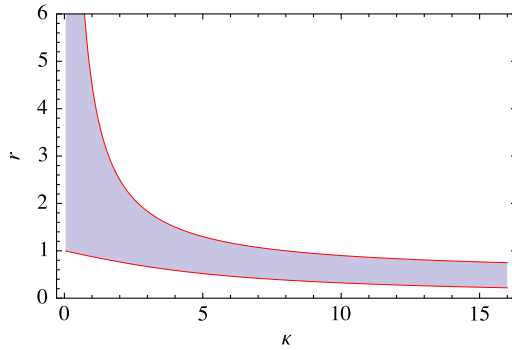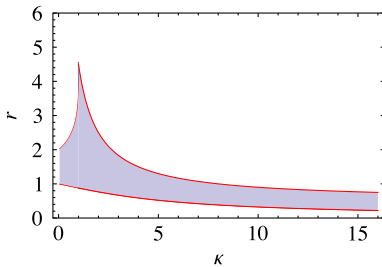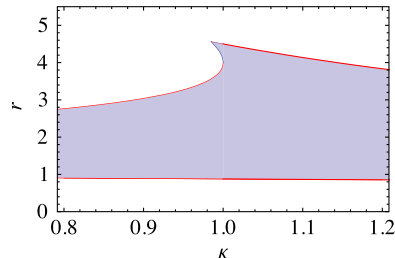
Figure 1. Admissible r in the sense of $r \in I$, as function of κ.

(a) $\kappa \in [0, 16]$

(b) As on the left, zoomed into region $[0.8, 1.2]$

Figure 2. Admissible r in the sense of $r \in I \cap J_1$, as function of κ.

For part (iii), the case $\kappa \in (0, 1]$ follows from the fact that

$$1 < j_{1-} < r_c.$$

The other cases are straightforward. $\qquad\square$

THEOREM 5.2. *If $r \in I \cap J_1 \cap J_2$, $q = q(r)$, then for all δ as in* (5.1) *and $p := 1/\delta$, we have*

$$\mathbb{E}\|\gamma\|^q_{p\text{-var};[0,1]} < \infty.$$

Optimization over the range of admissible r, shows that one can take any

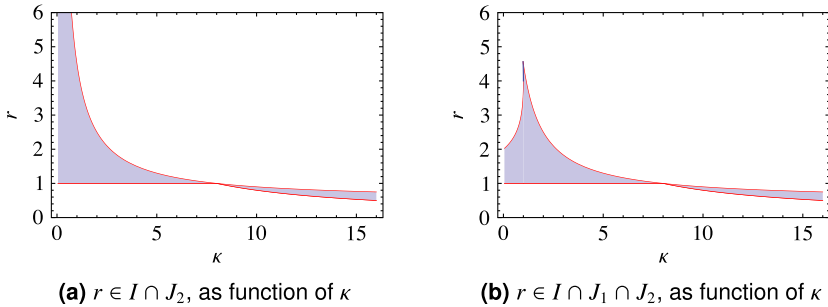$$p > p_* := q(\min(1, 8/\kappa)) = \min(1 + \kappa/8, 2).$$

(a) $r \in I \cap J_2$, as function of κ **(b)** $r \in I \cap J_1 \cap J_2$, as function of κ

Figure 3. Admissible r for Theorems 5.2 and 6.1. Note $I \cap J_2 = \emptyset$ when $\kappa = 8$.

Proof. The first statement follows immediately from (1.1), Theorem 4.2 and by noting that

$$\left(\frac{1}{q}, \frac{\zeta + q}{2q} \right) \subset (0, 1)$$

when $r \in I$. The infimum of all possible p is

$$p_* = \inf_{r \in I \cap J_1 \cap J_2} \frac{2q}{\zeta + q}.$$

One sees that

$$\frac{\zeta}{q} = 1 + \frac{\frac{\kappa}{4}}{\frac{\kappa}{8} r - (1 + \frac{\kappa}{4})}$$

is a decreasing function and so

$$\frac{2q}{\zeta + q} = \frac{2}{\zeta / q + 1}$$

is small when r is small. With Lemma 5.1, it is then easy to see that the 'optimal r' is given by

$$r_{\min} := \inf(I \cap J_1 \cap J_2) = \min(1, 8/\kappa). \tag{5.2}$$

Hence, when $\kappa \in (0, 8)$,

$$p_* = \frac{2q(r)}{\zeta(r) + q(r)} \bigg|_{r=1} = q(r)|_{r=1} = 1 + \frac{\kappa}{8}.$$

When $\kappa \in (8, \infty)$,

$$p_* = \frac{2q(r)}{\zeta(r) + q(r)} \bigg|_{r=\frac{8}{\kappa}} = q\left(\frac{8}{\kappa} \right) = 2. \qquad \square$$

COROLLARY 5.3 (Hausdorff dimension upper bound; [**RS05**]). $\dim_H(\gamma|_{[0,1]}) \leqslant \min(1 + \kappa/8, 2)$.

Proof. By a property of p-variation, the map $\gamma|_{[0,1]}$ can be reparametrized to a δ-Hölder map $\tilde{\gamma}$, with $\delta = 1/p$, so that by basic facts of Hausdorff dimension of sets under Hölder maps,

$$\dim_H(\gamma|_{[0,1]}) = \dim_H(\tilde{\gamma}) \leqslant \frac{1}{\delta} \dim_H([0,1]) = p.$$

Take $p \downarrow p_* = \min(1 + \kappa/8, 2)$ to recover the stated upper bound on the Hausdorff dimension of SLE_κ. $\square$

This upper bound was first derived by Rohde–Schramm [**RS05**]; equality was later established by Beffara [**Bef08**] which in turn shows that our p-variation result, any $p > p_*$, is indeed optimal.

6. Optimal Hölder exponent

Recall that for each $\alpha \in (0, 1]$, the α-Hölder seminorm of a continuous function ϕ defined on an interval $[a, b]$ is

$$\|\phi\|_{\alpha\text{-Höl};[a,b]} = \sup_{s \neq t \in [a,b]} \frac{|\phi(s) - \phi(t)|}{|s - t|^\alpha}.$$

We follow the same logic as in the previous section, again apply the embedding in (1.1), which is possible exactly when $r \in I \cap J_1 \cap J_2$. As a consequence, we recover the (optimal) SLE Hölder regularity of [**JVL11**, Theorem 1.1], with the novelty of having some control over moments.

THEOREM 6.1. *If $r \in I \cap J_2$, $q = q(r)$, then for all δ as in (5.1), and $\alpha := 1/\delta - q$, we have*

$$\mathbb{E}\|\gamma\|_{\alpha\text{-Höl};[\varepsilon,1]}^q < \infty$$

for any $\varepsilon \in (0, 1]$. Optimization over the range of admissible r shows that, in the capacity parametrization of γ, one can take any Hölder exponent

$$\alpha < \alpha_*(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{\kappa + 8}}.$$

If $r \in I \cap J_1 \cap J_2$, everything else as above, then

$$\mathbb{E}\|\gamma\|_{\alpha\text{-Höl};[0,1]}^q < \infty$$

and here one can take any Hölder exponent $\alpha < \min(\alpha_*, 1/2)$.

(a) $r \in I \cap J_1 \cap J_2$, with $\kappa \in [0, 9]$ **(b)** As on the left, $\kappa \in [6, 16]$
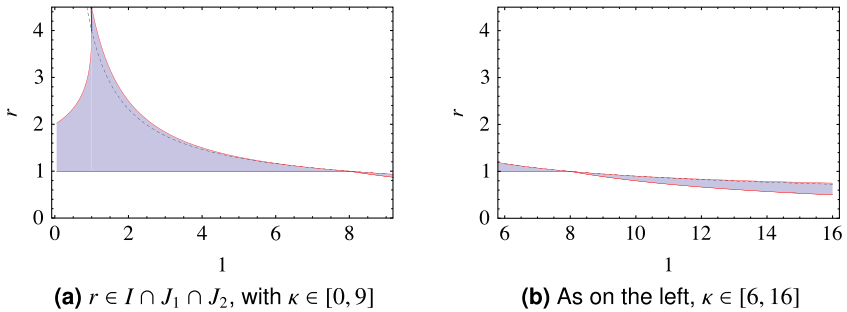
Figure 4. Dashed line for $r = r(\kappa) \in I \cap J_2$ which maximizes Hölder exponent.

Proof. The statements about finiteness of moments are immediate by Theorem 4.2 and the Besov–Hölder embedding (1.2). We can take any exponent $\alpha < \hat{\alpha}$, where $\hat{\alpha}$ is the supremum of $1/\delta - q$ with δ as in (5.1) and with $r \in I \cap J_2$ or $r \in I \cap J_1 \cap J_2$ depending on whether we consider $\|\gamma\|_{\alpha\text{-Höl};[\varepsilon,1]}$ or $\|\gamma\|_{\alpha\text{-Höl};[0,1]}$. Thus,

$$\hat{\alpha} = \sup_r \frac{\zeta + q - 2}{2q}.$$

Observe that the function $\phi(r) = (\zeta + q - 2)/2q$ satisfies

$$\phi'(r) \geqslant 0 \Leftrightarrow r \in [r_-, r_+]$$

where $r_{\pm} = (4(-2 \pm \sqrt{8 + \kappa}))/\kappa$.

Consider the case $\kappa \in (1, \infty) \setminus \{8\}$. By Lemma 5.1,

$$I \cap J_1 \cap J_2 = I \cap J_2 = \begin{cases} (1, r_c) & \text{when } \kappa < 8, \\ (8/\kappa, r_c) & \text{when } \kappa > 8. \end{cases}$$

One can check that $r_- < 0$ and $r_+ \in I \cap J_2$. See the dashed line in Figure 4 for an illustration of r_+. Hence

$$\hat{\alpha} = \phi(r_+) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{\kappa + 8}}.$$

Consider the case $\kappa \in (0, 1]$. Concerning $\|\gamma\|_{\alpha\text{-Höl};[\varepsilon,1]}$, the conclusion does not change:

$$\sup_{r \in I \cap J_2} \phi(r) = \sup_{r \in (1, r_c)} \phi(r) = \phi(r_+) = \alpha_*(\kappa).$$

Concerning $\|\gamma\|_{\alpha\text{-Höl};[0,1]}$, note that

$$1 < j_{1-} \leqslant r_+ < \min(j_{1+}, r_c) \leqslant r_c$$

and that by Lemma 5.1,

$$I \cap J_1 \cap J_2 = (1, j_{1-}) \cup (\min(j_{1+}, r_c), r_c).$$

Therefore,

$$\sup_{r \in I \cap J_1 \cap J_2} \phi(r) = \max\{\phi(j_{1-}), \phi(j_{1+})\} = \frac{2 + q - 2}{2q} = \frac{1}{2} = \min(\alpha_*, 1/2). \qquad \square$$

7. Further discussion

Quantified finite q-moments, p-variation case. Fix $p > p_* = \min(1 + \kappa/8, 2)$ so that, according to Theorem 5.2, there exists $q > 1$ so that

$$\mathbb{E}\|\gamma\|^q_{p\text{-var};[0,1]} < \infty.$$

How large can we take q? Our method allow here to identify a range of finite q-moments, with $q \in [1, Q)$ with $Q = Q(p, \kappa)$.

Since q is strictly increasing, for any $p > p_*$, a possible choice is $Q = Q_* := q(\min(1, 8/\kappa)) = \min(1 + \kappa/8, 2)$. Giving up on pleasant formulae, one can do better. Fixing $p > p_*$, we can take

$$Q = \sup_r q(r)$$

where r satisfies $r \in I \cap J_1 \cap J_2$ and that $(2q(r))/(\zeta(r) + q(r)) < p < q(r)$. We let $Q = 0$ if there is no such r.

Let $\phi(r) = (2q(r))/(\zeta(r) + q(r))$ and note

- $\phi(r)$ and $q(r)$ are strictly increasing on I,

- $p_* = \inf_{r \in I \cap J_1 \cap J_2} \phi(r) = \phi(r_{\min})$,

- $\phi(r) < q(r)$.

If $p \geqslant \phi(r_c)$, then $Q = 0$. Consider $p \in (\phi(r_{\min}), \phi(r_c))$. There exists $\hat{r} \in (r_{\min}, r_c)$ such that $\hat{r} = \sup\{r \in I \cap J_1 \cap J_2 : \phi(r) < p\}$. Thus,

$$\phi(r) < p \Leftrightarrow r < \hat{r}$$

and, therefore,

$$Q = \sup_{r \in I \cap J_1 \cap J_2 : r < \hat{r}, p < q(r)} q(r) = q(\hat{r}).$$

For the value of $\hat{r}$ we have

$$\hat{r} = \begin{cases} j_{1-} & \text{when } \kappa \in (0, 1] \text{ and } p \in \phi(I \cap J_1 \cap J_2), \\ \phi^{-1}(p) = \dfrac{(8+\kappa)p - (8+2\kappa)}{\kappa(p-1)} & \text{otherwise.} \end{cases}$$

Note that $Q = Q(p, \kappa) > Q_* = Q_*(\kappa)$. On the other hand, as $p \downarrow p_*$, $\hat{r}$ approaches $r_{\min}$, so that $Q \to Q_*$.

(A similar discussion about q-moments for the α-Hölder case is left to the reader.)

Beyond Hölder and variation. At last, we note that it is possible to regard Hölder and variation regularity as extreme points of a scale of Riesz-type variation spaces, recently related to a scale of Nikolskii spaces; see [**FP16**].

As $W^{\delta, p}$ embeds into these spaces, this would allow for another family of SLE regularity statements.

Acknowledgements

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