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CONGRUENCE LATTICES OF REGULAR SEMIGROUPS RELATED TO KERNELS AND TRACES

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The kernel–trace approach to congruences on a regular semigroup S can be refined by introducing the left and right traces. This induces eight operators on the lattice of congruences on S: t_l, k, t_r; T_l, K, T_r; t, T. We describe the lattice of congruences on S generated by six 3-element subsets of the set $\{\omega t_l, \omega k, \omega t_r, \varepsilon T_l, \varepsilon K, \varepsilon T_r\}$ where ω and ε denote the universal and the equality relations. This is effected by means of a diagram and in terms of generators and relations on a free distributive lattice, or a homomorphic image thereof. We perform the same analysis for the lattice of congruences on S generated by the set $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$.

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1. Introduction and summary

Let S be a regular semigroup, $\mathscr{C}(S)$ the lattice of its congruences and $E(S)$ the set of its idempotents. For $\rho \in \mathscr{C}(S)$,

$$\ker \rho = \{a \in S \,|\, a\rho e \text{ for some } e \in E(S)\}, \qquad \operatorname{tr} \rho = \rho|_{E(S)}$$

are the *kernel* and the *trace* of ρ, respectively. The study of congruences on S via their kernels and traces is known as the kernel–trace approach. The fundamental result here is that ρ is uniquely determined by the pair $(\ker \rho, \operatorname{tr} \rho)$, called a *congruence pair*. This approach can be refined by introducing the *left* and the *right* traces as follows

$$\operatorname{l\,tr} \rho = \operatorname{tr}(\rho \vee \mathscr{L})^0, \qquad \operatorname{r\,tr} \rho = \operatorname{tr}(\rho \vee \mathscr{R})^0,$$

where the join is taken in the lattice of equivalence relations on S and θ^0 means the greatest congruence on S contained in an equivalence relation θ on S. The triple $(\operatorname{l\,tr} \rho, \ker \rho, \operatorname{r\,tr} \rho)$ uniquely determines the congruence ρ. By

$$\rho K, \rho T, \rho T_l, \rho T_r; \qquad \rho k, \rho t, \rho t_l, \rho t_r$$

we denote the greatest congruences on S having the same kernel, trace, left trace and right trace, respectively, as ρ; and the same with least replacing greatest. This provides eight operators on $\mathscr{C}(S)$ which we denote by $K, T, \ldots$. Then K and k, T and t, T_l and t_l, T_r and t_r induce equivalence relations $\mathscr{K}$, $\mathscr{T}$, $\mathscr{T}_l$, $\mathscr{T}_r$, respectively, on $\mathscr{C}(S)$. The first one of these is a complete $\wedge$-congruence, the remaining ones are complete congruences. Note that

$$\mathscr{T} = \mathscr{T}_l \wedge \mathscr{T}_r, \qquad \rho T = \rho T_l, \wedge \rho T_r, \qquad \rho t = \rho t_l \vee \rho t_r.$$

In general, these operators, with each congruence on S, produce a further eight congruences (some of which may of course coincide). The operators themselves, through their various properties, provide useful information about congruences on a regular semigroup. In addition, one may consider the lattice generated by their values on a single congruence ρ (or a set of congruences), especially if the congruence ρ is a remarkable one. Another possibility is to iterate some (or all) of these operators on a single congruence thereby obtaining various networks of congruences. Finally, one can combine both of these procedures by iterating the operators and determining the lattice generated by their values.

Denoting by ω and ε the universal and equality congruences on S, respectively, we attempt to describe the lattice generated by the set

$$\{\omega t_l, \omega k, \omega t_r, \varepsilon T_l, \varepsilon K, \varepsilon T_r\}.$$

Observe that we may skip the operators T and t in view of the above identities. These congruences have appeared under different notation and description as follows:

$$(\leqq_r)^*, \beta, (\leqq_l)^*, \mu_l, \tau, \mu_r,$$

that is the congruence generated by the right partial order, the least band congruence, the congruence generated by the left partial order, the greatest congruence contained in $\mathscr{L}$, that is $\mu_l = \mathscr{L}^0$, the greatest idempotent pure congruence, and the greatest congruence contained in $\mathscr{R}$, that is $\mu_r = \mathscr{R}^0$, respectively. (Recall that $e \leqq_r f$ if and only if $e = fe$.) These descriptions destroy our systematic notation but they put things into a familiar garb thereby incidentally indicating that the task formulated above is a formidable one.

It is therefore not at all surprising that we are not able to describe the lattice generated by these congruences. There is another difficulty here which, on first sight, does not seem germane to the problem; namely, that $\mathscr{K}$ is not in general a congruence. We are thus forced to consider not only certain special cases of the above problem, but also to assume in almost all our considerations that $\mathscr{K}$ is a congruence. This is a strong restriction, but primitive regular semigroups have this property (Petrich [**9**, Theorem 3.6]). For strong semilattices of simple regular semigroups, for Reilly semigroups and for retract extensions of one Brandt semigroup by another, we gave necessary and sufficient conditions for the kernel relation to be a congruence (Petrich [**9**, Theorems 4.3, 4.7, 5.5] and Petrich [**10**, Theorem 5.7]). In addition to this restriction, we are forced to consider the lattices generated by relatively small subsets of the set of congruences above; namely, in Section 3 we describe the congruences generated by each of the sets

$$\{\omega t_l, \omega k, \omega t_r\}, \quad \{\varepsilon T_l, \omega k, \omega t_r\}, \quad \{\omega t_l, \varepsilon K, \omega t_r\},$$
$$\{\varepsilon T_l, \omega k, \varepsilon T_r\}, \quad \{\varepsilon T_l, \varepsilon K, \omega t_r\}, \quad \{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}. \tag{1}$$

Note that in each one of these sets there is exactly one representative of the left trace, the kernel and the right trace.

We can get some insight into the structure of the entire lattice we are interested in by describing the lattice generated by the set $\{\omega t, \varepsilon T, \omega k, \varepsilon K\}$, which we do in Section 4. Note that $\omega t = \sigma$—the least group congruence and $\varepsilon T = \mu$—the greatest idempotent separating congruence on S.

The description of the lattice in question consists of the assertion that a given lattice is a homomorphic image of a lattice given by a diagram. The latter lattice is then shown, in most cases, to occur itself as the lattice of such congruences. We also describe the latter lattices in terms of generators and relations.

For congruence lattices generated by various other special congruences on regular semigroups, we mention only Pastijn and Petrich [**5**]. Related to our subject are the semigroups generated by certain operators on the lattice of varieties of completely regular semigroups studied by Petrich and Reilly [**13**]. Various networks of congruences, that is systems of congruences resulting by iteration of some of the above operators, were considered by Petrich and Reilly [**12**], Pastijn and Trotter [**6**] and the author [**11**].

2. Preliminaries

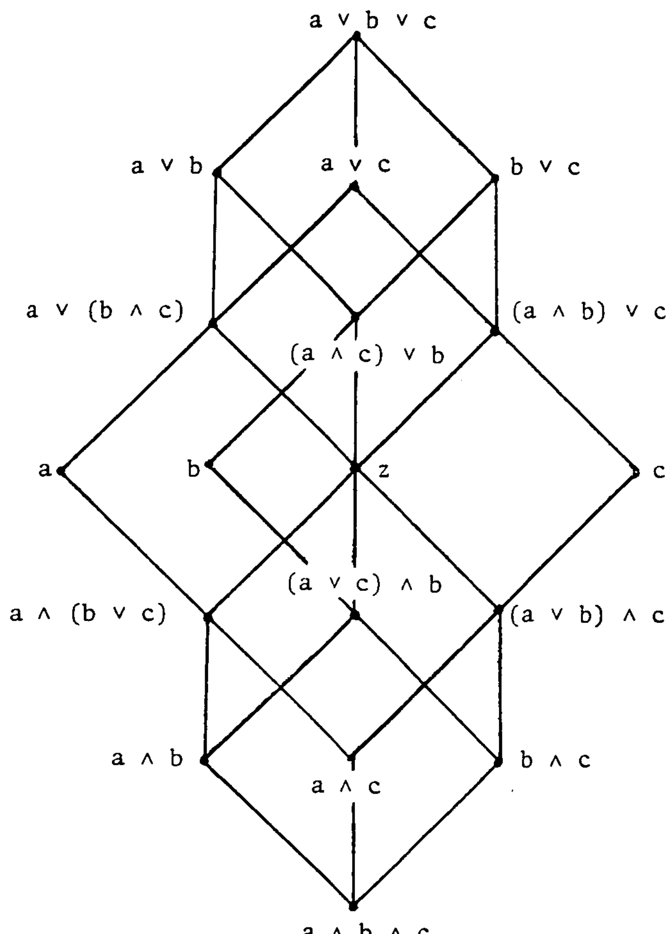
We will use the notation and terminology of Howie [**2**] and Petrich [**7**]. In addition to those in the introduction, we state a few frequently occurring symbols and concepts.

A regular semigroup S is E-unitary if $E(S)$ is a unitary subset of S, equivalently for $a \in S$, $e, ae \in E(S)$ we always have $a \in E(S)$. For any (quasi) variety $\mathcal{V}$, we denote by $\theta_{\mathcal{V}}$ the least congruence on S whose quotient is in $\mathcal{V}$. A completely regular semigroup S is a cryptogroup if the Green relation $\mathcal{H}$ on S is a congruence.

Diagram 1 represents a free distributive lattice on the generators a, b and c; we denote it by $\mathcal{FDL}(a,b,c)$, see Grätzer [**1**, I.5, Theorem 10]. If $\{\rho_\alpha\}_{\alpha \in A}$ is a family of relations on $L = \mathcal{FDL}(a,b,c)$, we denote the quotient lattice L divided by the congruence generated by $\{\rho_\alpha\}_{\alpha \in A}$ by $\mathcal{FDL}(a,b,c)/(\rho_\alpha)_{\alpha \in A}$.

We will use the following notation for a some (quasi) varieties of completely regular semigroups

$\mathcal{T}$ — trivial semigroups,
$\mathcal{LZ}$ — left zero semigroups,
$\mathcal{RZ}$ — right zero semigroups,
$\mathcal{RB}$ — rectangular bands,
$\mathcal{G}$ — groups,
$\mathcal{LG}$ — left groups,
$\mathcal{RG}$ — right groups,



where $z=[a \wedge (b \vee c)] \vee [(a \vee b) \wedge c]$.

DIAGRAM 1. The free distributive lattice $\mathcal{FDL}(a, b, c)$.

$\mathcal{R}e\mathcal{G}$ — rectangular groups,
$\mathcal{B}$ — bands,
$\mathcal{UCG}$ — E-unitary cryptogroups.

Lemma 2.1. *In any regular semigroup S, the following hold.*

(i) $\omega t_l = (\leqq_r)^* = \theta_{\mathcal{LG}}$.

(ii) $\omega t = \theta_{\mathcal{G}}$.

(iii) $\omega k = \theta_{\mathcal{B}}$.

Proof. (i) The first equality is a consequence of Pastijn and Petrich [4, Theorem

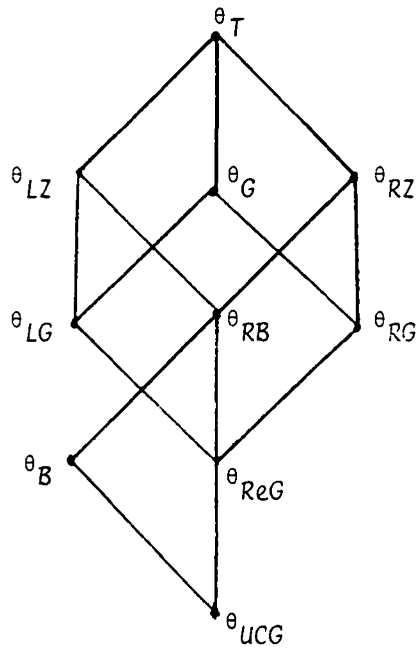


DIAGRAM 2. The lattice generated by $\{\theta_{\mathscr{L}\mathscr{G}}, \theta_{\mathscr{B}}, \theta_{\mathscr{R}\mathscr{G}}\}$.

4.12]. To prove the second equality, we let $\rho=(\leqq_r)^*$ and verify first that ρ is an $\mathscr{L}\mathscr{G}$-congruence. Let $e, f\in E(S)$ and let x be an inverse of fe. Then $exf\in E(S)$ and $exf=e(exf)$ gives $exf\leqq_r e$. It follows that $exf\rho e$ which evidently implies that $ef\rho e$. Consequently $E(S/\rho)$ is a left zero semigroup and thus S/ρ is a left group. Therefore ρ is an $\mathscr{L}\mathscr{G}$-congruence.

Let λ be an $\mathscr{L}\mathscr{G}$-congruence on S and let $e\leqq_r f$. Then $e=fe$ and hence $e\lambda fe$ whence $e\lambda f$ since $fe\lambda f$. Thus $\leqq_r\subseteq\lambda$ and therefore $\rho\subseteq\lambda$, proving the minimality of ρ, whence $\rho=\theta_{\mathscr{L}\mathscr{G}}$.

(ii) This follows from an obvious fact that a congruence ρ on S is a group congruence if and only if $\operatorname{tr}\rho=\omega$.

(iii) Similarly, a congruence ρ on S is a band congruence if and only if $\ker\rho=S$.

Lemma 2.2. *Let S be a regular semigroup. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\theta_{\mathscr{L}\mathscr{G}}, \theta_{\mathscr{B}}, \theta_{\mathscr{R}\mathscr{G}}\}$ is a homomorphic image of the lattice depicted in Diagram 2.*

Proof. With the vertices as labelled, Diagram 2 clearly represents a partially ordered set. That the joins in Diagram 2 are correct follows from $\theta_{\mathscr{A}}\vee\theta_{\mathscr{B}}=\theta_{\mathscr{A}\wedge\mathscr{B}}$ for any quasivarieties $\mathscr{A}$ and $\mathscr{B}$. We now verify the correctness of meets.

1. $\theta_{\mathscr{L}\mathscr{Z}}\wedge\theta_{\mathscr{R}\mathscr{Z}}=\theta_{\mathscr{R}\mathscr{B}}$ was proved in Petrich [8, II.1.11] for arbitrary semigroups.

$\theta_{\mathscr{L}\mathscr{Z}}\wedge\theta_{\mathscr{G}}=\theta_{\mathscr{L}\mathscr{G}}$. Both $\theta_{\mathscr{L}\mathscr{G}}$ and $\theta_{\mathscr{G}}$ are $\mathscr{L}\mathscr{G}$-congruences and thus so is their meet. Let

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$\rho\in\mathscr{C}(S)$ be an $\mathscr{L}\mathscr{G}$-congruence on S. Then $T=S/\rho\in\mathscr{L}\mathscr{G}$ and hence $T\cong L\times G$ for some $L\in\mathscr{L}\mathscr{Z}$ and $G\in\mathscr{G}$. This direct decomposition shows that there exist $\lambda,\sigma\in\mathscr{C}(T)$ such that $\lambda\wedge\sigma=\varepsilon$, λ is an $\mathscr{L}\mathscr{Z}$-congruence and σ is a $\mathscr{G}$-congruence. Lift λ and σ to congruences on S, say $\bar{\lambda}$ and $\bar{\sigma}$. Then $\bar{\lambda}$ is an $\mathscr{L}\mathscr{Z}$-congruence, $\bar{\sigma}$ is a $\mathscr{G}$-congruence and thus $\theta_{\mathscr{L}\mathscr{Z}}\subseteq\bar{\lambda}$ and $\theta_{\mathscr{G}}\subseteq\bar{\sigma}$. But then $\theta_{\mathscr{L}\mathscr{Z}}\wedge\theta_{\mathscr{G}}\subseteq\bar{\lambda}\wedge\bar{\sigma}=\rho$ which proves the minimality of $\theta_{\mathscr{L}\mathscr{Z}}\wedge\theta_{\mathscr{G}}$ and establishes the desired relation.

$\theta_{\mathscr{L}\mathscr{G}}\wedge\theta_{\mathscr{R}\mathscr{G}}=\theta_{\mathscr{R}_e\mathscr{G}}$. The proof runs along the same lines as the preceding one using the fact that every rectangular group is a direct product of a left zero semigroup and a right group.

2. $\theta_{\mathscr{L}\mathscr{G}}\wedge\theta_{\mathscr{R}\mathscr{G}}=\theta_{\mathscr{R}_e\mathscr{G}}$. Again the proof runs along the same lines as above using the fact that every rectangular group is a direct product of a left group and a right group.

3. $\theta_{\mathscr{B}}\wedge\theta_{\mathscr{G}}=\theta_{\mathscr{U}\mathscr{C}\mathscr{G}}$. This was proved in Petrich [7, Theorem 4.1].

4. $\theta_{\mathscr{R}\mathscr{B}}\wedge\theta_{\mathscr{G}}=\theta_{\mathscr{R}_e\mathscr{G}}$. Again the proof runs along the same lines as above using the fact that every rectangular group is a direct product of a rectangular band and a group.

The remaining cases follow by either symmetry or monotonicity. It is clear that the above lattice is generated by the set $\{\theta_{\mathscr{L}\mathscr{G}},\theta_{\mathscr{B}},\theta_{\mathscr{R}\mathscr{G}}\}$. Since some of the vertices may coincide, our lattice is a homomorphic image of the one depicted in Diagram 2.

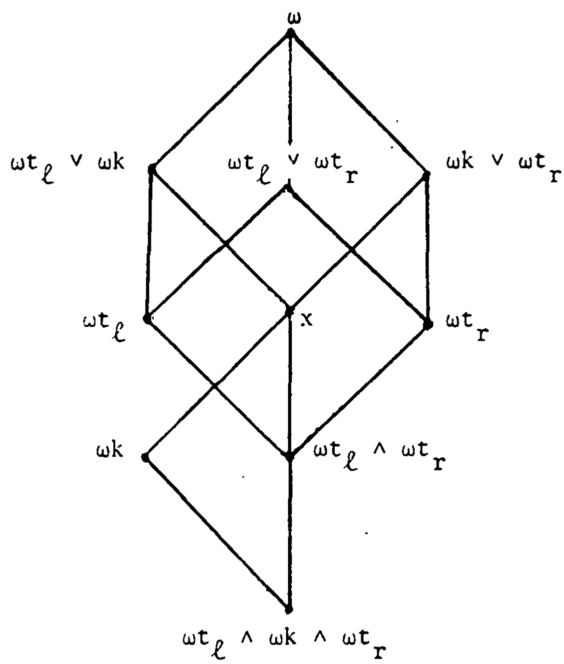
Example 2.3. Let S be a free object in $\mathscr{U}\mathscr{C}\mathscr{G}$ on a countably infinite set of generators. Then for S, all the vertices in Diagram 2 are distinct.

As another example we may take a free completely regular semigroup S on a countably infinite set of generators. In this case, acccording to Pastijn [3, Theorem 11] the restriction of the kernel relation $\mathscr{K}$ to the lattice of fully invariant congruences on S is a congruence. That $\mathscr{K}$ be a congruence is not needed in the next theorem. However, all the remaining results are under the hypothesis that $\mathscr{K}$ be a congruence on the entire lattice $\mathscr{C}(S)$ but the discussion there pertains only to the lattice generated by the given congruences, so only the congruence property of $\mathscr{K}$ on that lattice may be required, as is the case in the example of a free completely regular semigroup.

3. Lattices generated by the sets (1)

For each of these sets, we obtain the lattice generated by it as a homomorphic image of a lattice given by a diagram. In four out of six cases, we also provide an example of a regular semigroup whose corresponding lattice is isomorphic to the one given by the diagram. In three of these examples we also have that the kernel relation is a congruence as required by the corresponding result.

Theorem 3.1. *Let S be a regular semigroup. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\omega t_l,\omega k,\omega t_r\}$ is a homomorphic image of the lattice depicted in Diagram 3. The latter lattice is isomorphic to the lattice*



where $x=(\omega t_l \wedge \omega t_r) \vee \omega k$.

DIAGRAM 3. The lattice generated by $\{\omega t_l, \omega k, \omega t_r\}$.

$$\mathcal{FDL}(\omega t_l, \omega k, \omega t_r)/(\omega t_l \wedge \omega k \leqq \omega t_r, \omega k \wedge \omega t_r \leqq \omega t_l)$$

and none of these relations may be omitted.

Proof. We will reduce the discussion of Diagram 3 to Diagram 2. By Lemma 2.1(i), we have $\omega t_l = \theta_{\mathcal{LG}}$ and dually $\omega t_r = \theta_{\mathcal{RG}}$. Also $\omega t_l \vee \omega t_r = \omega t = \theta_{\mathcal{G}}$ by Lemma 2.1(ii). Finally, by Lemma 2.1(iii) we have $\omega k = \theta_{\mathcal{B}}$. This together with $\theta_{\mathcal{T}} = \omega$ identifies five vertices in Diagram 3 with the corresponding ones in Diagram 2. Further, using Lemma 2.2, we have

$$\omega t_l \vee \omega k = \theta_{\mathcal{LG}} \vee \theta_{\mathcal{B}} = \theta_{\mathcal{LZ}},$$

$$x = (\omega t_l \wedge \omega t_r) \vee \omega k = (\theta_{\mathcal{LG}} \wedge \theta_{\mathcal{RG}}) \vee \theta_{\mathcal{B}} = \theta_{\mathcal{R} \circ \mathcal{G}} \vee \theta_{\mathcal{B}} = \theta_{\mathcal{RB}},$$

$$\omega t_l \wedge \omega t_r = \theta_{\mathcal{LG}} \wedge \theta_{\mathcal{RG}} = \theta_{\mathcal{R} \circ \mathcal{G}},$$

$$\omega t_l \wedge \omega k \wedge \omega t_r = \theta_{\mathcal{LG}} \wedge \theta_{\mathcal{B}} \wedge \theta_{\mathcal{RG}} = \theta_{\mathcal{UCG}}.$$

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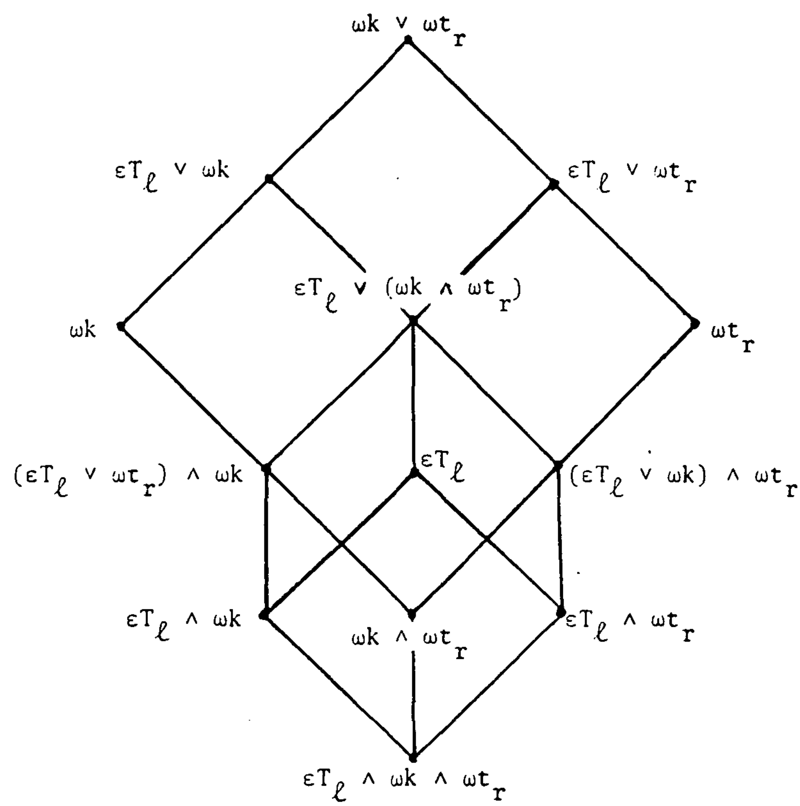

DIAGRAM 4. The lattice generated by $\{\varepsilon T_l, \omega k, \omega t_r\}$.

It follows that Diagram 3 represents the sublattice of $\mathscr{C}(S)$ generated by the set $\{\omega t_l, \omega k, \omega t_r\}$ with some possible coincidence of vertices, so it is a homomorphic image of the lattice depicted in Diagram 3. The last two assertions of the theorem can be easily deduced from Diagram 1.

Example 2.3 provides an instance of a regular semigroup S for which the lattice generated by the set $\{\omega t_l, \omega k, \omega t_r\}$ is isomorphic to the one depicted in Diagram 3.

Theorem 3.2. *Let S be a regular semigroup for which* $\mathscr{H}$ *is a congruence. Then the sublattice of* $\mathscr{C}(S)$ *generated by the set* $\{\varepsilon T_l, \omega k, \omega t_r\}$ *is a homomorphic image of the lattice depicted in Diagram* 4. *The latter lattice is isomorphic to*

$$\mathscr{F}\mathscr{D}\mathscr{L}(\varepsilon T_l, \omega k, \omega t_r)/(\varepsilon T_l \leqq \omega k \vee \omega t_r).$$

Proof. We will verify first that Diagram 4 depicts the lattice generated by the set $\{\varepsilon T_l, \omega k, \omega t_r\}$ (with some possible coincidence of vertices). To this end, we will systematically perform the join and the meet of each fixed vertex with all the others, omitting trivial ones as well as those following from some other case either by monotonicity or by symmetry. These we list below and now observe that they follow directly by showing their $\mathscr{T}_l$-, $\mathscr{K}$- and $\mathscr{T}_r$-equivalence in a routine manner. In this, the congruence properties of $\mathscr{T}_l$, $\mathscr{K}$ and $\mathscr{T}_r$ will be used repeatedly.

That Diagram 4, as labelled, represents a partially ordered set is clear for all pairs except

$$(\varepsilon T_l \vee \omega t_r) \wedge \omega k \leqq \varepsilon T_l \vee (\omega k \wedge \omega t_r)$$

and its dual, both of which are easily verified as indicated above. And now for the joins and meets.

1. $(\varepsilon T_l \vee \omega k) \wedge (\varepsilon T_l \vee \omega t_r) = \varepsilon T_l \vee (\omega k \wedge \omega t_r)$.
2. $wk \vee (\varepsilon T_l \wedge \omega t_r) = \varepsilon T_l \vee \omega k$.
3. $[(\varepsilon T_l \vee \omega t_r) \wedge \omega k] \vee \omega t_r = \varepsilon T_l \vee \omega t_r$,
   $[(\varepsilon T_l \vee \omega t_r) \wedge \omega k] \vee (\varepsilon T_l \wedge \omega t_r) = \varepsilon T_l \vee (\omega k \wedge \omega t_r)$,
   $[(\varepsilon T_l \vee \omega t_r) \wedge \omega k] \wedge \omega t_r = \omega k \wedge \omega t_r$,
   $[(\varepsilon T_l \vee \omega t_r) \wedge \omega k] \wedge \varepsilon T_l = \varepsilon T_l \wedge \omega k$.
4. $(\varepsilon T_l \wedge \omega k) \vee (\varepsilon T_l \wedge \omega t_r) = \varepsilon T_l$,
   $(\varepsilon T_l \wedge \omega t) \vee \omega t_r = \varepsilon T_l \vee \omega t_r$.
5. $(\omega k \wedge \omega t_r) \vee (\varepsilon T_l \wedge \omega t_r) = (\varepsilon T_l \vee \omega k) \wedge \omega t_r$.
6. $\varepsilon T_l \vee [(\varepsilon T_l \vee \omega k) \wedge \omega t_r] = \varepsilon T_l \vee (\omega k \wedge \omega t_r)$.
7. $[\varepsilon T_l \vee (\omega k \wedge \omega t_r)] \vee \omega t_r = \varepsilon T_l \vee \omega t_r$.

Simple inspection will show that, except for trivial cases following by symmetry, we have covered all possibilities. Therefore Diagram 4 depicts a lattice with the vertices as labelled. In any particular semigroup S, some of the vertices may coincide which shows that the lattice generated by $\{\varepsilon T_l, \omega k, \omega t_r)$ is a homomorphic image of the one pictured in Diagram 4.

That the lattice depicted in Diagram 4 admits the representation in terms of generators and relations as indicated in the theorem can be checked easily on the free distributive lattice pictured in Diagram 1.

We have no example of a regular semigroup S for which $\mathscr{K}$ is a congruence and the lattice generated by the set $\{\varepsilon T_l, \omega k, \omega t_r\}$ is isomorphic to the lattice depicted in Diagram 4.

The proofs of the next four theorems follow the general pattern of the proof of Theorem 3.2. We will thus restrict arguments in these proofs to the bare minimum omitting the relevant discussions.

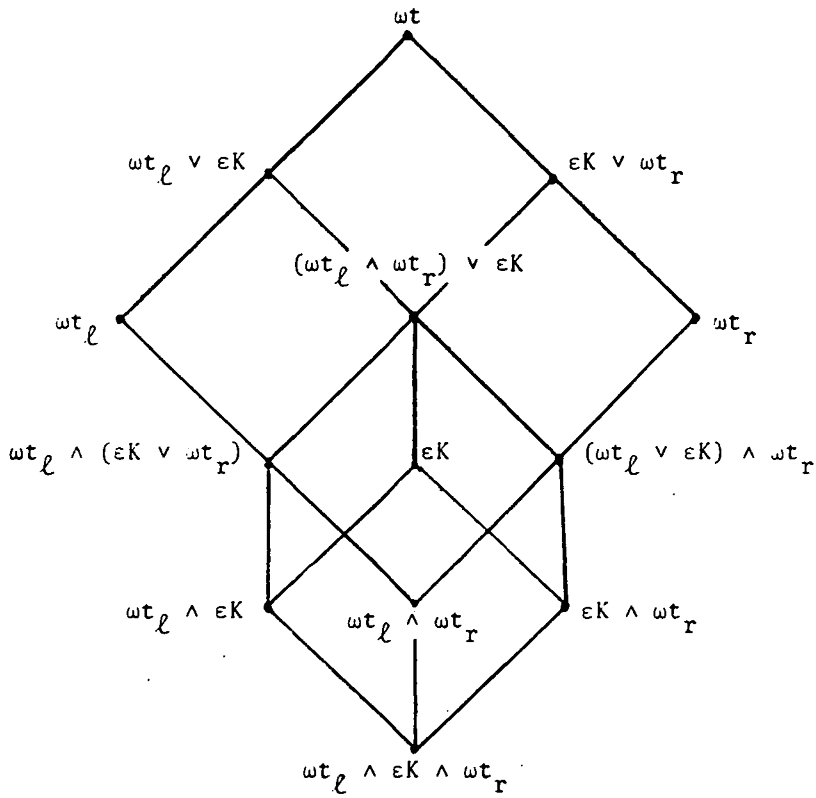

DIAGRAM 5. The lattice generated by $\{\omega t_l, \varepsilon K, \omega t_r\}$.

Theorem 3.3. *Let S be a regular semigroup for which $\mathscr{K}$ is a congruence. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\omega t_l, \varepsilon K, \omega t_r\}$ is a homomorphic image of the lattice depicted in Diagram 5. The latter lattice is isomorphic to*

$$\mathscr{F}\mathscr{D}\mathscr{L}(\omega t_l, \varepsilon K, \omega t_r)/(\varepsilon K \leqq \omega t_l \vee \omega t_r).$$

Proof. We follow the pattern of the proof of Theorem 3.2. That Diagram 5, with vertices as labelled, represents a partially ordered set is clear for all pairs of elements except

$$\omega t_l \wedge (\varepsilon K \vee \omega t_r) \leqq (\omega t_l \wedge \omega t_r) \vee \varepsilon K$$

and its dual, both of which are easily verified. Key joins and meets follow.

1. $(\omega t_l \vee \varepsilon K) \wedge (\varepsilon K \vee \omega t_r) = (\omega t_l \wedge \omega t_r) \vee \varepsilon K$.

2. $\omega t_l \vee (\varepsilon K \wedge \omega t_r) = \omega t_l \vee \varepsilon K$.

3. $[\omega t_l \wedge (\varepsilon K \vee \omega t_r)] \vee (\varepsilon K \wedge \omega t_r) = (\omega t_l \wedge \omega t_r) \vee \varepsilon K$,
   $[\omega t_l \wedge (\varepsilon K \vee \omega t_r)] \vee \omega t_r = \varepsilon K \vee \omega t_r$,
   $[\omega t_l \wedge (\varepsilon K \vee \omega t_r)] \wedge \varepsilon K = \omega t_l \wedge \varepsilon K$,
   $[\omega t_l \wedge (\varepsilon K \vee \omega t_r)] \wedge \omega t_r = \omega t_l \wedge \omega t_r$.
4. $(\omega t_l \wedge \varepsilon K) \vee \omega t_r = \varepsilon K \vee \omega t_r$,
   $(\omega t_l \wedge \varepsilon K) \vee [(\omega t_l \vee \varepsilon K) \wedge \omega t_r] = (\omega t_l \wedge \omega t_r) \vee \varepsilon K$.
5. $[(\omega t_l \wedge \omega t_r) \vee \varepsilon K] \vee \omega t_r = \varepsilon K \vee \omega t_r$,
   $[(\omega t_l \wedge \omega t_r) \vee \varepsilon K] \wedge \omega t_r = (\omega t_l \vee \varepsilon K) \wedge \omega t_r$.
6. $(\omega t_l \wedge \omega t_r) \vee (\varepsilon K \wedge \omega t_r) = (\omega t_l \vee \varepsilon K) \wedge \omega t_r$.

As in the proof of Theorem 3.2, we conclude that the lattice generated by the set $\{\omega t_l, \varepsilon K, \omega t_r\}$ is a homomorphic image of the one depicted in Diagram 5. The proof of the last assertion of the theorem follows similarly as in the proof of Theorem 3.2.

An example of a completely simple semigroup S for which the lattice generated by the set $\{\omega t_l, \varepsilon K, \omega t_r\}$ is isomorphic to the one depicted by Diagram 5 is provided by any Rees matrix semigroup with normalized sandwich matrix P in which at least two but not all rows (respectively columns) are identical.

Theorem 3.4. *Let S be a regular semigroup for which $\mathscr{K}$ is a congruence. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\varepsilon T_l, \omega k, \varepsilon T_r\}$ is a homomorphic image of the lattice depicted in Diagram 6. The latter lattice is isomorphic to*

$$\mathscr{F}\mathscr{D}\mathscr{L}(\varepsilon T_l, \omega k, \varepsilon T_r)/(\varepsilon T_l \wedge \varepsilon T_r \leqq \omega k).$$

Proof. We follow the pattern of the proof of Theorem 3.2. That Diagram 6, with vertices as labelled, represents a partially order set is clear for all pairs of elements except

$$\varepsilon T_l \vee (\omega k \wedge \varepsilon T_r) \geqq (\varepsilon T_l \vee \varepsilon T_r) \vee \omega k$$

and its dual, both of which are easily verified. Key joins and meets follow.

1. $(\varepsilon T_l \wedge \omega k) \vee (\omega k \wedge \varepsilon T_r) = (\varepsilon T_l \vee \varepsilon T_r) \wedge \omega k$.
2. $\varepsilon T_l \wedge (\omega k \vee \varepsilon T_r) = \varepsilon T_l \wedge \omega k$.
3. $[\varepsilon T_l \vee (\omega k \wedge \varepsilon T_r)] \wedge (\omega k \vee \varepsilon T_r) = (\varepsilon T_l \vee \varepsilon T_r) \wedge \omega k$,
   $[\varepsilon T_l \vee (\omega k \wedge \varepsilon T_r)] \wedge \varepsilon T_r = \omega k \vee \varepsilon T_r$,
   $[\varepsilon T_l \vee (\omega k \wedge T_r)] \vee \varepsilon T_r = \varepsilon T_l \vee \varepsilon T_r$.
4. $(\omega t_l \vee \omega k) \wedge \varepsilon T_r = \omega k \wedge \varepsilon T_r$,

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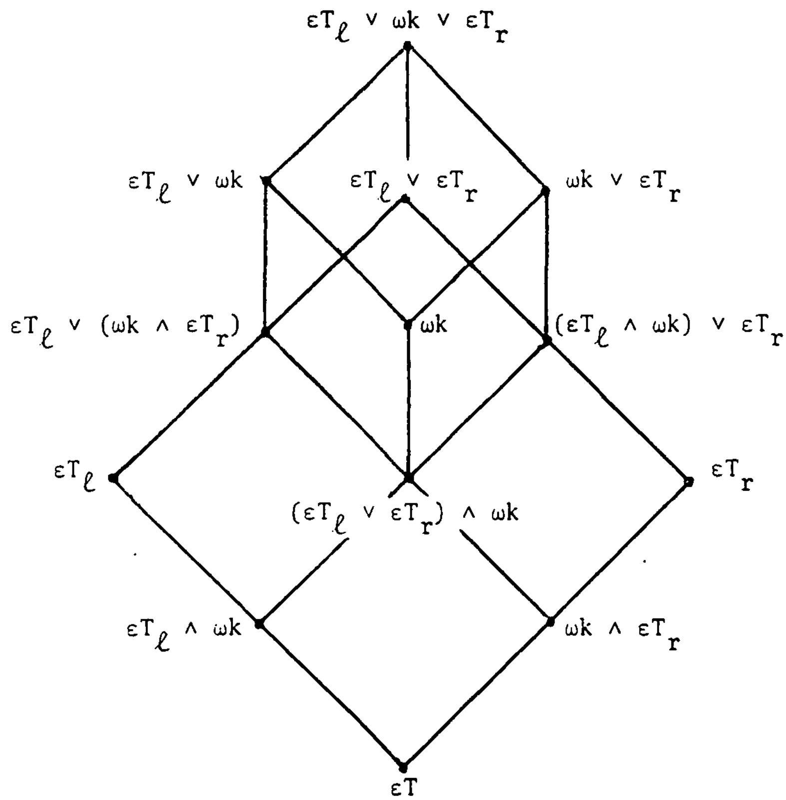

DIAGRAM 6. The lattice generated by $\{\varepsilon T_l, \omega k, \varepsilon T_t\}$.

$(\omega t_l \vee \omega k) \wedge [(\varepsilon T_l \wedge \omega k) \vee \varepsilon T_r] = (\varepsilon T_l \vee \varepsilon T_r) \wedge \omega k.$

5. $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \omega k] \wedge \varepsilon T_r = \omega k \vee \varepsilon T_r,$
   $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \omega k] \vee \varepsilon T_r = (\varepsilon T_l \wedge \omega k) \vee \varepsilon T_r.$

6. $(\varepsilon T_l \vee \varepsilon T_r) \wedge (\omega k \vee \varepsilon T_r) = (\varepsilon T_l \wedge \omega k) \vee \varepsilon T_r.$

As in the proof of Theorem 3.2, we conclude that the lattice generated by $\{\varepsilon T_l, \omega k, \varepsilon T_r\}$ is a homomorphic image of the one depicted in Diagram 6. The proof of the last assertion of the theorem follows similarly as in the proof of Theorem 3.2.

We now present an example of a primitive regular semigroup for which the lattice generated by the set $\{\varepsilon T_l, \omega k, \varepsilon T_r\}$ is isomorphic to the one depicted in Diagram 6.

Example 3.5. First let $B = \mathscr{M}^0(I, e, I; P)$ where $I = \{1, 2, 3\}$, e stands for the trivial group and

$$\mathrm{P}=\begin{bmatrix} e & e & 0 \\ 0 & 0 & e \\ 0 & 0 & e \end{bmatrix}.$$

Since the structure group of B is trivial, we write its nonzero elements as (i,j) and present them as an array:

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)

where full lines mean $\mathscr{R}^0$-classes and broken lines $\mathscr{L}^0$-classes. Hence $\varepsilon \neq \mathscr{L}^0 \neq \omega$, $\varepsilon \neq \mathscr{R}^0 \neq \omega$, $\mathscr{L}^0 \wedge \mathscr{R}^0 = \varepsilon$, $\mathscr{L}^0 \vee \mathscr{R}^0 \neq \omega$.

Next let A be a 2×2 rectangular band with a zero adjoined and let S be the orthogonal sum of A and B. In the first column below we list all the congruences occurring in Diagram 6 noting that $\varepsilon T_l = \mathscr{L}^0$, $\varepsilon T_r = \mathscr{R}^0$, $\omega k = \beta$—the least band congruence, and omitting the symmetric ones. In the second column we put their intersections with A and in the third their intersections with B.

$\mathscr{L}^0$	$\mathscr{L}$	$\mathscr{L}^0$
β	ε	ω
$\mathscr{L}^0 \vee \mathscr{R}^0$	$\mathscr{D}$	$\mathscr{L}^0 \vee \mathscr{R}^0$
$\mathscr{L}^0 \wedge \mathscr{R}^0$	ε	ε
$\mathscr{L}^0 \vee \beta$	$\mathscr{L}$	ω
$\mathscr{L}^0 \wedge \beta$	ε	$\mathscr{L}^0$
$\mathscr{L}^0 \vee (\beta \wedge \mathscr{R}^0)$	$\mathscr{L}$	$\mathscr{L}^0 \vee \mathscr{R}^0$
$\mathscr{L}^0 \vee \beta \vee \mathscr{R}^0$	$\mathscr{D}$	ω
$(\mathscr{L}^0 \vee \mathscr{R}^0) \wedge \beta$	ε	$\mathscr{L}^0 \vee \mathscr{R}^0$

For the lattice operations can be performed componentwise. Comparing these among themselves and with their duals, we see that the congruences in Diagram 6 are all distinct in this case. By Petrich [9, Theorem 3.6], $\mathscr{H}$ is a congruence for S.

Theorem 3.6. *Let S be a regular semigroup for which $\mathscr{H}$ is a congruence. Then the*

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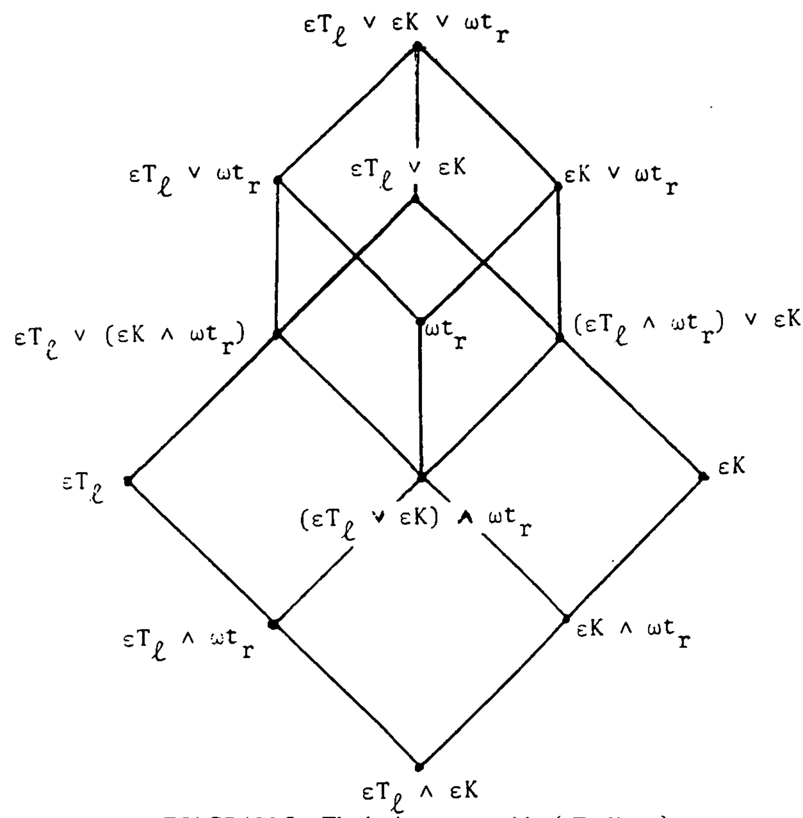


DIAGRAM 7. The lattice generated by $\{\varepsilon T_l, \varepsilon K, \omega t_r\}$.

sublatttice of $\mathscr{C}(S)$ generated by the set $\{\varepsilon T_l, \varepsilon K, \omega t_r\}$ is a homomorphic image of the lattice depicted in Diagram 7. The latter lattice is isomorphic to

$$\mathscr{F}\mathscr{D}\mathscr{L}(\varepsilon T_l, \varepsilon K, \omega t_r)/(\varepsilon T_l \wedge \varepsilon K \leqq \omega t_r).$$

Proof. We follow the pattern of the proof of Theorem 3.2. That Diagram 7, with vertices as labelled, represents a partially ordered set is clear for all pairs of elements except

$$(\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r \leqq (\varepsilon T_l \wedge \omega t_r) \vee \varepsilon K$$

and its dual, both of which are easily verified. Key joins and meets follow.

1. $(\varepsilon T_l \wedge \omega t_r) \vee (\varepsilon K \wedge \omega t_r) = (\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r$.

2. $\varepsilon T_l \wedge (\varepsilon K \vee \omega t_r) = \varepsilon T_l \wedge \omega t_r$.

3. $[\varepsilon T_l \vee (\varepsilon K \wedge \omega t_r)] \vee \omega t_r = \varepsilon T_l \vee \omega t_r$,

$[\varepsilon T_l \vee (\varepsilon K \wedge \omega t_r)] \vee \varepsilon K = \varepsilon T_l \vee \varepsilon K,$

$[\varepsilon T_l \vee (\varepsilon K \wedge \omega t_r)] \wedge (\varepsilon K \vee \omega t_r) = (\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r,$

$[\varepsilon T_l \vee (\varepsilon K \wedge \omega t_r)] \wedge \varepsilon K = \varepsilon K \wedge \omega t_r.$

4. $(\varepsilon T_l \vee \omega t_r) \wedge (\varepsilon T_l \vee \varepsilon K) = \varepsilon T_l \vee (\varepsilon K \wedge \omega t_r),$
   $(\varepsilon T_l \vee \omega t_r) \wedge (\varepsilon K \vee \omega t_r) = \omega t_r,$
   $(\varepsilon T_l \vee \omega t_r) \wedge [(\varepsilon T_l \wedge \omega t_r) \vee \varepsilon K] = (\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r,$
   $(\varepsilon T_l \vee \omega t_r) \wedge \varepsilon K = \varepsilon K \wedge \omega t_r.$
5. $[(\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r] \wedge \varepsilon K = \varepsilon K \wedge \omega t_r.$
6. $\omega t_r \wedge [(\varepsilon T_l \wedge \omega t_r) \vee \varepsilon K] = (\varepsilon T_l \vee \varepsilon K) \wedge \omega t_r.$
7. $(\varepsilon T_l \vee \varepsilon K) \wedge (\varepsilon K \vee \omega t_r) = (\varepsilon T_l \wedge \omega t_r) \vee \varepsilon K.$

As in the proof of Theorem 3.2, we conclude that the lattice generated by $\{\varepsilon T_l, \varepsilon K, \omega t_r\}$ is a homomorphic image of the one depicted in Diagram 7. The proof of the last assertion of the theorem follows similarly as in the proof of Theorem 3.2.

We have no example of a regular semigroup S for which $\mathscr{K}$ is a congruence and the lattice generated by the set $\{\varepsilon T_l, \varepsilon K, \omega t_r\}$ is isomorphic to the lattice depicted in Diagram 7.

Theorem 3.7. *Let S be a regular semigroup for which $\mathscr{K}$ is a congruence. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}$ is a homomorphic image of the lattice depicted in Diagram 8. The latter lattice is a free distributive lattice on the set $\{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}$.*

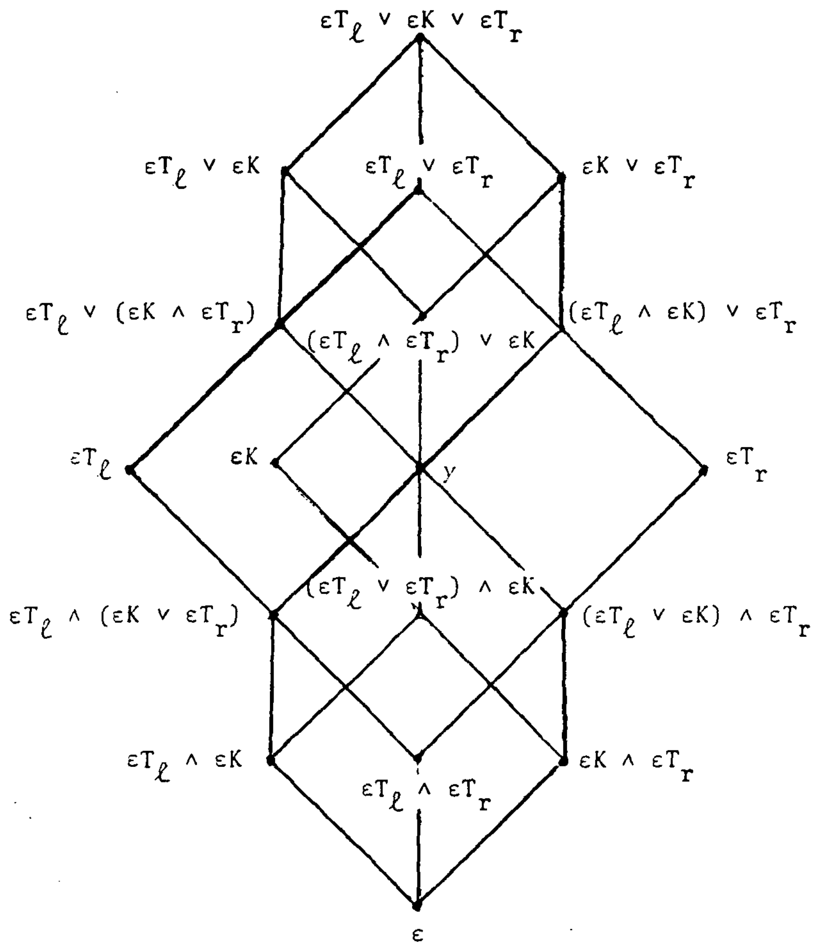
Proof. We follow the pattern of the proof of Theorem 3.2. That Diagram 8, with vertices as labelled, represents a partially ordered set is clear for all pairs of elements except

$$(\varepsilon T_l \vee \varepsilon T_r\} \wedge \varepsilon K \leqq y \leqq (\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K,$$

$$\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r) \leqq y, \qquad (\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r \leqq y$$

all of which can be easily verified. Key joins and meets follow.

1. $(\varepsilon T_l \wedge \varepsilon K) \vee [(\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r] = y,$
   $(\varepsilon T_l \wedge \varepsilon K) \vee (\varepsilon K \wedge \varepsilon T_r) = (\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K,$
   $(\varepsilon T_l \wedge \varepsilon K) \vee (\varepsilon T_l \wedge \varepsilon T_r) = \varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r).$
2. $[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \vee (\varepsilon K \wedge \varepsilon T_r) = y,$
   $[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \vee \varepsilon T_r = (\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r,$



where $y=[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \vee [(\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r]$

DIAGRAM 8. The lattice generated by $\{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}$.

$$[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \wedge \varepsilon T_r = \varepsilon T_l \wedge \varepsilon T_r,$$
$$[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \wedge [(\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K] = \varepsilon T_l \wedge \varepsilon K,$$
$$[\varepsilon T_l \wedge (\varepsilon K \vee \varepsilon T_r)] \wedge (\varepsilon K \wedge \varepsilon T_r) = \varepsilon T_l \wedge \varepsilon K \wedge \varepsilon T_r.$$

3. εT_l as labelled.

4. $[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \vee \varepsilon T_r = \varepsilon T_l \vee \varepsilon T_r,$
$$[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \vee \varepsilon K = \varepsilon T_l \vee \varepsilon K,$$
$$[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \vee (\varepsilon K \vee \varepsilon T_r) = \varepsilon T_l \vee \varepsilon K \vee \varepsilon T_r,$$

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$[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \wedge \varepsilon T_r = (\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r,$

$[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \wedge \varepsilon K = \varepsilon T_l \vee \varepsilon K,$

$[\varepsilon T_l \vee (\varepsilon K \wedge \varepsilon T_r)] \wedge (\varepsilon K \vee \varepsilon T_r) = y.$

5. $(\varepsilon T_l \vee \varepsilon K) \wedge (\varepsilon K \vee \varepsilon T_r) = (\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K,$

   $(\varepsilon T_l \vee \varepsilon K) \wedge [(\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r] = y.$

6. $\varepsilon K \vee [(\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r] = (\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K,$

   $\varepsilon K \wedge [(\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r] = (\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K.$

7. $(\varepsilon T_l \wedge \varepsilon T_r) \vee (\varepsilon K \wedge \varepsilon T_r) = (\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r.$

8. $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K] \vee (\varepsilon T_l \wedge \varepsilon T_r) = y,$

   $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K] \vee \varepsilon T_r = (\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r,$

   $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K] \wedge (\varepsilon T_l \wedge \varepsilon T_r) = \varepsilon T_l \wedge \varepsilon K \wedge \varepsilon T_r,$

   $[(\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K] \wedge \varepsilon T_r = \varepsilon K \wedge \varepsilon T_r.$

9. $y \vee \varepsilon K = (\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K,$

   $y \vee \varepsilon T_r = (\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r,$

   $y \wedge \varepsilon K = (\varepsilon T_l \vee \varepsilon T_r) \wedge \varepsilon K,$

   $y \wedge \varepsilon T_r = (\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r.$

10. $[(\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K] \vee (\varepsilon T_l \vee \varepsilon T_r) = \varepsilon T_l \vee \varepsilon K \vee \varepsilon T_r,$

    $[(\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K] \vee \varepsilon T_r = \varepsilon K \vee \varepsilon T_r,$

    $[(\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K] \wedge (\varepsilon T_l \vee \varepsilon T_r) = y,$

    $[(\varepsilon T_l \wedge \varepsilon T_r) \vee \varepsilon K] \wedge \varepsilon T_r = (\varepsilon T_l \vee \varepsilon K) \wedge \varepsilon T_r.$

11. $(\varepsilon T_l \vee \varepsilon T_r) \wedge (\varepsilon K \vee \varepsilon T_r) = (\varepsilon T_l \wedge \varepsilon K) \vee \varepsilon T_r.$

As in the proof of Theorem 3.2, we conclude that the lattice generated by $\{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}$ is a homomorphic image of the one depicted in Diagram 8. The proof of the last assertion of the theorem follows by a direct comparison of Diagrams 8 and 1.

We now present an example of a primitive regular semigroup for which the lattice generated by the set $\{\varepsilon T_l, \varepsilon K, \varepsilon T_r\}$ is isomorphic to the one depicted in Diagram 8.

Example 3.8. First let $B = \mathscr{M}^0(I, G, I; P)$ where $I = \{1, 2, 3, 4\}$, G is a group with identity e, $a \in G$, $a \neq e$ and

$$P = \begin{bmatrix} e & a^{-1} & 0 & 0 \\ 0 & 0 & e & e \\ 0 & 0 & e & a \\ e & a & 0 & 0 \end{bmatrix}.$$

We represent partitions of I by grouping the elements which belong to their classes. In terms of admissible triples we easily obtain

$$\varepsilon T_l = \mathscr{L}^0 \sim ((12)(34), G, \varepsilon),$$

$$\varepsilon T_r = \mathscr{R}^0 \sim (\varepsilon, G, (14)(23)),$$

$$\varepsilon K = \tau \sim ((12)(3)(4), e, (14)(2)(3)).$$

Hence $\mathscr{H} \subset \mathscr{L}^0 \subset \omega$, $\mathscr{H} \subset \mathscr{R}^0 \subset \omega$, $\mathscr{L}^0 \wedge \mathscr{R}^0 = \mathscr{H} \neq \varepsilon$, $\mathscr{L}^0 \vee \mathscr{R}^0 \neq \omega$.

Next let A be a 2×2 rectangular band with a zero adjoined and let S be the orthogonal sum of A and B. In the first column below we list all the congruences occurring in Diagram 8. In the second column we put their intersections with A and in the third their intersections with B.

$\mathscr{L}^0$	$\mathscr{L}$	$\mathscr{L}^0 \sim ((12)(34), G, \varepsilon)$
$\mathscr{R}^0$	$\mathscr{R}$	$\mathscr{R}^0 \sim (\varepsilon, G, (14)(23))$
$\mathscr{L}^0 \vee \mathscr{R}^0$	$\mathscr{D}$	$\mathscr{L}^0 \vee \mathscr{R}^0 \sim ((12)(34), G, (14)(23))$
$\mathscr{L}^0 \wedge \mathscr{R}^0$	ε	$\mathscr{H} \sim (\varepsilon, G, \varepsilon)$
τ	ω	$\tau \sim ((12)(3)(4), e, (14)(2)(3))$
$\mathscr{L}^0 \vee \tau$	ω	$\mathscr{L}^0 \vee \tau \sim ((12)(34), G, (14)(2)(3))$
$\tau \vee \mathscr{R}^0$	ω	$\tau \vee \mathscr{R}^0 \sim ((12)(3)(4), G, (14)(23))$
$\mathscr{L}^0 \wedge \tau$	$\mathscr{L}$	$\mathscr{L}^0 \wedge \tau \sim ((12)(3)(4), e, \varepsilon)$
$\tau \wedge \mathscr{R}^0$	$\mathscr{R}$	$\tau \wedge \mathscr{R}^0 \sim (\varepsilon, e, (14)(2)(3))$
$\mathscr{L}^0 \vee \tau \vee \mathscr{R}^0$	ω	$\mathscr{L}^0 \vee \tau \vee \mathscr{R}^0 \sim ((12)(34), G, (14)(23))$
$(\mathscr{L}^0 \wedge \mathscr{R}^0) \vee \tau$	ω	$\mathscr{H} \vee \tau \sim ((12)(3)(4), G, (14)(2)(3))$
$(\mathscr{L}^0 \vee \mathscr{R}^0) \wedge \tau$	$\mathscr{D}$	$(\mathscr{L}^0 \vee \mathscr{R}^0) \wedge \tau \sim ((12)(3)(4), e, (14)(2)(3))$
$\mathscr{L}^0 \vee (\tau \wedge \mathscr{R}^0)$	$\mathscr{D}$	$\mathscr{L}^0 \vee (\tau \wedge \mathscr{R}^0) \sim ((12)(34), G, (14)(2)(3))$
$(\mathscr{L}^0 \wedge \tau) \vee \mathscr{R}^0$	$\mathscr{D}$	$(\mathscr{L}^0 \wedge \tau) \vee \mathscr{R}^0 \sim ((12)(3)(4), G, (14)(23))$
$\mathscr{L}^0 \wedge (\tau \vee \mathscr{R}^0)$	$\mathscr{L}$	$\mathscr{L}^0 \wedge (\tau \vee \mathscr{R}^0) \sim ((12)(3)(4), G, \varepsilon)$
$(\mathscr{L}^0 \vee \tau) \wedge \mathscr{R}^0$	$\mathscr{R}$	$(\mathscr{L}^0 \vee \tau) \wedge \mathscr{R}^0 \sim (\varepsilon, G, ((14)(2)(3))$
y	$\mathscr{D}$	$y \sim ((12)(3)(4), G, (14)(2)(3))$

For the lattice operations can be performed componentwise. Comparing these among themselves and with their duals, we see that the congruences in Diagram 8 are all distinct in this case. By Petrich [9, Theorem 3.6], $\mathscr{K}$ is a congruence for S.

4. The lattice generated by $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$

As in the preceding section, we represent this lattice as a homomorphic image of a lattice depicted by a diagram.

Lemma 4.1. *Let S be a regular semigroup and let $\lambda, \rho \in \mathscr{C}(S)$.*

(i) $\operatorname{tr}\lambda \subseteq \operatorname{tr}\rho \Leftrightarrow \operatorname{tr}(\lambda \wedge \rho) = \operatorname{tr}\lambda \Leftrightarrow \operatorname{tr}(\lambda \vee \rho) = \operatorname{tr}\rho$.

(ii) $\ker\lambda \subseteq \ker\rho \Leftrightarrow \ker(\lambda \wedge \rho) = \ker\rho$.

(iii) *If $\mathscr{K}$ is a congruence for S, then* $\ker\lambda \subseteq \ker\rho \Leftrightarrow \ker(\lambda \vee \rho) = \ker\rho$.

Proof. Straightforward using Pastijin and Petrich [4, Lemma 2.5(i) and Corollary 4.9].

Theorem 4.2. *Let S be a regular semigroup for which $\mathscr{K}$ is a congruence. Then the sublattice of $\mathscr{C}(S)$ generated by the set $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$ is a homomorphic image of the lattice depicted in Diagram 9. The latter lattice is isomorphic to*

$$\mathscr{F}\mathscr{D}\mathscr{L}(\varepsilon K, \omega k, \varepsilon T, \omega t\}/(\varepsilon K \leqq \omega t, \varepsilon T \leqq \omega k\}$$

and none of these relations may be omitted.

Proof. Since $\operatorname{tr}\varepsilon \subseteq \operatorname{tr}(\varepsilon K \wedge \omega k) \subseteq \operatorname{tr}\varepsilon K$ and $\ker\varepsilon \subseteq \ker(\omega t \wedge \varepsilon T) \subseteq \ker\varepsilon T$, Lemma 4.1 gives that the joins and meets in the interval $[\varepsilon, \varepsilon K \vee \varepsilon T]$ are as indicated in Diagram 9 for the left side down to right side up lines contain $\mathscr{T}$-related congruences and the right side down to left side up lines contain $\mathscr{K}$-related congruences. The same type of argument goes through for the interval $[\omega k \wedge \omega t, \omega]$. In view of symmetry of the diagram, it suffices to verify the following cases.

1. $(\varepsilon K \vee \varepsilon T) \vee [(\omega t \vee \varepsilon T) \wedge \omega k] \mathscr{K} \varepsilon T \vee (\omega t \vee \varepsilon T) = \omega t \vee \varepsilon T$,
$(\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k) \mathscr{K} (\omega t \vee \varepsilon T) \wedge (\varepsilon \vee \omega) = \omega t \vee \varepsilon T$,
$(\varepsilon K \vee \varepsilon T) \vee [(\omega t \vee \varepsilon T) \wedge \omega k] \mathscr{T} \varepsilon K \vee (\omega \wedge \omega k) = \varepsilon K \vee \omega k$,
$(\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k) \mathscr{T} (\omega \vee \varepsilon) \wedge (\varepsilon K \vee \omega k) = \varepsilon K \vee \omega k$,

and therefore

$$(\varepsilon K \vee \varepsilon T) \vee [(\omega t \vee \varepsilon T) \wedge \omega k] = (\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k). \tag{2}$$

2. $(\varepsilon K \vee \varepsilon T) \wedge [(\omega t \vee \varepsilon T) \wedge \omega k] \mathscr{K} \varepsilon T \wedge (\omega t \vee \varepsilon T) = \varepsilon T$,
$(\varepsilon K \wedge \omega k) \vee \varepsilon T \mathscr{K} (\varepsilon \wedge \omega) \vee \varepsilon T = \varepsilon T$,

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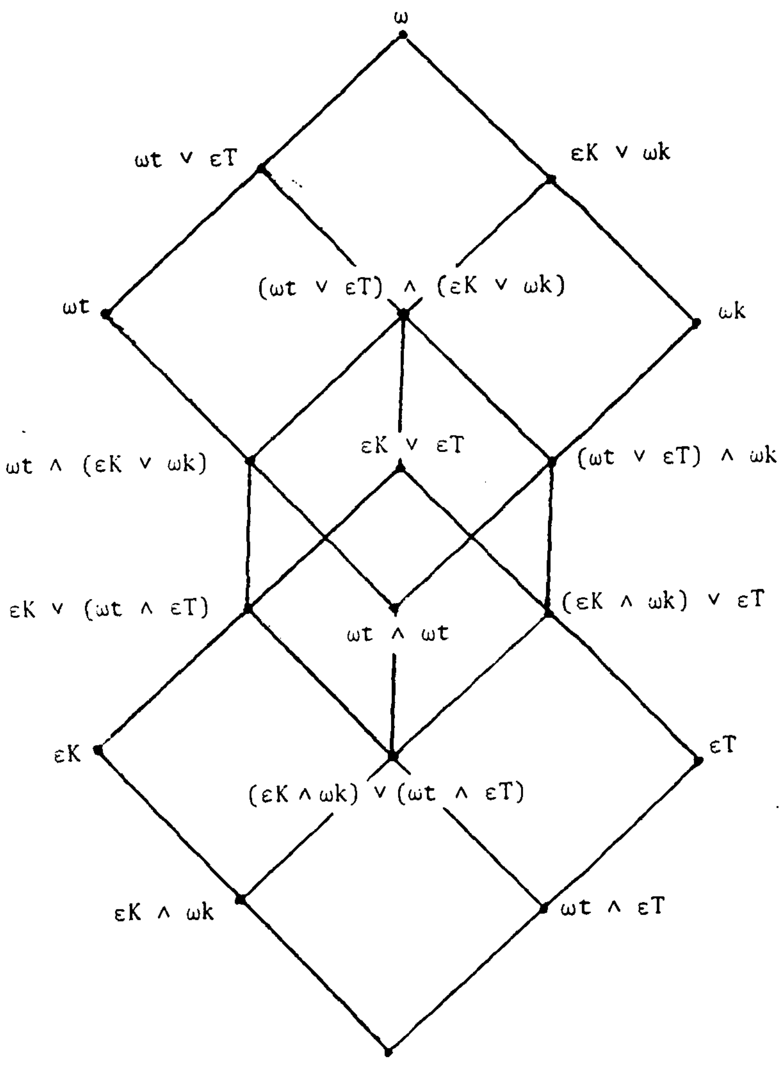

DIAGRAM 9. The lattice generated by $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$.

$$(\varepsilon K \vee \varepsilon T) \wedge [(\omega t \vee \varepsilon T) \wedge \omega k] \mathscr{T} \varepsilon K \wedge [(\omega \vee \varepsilon) \wedge \omega k] = \varepsilon K \wedge \omega k,$$
$$(\varepsilon K \wedge \omega k) \vee \varepsilon T \mathscr{T} \varepsilon K \wedge \omega k$$

and therefore

$$(\varepsilon K \vee \varepsilon T) \wedge [(\omega t \vee \varepsilon T) \wedge \omega k] = (\varepsilon K \wedge \omega k) \vee \varepsilon T. \tag{3}$$

3. The equalities

$$(\omega t \wedge \omega k) \wedge [(\varepsilon K \wedge \omega k) \vee \varepsilon T] = (\varepsilon K \wedge \omega k) \vee (\omega t \wedge \varepsilon T), \tag{4}$$

$$(\omega t \wedge \omega k) \vee [(\varepsilon K \wedge \omega k) \vee \varepsilon T] = (\omega t \vee \varepsilon T) \wedge \omega k \tag{5}$$

can be established similarly as equalities (2) and (3) above. In fact, (4) can be obtained from (2) and (5) from (3) by the following transformation

$$\omega \leftrightarrow \varepsilon, \qquad t \leftrightarrow K, \qquad k \leftrightarrow T, \qquad \vee \leftrightarrow \wedge. \tag{6}$$

4. $(\varepsilon K \vee \varepsilon T) \vee (\omega t \wedge \omega k) \mathscr{K} \varepsilon T \vee \omega t$,
   $(\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k) \mathscr{K} \omega t \vee \varepsilon t$, as computed above
   $(\varepsilon K \vee \varepsilon T) \vee (\omega t \wedge \omega k) \mathscr{T} \varepsilon K \vee \omega k$,
   $(\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k) \mathscr{T} \varepsilon K \vee \omega k$, as computed above

and therefore

$$(\varepsilon K \vee \varepsilon T) \vee (\omega t \wedge \omega k) = (\omega t \vee \varepsilon T) \wedge (\varepsilon K \vee \omega k).$$

5. $(\varepsilon K \vee \varepsilon T) \wedge (\omega t \wedge \omega k) \mathscr{K} \varepsilon T \wedge \omega t$,
   $(\varepsilon K \wedge \omega k) \vee (\omega t \wedge \varepsilon T) \mathscr{K} (\varepsilon \wedge \omega) \vee (\omega t \vee \varepsilon T) = \omega t \vee \varepsilon T$,
   $(\varepsilon K \vee \varepsilon T) \wedge (\omega t \wedge \omega k) \mathscr{T} \varepsilon K \wedge \omega k$,
   $(\varepsilon K \wedge \omega k) \vee (\omega t \wedge \varepsilon T) \mathscr{T} \varepsilon K \wedge \omega k \vee (\omega \wedge \varepsilon) = \varepsilon K \wedge \omega k$,

and therefore

$$(\varepsilon K \vee \varepsilon T) \wedge (\omega t \wedge \omega k) = (\varepsilon K \wedge \omega k) \vee (\omega t \wedge \varepsilon T).$$

We have proved that Diagram 9 depicts the lattice generated by the set $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$ with a possible coincidence of vertices. But this means that our lattice is a homomorphic image of the one depicted in Diagram 9.

We now prove the next assertion of the theorem. To facilitate our notation, we let

$$a = \varepsilon K, \qquad b = \omega t, \qquad c = \varepsilon T, \qquad d = \omega k.$$

Our task is to construct the free distributive lattice L on the generators a, b, c, d subject to the relations $a \leqq b$ and $c \leqq d$. We consider L as a subdirect product of copies of the nontrivial subdirectly irreducible distributive lattice $Y = \{0, 1\}$. Since L is generated by four elements, all homomorphisms of L into Y may be represented by at most 2^4 quadruples of 0's and 1's. In the obvious notation, for such a string (x_1, x_2, x_3, x_4) to qualify as an endomorphism, it is necessary and sufficient that

$$x_1 = 1 \Rightarrow x_2 = 1, \qquad x_3 = 1 \Rightarrow x_4 = 1,$$

$$x_2 = 0 \Rightarrow x_1 = 0, \qquad x_4 = 0 \Rightarrow x_3 = 0.$$

By a simple selection, and omitting the trivial strings (0 0 0 0) and (1 1 1 1), which give no information, we arrive at the following list.

a	b	c	d
0	0	0	1
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	1
1	1	0	0
1	1	0	1

We now label the four columns by a, b, c and d and construct the lattice L generated by these elements with coordinatewise operations within the lattice $Y=\{0,1\}$. Since L is a distributive lattice, its elements can be written as joins of meets of generators. We thus first make a complete list of all meets of generators thereby obtaining

$$a, b, c, d; \quad a \wedge d, \quad b \wedge c, \quad b \wedge d \tag{7}$$

and $a \wedge c=(0\,0\,0\,0\,0\,0\,0)$, which is the zero of L and need not be taken into further consideration. All the joins of the first part of (7) are

$$a \vee c, \quad a \vee d, \quad b \vee c \tag{8}$$

and $b \vee d=(1\,1\,1\,1\,1\,1\,1)$, which is the identity of L and may be henceforth omitted. The only join of the second part of (7) is

$$(a \wedge d) \vee (b \wedge c). \tag{9}$$

For the joins of elements in the first and second parts of (7), we get

$$a \vee (b \wedge c), \quad a \vee (b \wedge d), \quad c \vee (a \wedge d), \quad c \vee (b \wedge d), \quad a \vee c \vee (b \wedge d), \tag{10}$$

where we have omitted the terms which have occurred earlier. Computing these meets and joins for the elements in (7)–(10), we obtain the lattice in Diagram 10.

Relabelling several vertices and going back to the original notation εK, ωt, εT and ωk, we see that Diagrams 10 and 9 essentially coincide. This establishes the isomorphism assertion of the theorem.

We are indebted to Jiří Sichler for an outline of this part of the proof.

In order to prove the last assertion of the theorem, it suffices to construct quadruples of 0's and 1's which satisfy exactly one of the given relations for both choices. Indeed,

	a	b	c	d
$a \leqq b$	1	0	0	0
$c \leqq d$	0	0	1	0

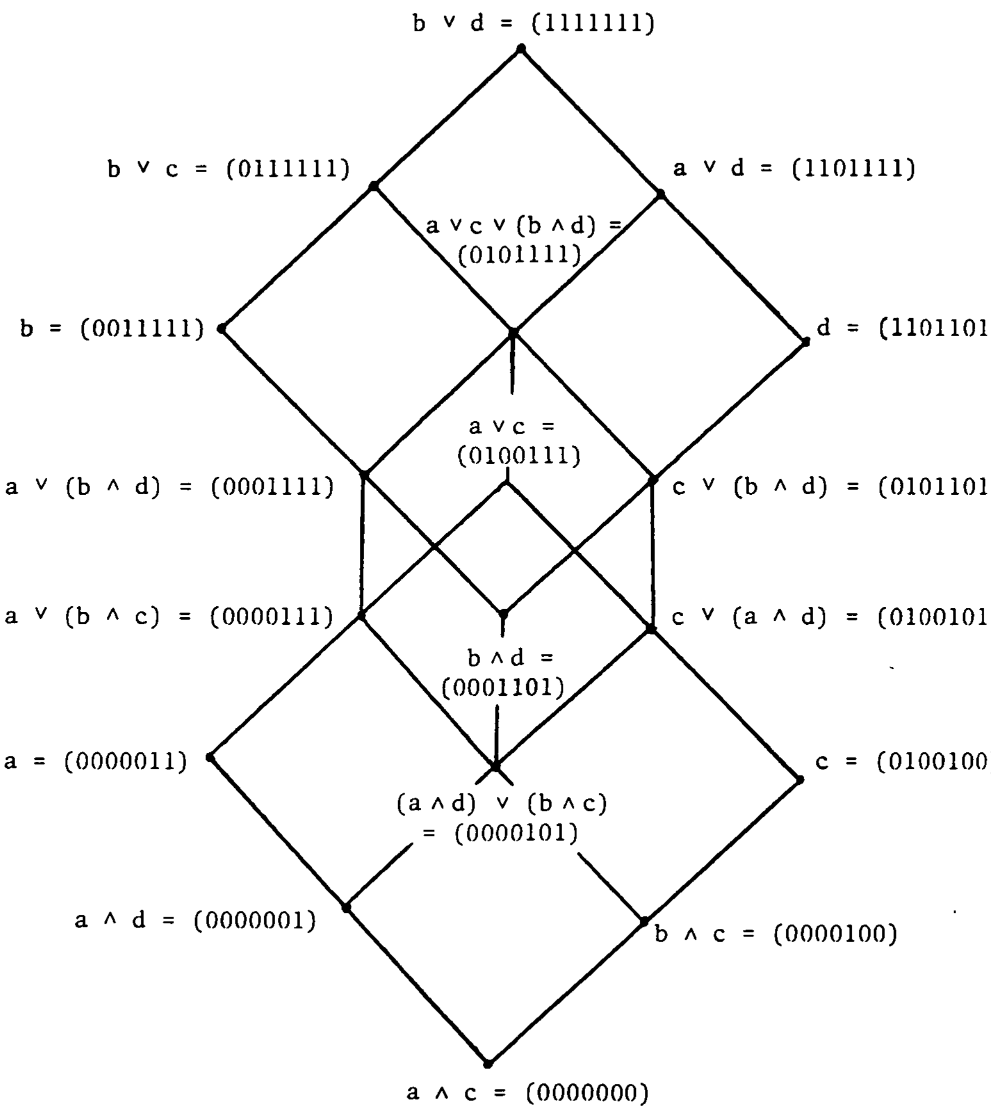

DIAGRAM 10

We have no example of a regular semigroup for which $\mathscr{K}$ is a congruence and the lattice generated by $\{\varepsilon K, \omega k, \varepsilon T, \omega t\}$ is isomorphic to the lattice depicted in Diagram 9.

5. Remarks

From Section 3 and by symmetry, we have the following relations

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$$\omega t_l \wedge \omega k \leqq \omega t_r, \qquad \omega k \wedge \omega t_r \leqq \omega t_l,$$

$$\varepsilon T_l \leqq \omega k \vee \omega t_r, \qquad \varepsilon K \leqq \omega t_l \vee \omega t_r, \qquad \varepsilon T_r \leqq \omega t_l \vee \omega k,$$

$$\varepsilon T_l \wedge \varepsilon K \leqq \omega t_r, \qquad \varepsilon T_l \wedge \varepsilon T_r \leqq \omega k, \qquad \varepsilon K \wedge \varepsilon T_r \leqq \omega t_l.$$

By a change of notation, we are dealing with a lattice generated by the set $\{\alpha_i, \beta_i \mid i=1,2,3\}$ where the above relations become

$$\begin{matrix} \beta_1 \wedge \beta_2 \leqq \beta_3, & \beta_2 \wedge \beta_3 \leqq \beta_1, \\ \alpha_i \leqq \beta_j \vee \beta_k, & \alpha_i \wedge \alpha_j \leqq \beta_k \end{matrix} \qquad \text{for } \{i,j,k\}=\{1,2,3\}.$$

Alternatively, we may introduce new relations by setting

$$\mathscr{K}_l = \mathscr{K} \wedge \mathscr{T}_l, \qquad \mathscr{T} = \mathscr{T}_l \wedge \mathscr{T}_r, \qquad \mathscr{K}_r = \mathscr{K} \wedge \mathscr{T}_r.$$

For any $\rho \in \mathscr{C}(S)$ and $\mathscr{P} \in \{\mathscr{K}_l, \mathscr{T}, \mathscr{K}_r\}$, we may write

ρP for the greatest congruence $\mathscr{P}$-related to ρ,

ρp for the least congruence $\mathscr{P}$-related to ρ.

Some of the above relations then become

$$\varepsilon T_l \leqq \omega k_r, \qquad \varepsilon K \leqq \omega t, \qquad \varepsilon T_r \leqq \omega k_l,$$

$$\varepsilon K_l \leqq \omega t_r, \qquad \varepsilon T \leqq \omega k, \qquad \varepsilon K_r \leqq \omega t_l.$$

The second relation in each of the preceding two lines also appears in Diagram 9.

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