# 4

# Stable Pairs over Reduced Base Schemes

So far we have identified stable pairs $(X, \Delta)$ as the basic objects of our moduli problem, defined stable and locally stable families of pairs over one-dimensional regular schemes in Chapter 2, and in Chapter 3 we treated families of varieties over reduced base schemes. Here we unite the two by discussing stable and locally stable families over reduced base schemes.

After stating the main results in Section 4.1, we give a series of examples in Section 4.2. The technical core of the chapter is the treatment of various notions of families of divisors given in Section 4.3. Valuative criteria are proved in Section 4.4 and the behavior of generically $\mathbb{R}$-Cartier divisors is studied in Section 4.5.

In Section 4.6, we finally define stable and locally stable families over reduced base schemes (4.7) and prove that local stability is a representable property. Families over a smooth base scheme are especially well behaved; their properties are discussed in the short Section 4.7.

The universal family of Mumford divisors is constructed in Section 4.8; this is probably the main technical result of the chapter. The correspondence between (not necessarily flat) families of Mumford divisors and flat families of Cayley–Chow hypersurfaces – established over reduced bases in Theorem 4.69 – leads to the fundamental notion of Cayley flatness in Chapter 7.

At the end, we have all the ingredients needed to treat the moduli functor $\mathcal{SP}^{\mathrm{red}}$, which associates to a reduced scheme $S$ the set of all stable families $f\colon (X, \Delta) \to S$, up to isomorphism. (Here the superscript $^{\mathrm{red}}$ indicates that we work with reduced base schemes only.)

To be precise, we fix the dimension $n$ of the fibers, a finite set of allowed coefficients $\mathbf{c} \subset [0, 1]$ and the volume $v$. Our families are $f : (X, \Delta) \to S$, where $X \to S$ is flat and projective, $\Delta$ is a Weil $\mathbb{R}$-divisor on $X$ whose coefficients are in $\mathbf{c}$, $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier, and the fibers $(X_s, \Delta_s)$ are stable pairs of dimension $n$ with $\mathrm{vol}(K_{X_s} + \Delta_s) := ((K_{X_s} + \Delta_s)^n) = v$. This gives the functor

$$\mathcal{SP}^{\rm red}(\mathbf{c}, n, v) \colon \{\text{reduced } S\text{-schemes}\} \to \{\text{sets}\}.$$

We can now state one of the main consequence of the results of this chapter.

**Theorem 4.1** (Moduli theory of stable pairs I) *Let $S$ be an excellent base scheme of characteristic 0 and fix $n, \mathbf{c}, v$. Then $\mathcal{SP}^{red}(\mathbf{c}, n, v)$ is a good moduli theory (6.10), which has a projective, coarse moduli space* $\mathrm{SP}^{red}(\mathbf{c}, n, v) \to S$.

Moreover, $\mathrm{SP}^{\rm red}(\mathbf{c}, n, v)$ is the reduced subscheme of the "true" moduli space $\mathrm{SP}(\mathbf{c}, n, v)$ of marked, stable pairs, to be constructed in Chapter 8.

**Assumptions** In the foundational Sections 4.1–4.5 we work with arbitrary schemes, but for Sections 4.6 and 4.7 we need to assume that the base scheme is over a field of characteristic 0.

## 4.1 Statement of the Main Results

In the study of locally stable families of pairs over reduced base schemes, the key step is to give the "correct" definition for the divisorial component

**Temporary Definition 4.2** A *family of pairs* (with $\mathbb{Z}$-coefficients) of dimension $n$ over a reduced scheme is an object

$$f \colon (X, D) \to S, \tag{4.2.1}$$

consisting of a morphism of schemes $f \colon X \to S$ and an effective Weil divisor $D$ satisfying the following properties.

4.2.2 (Flatness for $X$) *The morphism $f \colon X \to S$ is flat, of finite type, of pure relative dimension n, with geometrically reduced fibers.* This is the expected condition from the point of view of moduli theory, following the Principles (3.12) and (3.13).

4.2.3 (Equidimensionality for $\operatorname{Supp} D$) *Every irreducible component $D_i \subset \operatorname{Supp} D$ dominates an irreducible component of $S$ and all nonempty fibers of $\operatorname{Supp} D \to S$ have pure dimension $n-1$.* In particular, $\operatorname{Supp} D$ does not contain any irreducible component of any fiber of $f$. If $S$ is normal then $\operatorname{Supp} D \to S$ has pure relative dimension $n-1$ by (2.71.2), but in general our assumption is weaker. We noted in (2.41) that $D \to S$ need not be flat for locally stable families. So we start with this weak assumption and strengthen it later.

4.2.4 (Mumford condition) *The morphism $f$ is smooth at generic points of $X_s \cap \operatorname{Supp} D$ for every $s \in S$.* Equivalently, for each $s \in S$, none of the irreducible components of $X_s \cap \operatorname{Supp} D$ is contained in $\operatorname{Sing}(X_s)$.

This condition was first codified in Mumford's observation that, in order to get a good moduli theory of pointed curves $(C, P)$, the marked points $P = \{p_1, \dots, p_n\}$ should be smooth points of $C$; see Section 4.8 for details.

If $(X, \Delta)$ is an slc pair, then $X$ is smooth at all generic points of $\operatorname{Supp} \Delta$. So if $D$ is an effective divisor supported on $\operatorname{Supp} \Delta$, then this conditions is satisfied.

It turns out that such generic smoothness is a crucial condition technically. So we make it part of the definition for families of pairs.

A big advantage is that, if $S$ is reduced, then $X$ is regular at the generic points of $\operatorname{Supp} D$. Thus, as for normal varieties, we can harmlessly identify Mumford divisors with divisorial subschemes; see (4.16.6–7) for details.

Next we come to the heart of the matter: we would like the notion of families of pairs to give a functor. So, for any morphism $g : W \to S$, we need to define the pulled-back family. We have a fiber product diagram

$$\begin{array}{ccc} X \times_S W & \xrightarrow{q_X} & X \\ {\scriptstyle f_W}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{q} & S. \end{array} \tag{4.2.5}$$

It is clear that we should take $X_W := X \times_S W$, with morphism $f_W : X_W \to W$. The definition of the divisor part $D_W$ is less clear, since pull-backs of Cartier and of Weil divisors are not compatible in general.

4.2.6 (Weil-divisor pull-back) For any subscheme $Z \subset X$ and morphism $h : Y \to X$, define the *Weil-divisor pull-back* as the Weil divisor $\operatorname{Weil}(h^{-1}(Z))$ associated to the subscheme $h^{-1}(Z) \subset Y$; see (4.16.6) for formal definitions.

Let $D, X$ be as in (4.2.1) and $g : W \to S$ a morphism. Using the Mumford condition we can view $D$ as a subscheme of $X$. Then set

$$g^*_{\mathrm{Wdiv}}(D) := \operatorname{Weil}(g_X^{-1}(D)).$$

In particular, if $\tau : \{s\} \to S$ is a point, we get the *Weil-divisor fiber,* denoted by $\tau^*_{\mathrm{Wdiv}}(D)$.

If $H \subset X$ is a relative Cartier divisor and $g_X^* H$ does not contain any codimension $\leq 1$ associated points of $g_X^{-1}(D)$, then

$$g^*_{\mathrm{Wdiv}}(D \cap H) = g^*_{\mathrm{Wdiv}}(D) \cap g_X^* H.$$

*Warning* The Weil-divisor fiber is always defined, but frequently not functorial, not even additive. If $D', D''$ are two divisors on $X$ then $\tau^*_{\mathrm{Wdiv}}(D' + D'')$

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and $\tau^*_{\mathrm{Wdiv}}(D') + \tau^*_{\mathrm{Wdiv}}(D'')$ have the same support, but the multiplicities can be different, even in étale locally trivial families as in (4.14). If $D', D''$ satisfy (4.2.4), then $\tau^*_{\mathrm{Wdiv}}(D' + D'') \le \tau^*_{\mathrm{Wdiv}}(D') + \tau^*_{\mathrm{Wdiv}}(D'')$, but otherwise the inequality can go the other way; see (4.12) and (4.13).

4.2.7 (Generically Cartier divisor and pull-back) Assume that $D$ is a relative Cartier divisor (4.20) on an open subset $U \subset X$ such that $g_X^{-1}(U \cap D)$ is dense in $g_X^{-1}(D)$. We can then define the *generically Cartier pull-back* of $D$ as

$$g^{[*]}(D) := \text{ the closure of } g_X^{-1}(D|_U) \subset X_W.$$

If $f$ has $S_2$ fibers then $\mathscr{O}_{X_W}(-g^{[*]}(D))$ is the hull pull-back of $\mathscr{O}_X(-D)$ (3.27). The generically Cartier pull-back is clearly functorial, but not always defined. If it is defined, then $g^*_{\mathrm{Wdiv}}(D)$ is the Weil divisor corresponding to $g^{[*]}(D)$, so the two notions are equivalent; see (4.6).

4.2.8 (Well-defined pull-backs) We say that $f\colon (X, D) \to S$ has *well-defined Weil-divisor pull-backs* if it satisfies the assumptions (4.2.2–4) and the Weil-divisor pull-back (4.2.6) is a functor for reduced schemes. That is,

$$h^*_{\mathrm{Wdiv}}(g^*_{\mathrm{Wdiv}}(D)) = (g \circ h)^*_{\mathrm{Wdiv}}(D)$$

for all morphisms of reduced schemes $h\colon V \to W$ and $g\colon W \to S$.

In any concrete situation, the conditions (4.2.2–4) should be easy to check, but (4.2.8) requires computing $g^*_{\mathrm{Wdiv}}(D)$ for all morphisms $W \to S$. The following variant is much easier to verify.

4.2.9 (Well-defined specializations) We say that $f\colon (X, D) \to S$ has *well-defined specializations* if (4.2.8) holds whenever $W$ is the spectrum of a DVR.

The good news is that, over reduced schemes, the three versions (4.2.6–9) are equivalent to each other and also to other natural conditions. The common theme is that we need to understand only the codimension 1 behavior of $f\colon (X, D) \to S$.

**Theorem-Definition 4.3** (Well-defined families of pairs I) *Let $S$ be a reduced scheme. A family of pairs $f\colon (X, D) \to S$ satisfying (4.2.2–4) is* well defined *if the following equivalent conditions hold.*

(4.3.1) *The family has well-defined Weil-divisor pull-backs (4.2.8).*

(4.3.2) *The family has well-defined specializations (4.2.9).*

(4.3.3) *$D$ is a relative, generically Cartier divisor (4.2.7).*

(4.3.4) *$D \to S$ is flat at the generic points of $X_s \cap \operatorname{Supp} D$ for every $s \in S$.*

*If $f$ is projective then these are also equivalent to*

(4.3.5) *$s \mapsto \deg(X_s \cap D)$ is a locally constant function on $S$.*

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The theorem is proved in (4.25). The next result says that, if $S$ is normal, then the conditions (4.2.2–4) imply that $f\colon (X, D) \to S$ is well defined. It follows from (4.21) by setting $W := \operatorname{Sing} S$.

**Theorem 4.4** (Ramanujam (1963); Samuel (1962)) *Let $S$ be a normal scheme, $f\colon X \to S$ a smooth morphism and $D$ a Weil divisor on $X$. Assume that $D$ does not contain any irreducible component of a fiber. Then $D$ is a Cartier divisor, hence a relative Cartier divisor.*

Over nonnormal base schemes it is usually easy to check well-definedness using the normalization.

**Corollary 4.5** *Let $S$ be a reduced scheme with normalization $\bar{S} \to S$. Let $f\colon (X, D) \to S$ be a projective family of pairs satisfying the assumptions (4.2.2–4) and*

$$\bar{f} : (\bar{X}, \bar{D}) := (X, D) \times_S \bar{S} \to \bar{S}$$

*the corresponding family over $\bar{S}$. Then $D$ is a relative, generically Cartier divisor in either of the following cases.*

(4.5.1) $\tau^*_{Wdiv}(D) = \bar{\tau}^*_{Wdiv}(\bar{D}) = \bar{\tau}^{[*]}(\bar{D})$ *for every geometric point* $\tau\colon \{s\} \to S$ *and for every lifting* $\bar{\tau} : \{s\} \to \bar{S}$.

(4.5.2) $S$ *is weakly normal and* $\bar{\tau}^*_{Wdiv}(\bar{D}) = \bar{\tau}^{[*]}(\bar{D})$ *is independent of the lifting* $\bar{\tau} : s \to \bar{S}$ *for every geometric point* $\tau : \{s\} \to S$.

*Proof* Note first that $\bar{D}$ is a relative, generically Cartier divisor by (4.4), so $\bar{\tau}^*_{\mathrm{Wdiv}}(\bar{D}) = \bar{\tau}^{[*]}(\bar{D})$.

Let $g \in S$ be a generic point. Then $(\bar{D})_g = D_g$ and $\deg \bar{\tau}^*_{\mathrm{Wdiv}}(\bar{D}) = \deg(\bar{D})_g$ by (4.3) applied to $\bar{f} : (\bar{X}, \bar{D}) \to \bar{S}$. Together with (1) this shows that (4.3.5) holds for $f : (X, D) \to S$.

For (2), we explain in (4.25) how to reduce everything to the special case when $f$ has relative dimension 1. Then (10.64) shows that $D$ is flat over $S$. □

Next we turn to the case that we are really interested in, when the boundary $\Delta$ is a $\mathbb{Q}$ or $\mathbb{R}$-divisor. The right choice is to work with the relative, generically Cartier condition.

**Definition 4.6** (Divisorial pull-back) Let $S$ be a scheme, $f\colon X \to S$ a morphism and $\Delta$ a $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$-divisor on $X$. For $q\colon W \to S$, consider the fiber product as in (4.2.5). We define *relatively, generically Cartier* divisors and their *divisorial pull-backs,* denoted by $\Delta_W$, in three steps as follows.

(4.6.1) $\Delta$ is a relatively, generically Cartier $\mathbb{Z}$-divisor if it is Cartier at the generic points of $X_s \cap \operatorname{Supp} D$ for every $s \in S$. $\Delta_W$ is then defined as in (4.2.7).

(4.6.2) $\Delta$ is a relatively, generically $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor iff $m\Delta$ is a relatively, generically Cartier $\mathbb{Z}$-divisor for some $m > 0$. Then we set $\Delta_W := \frac{1}{m}((m\Delta)_W)$.

This is independent of $m$, but there is a subtle point. We prove in (4.39) that, if the characteristic is 0, then a $\mathbb{Z}$-divisor is relatively, generically $\mathbb{Q}$-Cartier iff it is relatively, generically Cartier. So we can choose $m$ to be the common denominator of the coefficients in $\Delta$. However, this is not true in positive characteristic; see (8.75–8.76).

(4.6.3) $\Delta$ is a relatively, generically $\mathbb{R}$-Cartier $\mathbb{R}$-divisor iff one can write $\Delta = \sum c_i \Delta_i$ where the $\Delta_i$ are relatively, generically $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. Then we set $\Delta_W := \sum c_i (\Delta_i)_W$.

This is independent of the choice of $c_i$ and $\Delta_i$. We may assume that the $c_i$ are $\mathbb{Q}$-linearly independent. Then $\Delta$ is relatively, generically $\mathbb{R}$-Cartier iff the $\Delta_i$ are relatively, generically $\mathbb{Q}$-Cartier by (11.43.2).

Let $f\colon (X, \Delta) \to S$ be a well-defined family of pairs as in (4.3). In (3.1) we gave seven equivalent definitions of locally stable families of varieties. Some of these extend to families of pairs. See (2.41) for some negative examples and Section 8.2 for some solutions.

**Definition–Theorem 4.7** Let $S$ be a reduced scheme, $f\colon X \to S$ a flat morphism of finite type and $f\colon (X, \Delta) \to S$ a well-defined family of pairs. Assume that $(X_s, \Delta_s)$ is slc for every $s \in S$. Then $f\colon (X, \Delta) \to S$ is *locally stable* or *slc* if the following equivalent conditions hold.

(4.7.1) $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier.

(4.7.2) For every spectrum of a DVR $T$ and morphism $q\colon T \to S$, the pullback $f_T\colon (X_T, \Delta_T) \to T$ is locally stable, as in (2.3).

(4.7.3) There is a closed subset $Z \subset X$ such that $\operatorname{codim}(Z \cap X_s, X_s) \geq 3$ for every $s \in S$ and $f|_{X \setminus Z}\colon (X \setminus Z) \to S$ satisfies the above (1–2).

Such a family is called *stable* if, in addition, $f$ is proper and $K_{X/S} + \Delta$ is $f$-ample.

*Proof* The arguments are essentially the same as in (3.37). It is clear that (4.7.1) $\Rightarrow$ (4.7.2). If (4.7.2) holds then $K_{X_T} + \Delta_T$ is $\mathbb{R}$-Cartier for every $q : T \to S$. Thus $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier by (4.35).

Finally, if any of the properties (4.7.1–2) holds for $X$, then it also holds for $X \setminus Z$. Using (4.7.2) both for $X$ and for $X \setminus Z$, reduces us to checking (4.7.3) $\Rightarrow$ (4.7.2) when $S$ is the spectrum of a DVR; which is (2.7). □

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Let $f: (X, \Delta) \to S$ be a family of pairs. It turns out that, starting in relative dimension 3, the set of points $\{s \in S : (X_s, \Delta_s) \text{ is slc}\}$ is neither open nor closed; see (3.41) for an example. Thus the strongest result one can hope for is the following.

**Theorem 4.8** (Local stability is representable) *Let $S$ be a reduced, excellent scheme over a field of characteristic 0 and $f: (X, \Delta) \to S$ a well-defined, projective family of pairs. Assume that $\Delta$ is an effective, relative, generically $\mathbb{R}$-Cartier divisor. Then there is a locally closed partial decomposition $j : S^{ls} \to S$ such that the following holds.*

*Let $W$ be any reduced scheme and $q: W \to S$ a morphism. Then the family obtained by base change $f_W: (X_W, \Delta_W) \to W$ is locally stable iff $q$ factors as $q: W \to S^{ls} \to S$.*

A stable morphism is locally stable and stability is an open condition for a locally stable morphism. Thus (4.8) implies the following.

**Corollary 4.9** (Stability is representable) *Using the notation and assumptions as in (4.8), there is a locally closed partial decomposition $j : S^{stab} \to S$ such that the following holds.*

*Let $W$ be any reduced scheme and $q : W \to S$ a morphism. Then the family obtained by base change $f_W : (X_W, \Delta_W) \to W$ is stable iff $q$ factors as $q : W \to S^{stab} \to S$.* □

## 4.2 Examples

We start with a series of examples related to (4.3).

**Example 4.10** Let $S = (xy = 0) \subset \mathbb{A}^2$ and $X = (xy = 0) \subset \mathbb{A}^3$. Consider the divisors $D_x := (y = z - 1 = 0)$ and $D_y := (x = z + 1 = 0)$. We get a family $f : (X, D_x + D_y) \to S$ that satisfies the assumptions (4.2.2–4).

We compute the "fiber" of the family over the origin in three different ways and get three different results.

First, restrict the family to the $x$-axis. The pull-back of $X$ becomes the plane $\mathbb{A}^2_{xz}$. The divisor $D_x$ pulls back to $(z-1 = 0)$, but the pull-back of the ideal sheaf of $D_y$ is the maximal ideal $(x, z + 1)$. It has no divisorial part, so restriction to the $x$-axis gives the pair $(\mathbb{A}^2_{xz}, (z - 1 = 0)) \to \mathbb{A}^1_x$. Similarly, restriction to the $y$-axis gives the pair $(\mathbb{A}^2_{yz}, (z + 1 = 0)) \to \mathbb{A}^1_y$. If we restrict these to the origin, we get $(\mathbb{A}^1_z, (z - 1 = 0))$ and $(\mathbb{A}^1_z, (z + 1 = 0))$.

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Finally, if we restrict to the origin of $S$ in one step then we get the pair $(\mathbb{A}^1_z, (z-1=0)+(z+1=0))$. Thus we have three different pairs that can claim to be the fiber of $f:(X, D_x+D_y)\to S$ over the origin.

In this example the problem is visibly set-theoretic, but there can be problems even when the set theory works out.

**Example 4.11** Set $C := (xy(x-y)=0)\subset \mathbb{A}^2_{xy}$ and $X := (xy(x-y)=0)\subset \mathbb{A}^3_{xyz}$. For any $c\in k$ consider the divisor

$$D_c := (x=z=0)+(y=z=0)+(x-y=z-cx=0).$$

The pull-back of $D_c$ to any of the irreducible components of $X$ is Cartier, it intersects the central fiber at the origin of the $z$-axis and with multiplicity 1. Nonetheless, we claim that $D_c$ is Cartier only for $c=0$.

Indeed, assume that $h(x,y,z)=0$ is a local equation of $D_c$. Then $h(x,0,z)=0$ is a local equation of the $x$-axis and $h(0,y,z)=0$ is a local equation of the $y$-axis. Thus $h=az+$ (higher terms). Restricting to the $(x-y=0)$ plane we get that $c=0$.

Note also that if $\operatorname{char} k=0$ and $c\neq 0$ then no multiple of $D_c$ is a Cartier divisor. To see this note that if $f(x,y,z)=0$ is a local defining equation of $mD_c$ on $X$ then $\partial^{m-1}f/\partial z^{m-1}$ vanishes on $D_c$. Its restriction to the $z$-axis vanishes at the origin with multiplicity 1. We proved above that this is not possible.

However, if $\operatorname{char} k=p>0$, then $z^p-c^p xy^{p-1}=0$ shows that $pD_c$ is a Cartier divisor.

**Example 4.12** Consider the cusp $C:=(x^2=y^3)\subset\mathbb{A}^2_{xy}$ and the trivial curve family $Y:=C\times\mathbb{A}^1_z\to C$. Let $D\subset Y$ be the Cartier divisor given by the equation $y=z^2$. Then $D\to C$ is flat of degree 2. Furthermore, $D$ is reducible with irreducible components $D^{\pm}:=$ image of $t\mapsto(t^3,t^2,\pm t)$.

Note that $D^{\pm}\simeq\mathbb{A}^1_t$ and the projections $D^{\pm}\to C$ corresponds to the ring extension $k[t^3,t^2]\hookrightarrow k[t]$. Thus the projections $D^{\pm}\to C$ are not flat and the Weil-divisorial fiber of $D^{\pm}\to C$ over the origin has length 2.

However, the Weil-divisorial fiber of $D=D^+\cup D^-\to C$ over the origin is again the point $(0,0,0)$ with multiplicity 2.

Arguing as in (4.11) shows that the $D^{\pm}$ are not $\mathbb{Q}$-Cartier in characteristic 0, but $pD^+=(xy^{(p-3)/2}=z^p)$ shows that it is $\mathbb{Q}$-Cartier in characteristic $p>0$.

The next example shows the importance of the Mumford condition.

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**Example 4.13** Set $X = (x^2 - y^2 = u^2 - v^2) \subset \mathbb{A}^4$,$D = (x - u = y - v = 0) \cup (x + u = y + v = 0)$ and $f : (X, D) \to \mathbb{A}^2_{uv}$ the coordinate projection. The fiber $X_{uv}$ is a pair of intersecting lines if $u^2 = v^2$ and a smooth conic otherwise.

The irreducible components of $D$ intersect only at the origin and $D$ is not Cartier there. The divisorial fiber $D_{uv}$ consists of 2 distinct smooth points if $(u, v) \neq (0, 0)$, but $D_{00}$ is the origin with multiplicity 3.

Let $L_c$ be the line $(v = cu)$ for some $c \neq \pm 1$. Restricting the family to $L_c$ we get $X_c = (x^2 - y^2 = (1 - c^2)u^2) \subset \mathbb{A}^3$ and the divisor becomes $D_c = (x - u = y - cu = 0) \cup (x + u = y + cu = 0)$. Observe that $D_c$ is a Cartier divisor with defining equation $cx = y$. (Note that base change does not commute with union, so $D \times_{\mathbb{A}^2} L_c$ has an embedded point at the origin.)

Thus although $D$ is not Cartier at the origin, after base change to a general line we get a Cartier divisor. For all of these base changes, $D_c$ has multiplicity 2 at the origin. (These also hold after base change to any smooth curve.)

However, the origin is a singular point of the fiber. If we restrict $D_c$ to the fiber over the origin, the resulting scheme structure varies with $c$.

This would be a very difficult problem to deal with, but for a stable pair $(X, \Delta)$ we are in a better situation since the irreducible components of $\Delta$ are not contained in $\operatorname{Sing} X$.

**Example 4.14** Let $B$ be a smooth projective curve of genus $\geq 1$ with an involution $\sigma$ and $b_1, b_2 \in B$ a pair of points interchanged by $\sigma$. Let $C'$ be another smooth curve with two points $c'_1, c'_2 \in C'$. Start with the trivial family $(B \times C', \{b_1\} \times C' + \{b_2\} \times C') \to C'$ and then identify $c'_1 \sim c'_2$ and $(b, c'_1) \sim (\sigma(b), c'_2)$ for every $b \in B$. We get an étale locally trivial stable morphism $(S, D_1 + D_2) \to C$. Here $C$ is a nodal curve with node $\tau : \{c\} \to C$. The fiber over the node is $(B, [b_1] + [b_2])$. However, the fiber of each $D_i$ over $c$ is $[b_1] + [b_2]$, hence

$$\tau^*_{\mathrm{Wdiv}}(D_1) + \tau^*_{\mathrm{Wdiv}}(D_1) = (B, 2[b_1] + 2[b_2]) \neq (B, [b_1] + [b_2]) = \tau^*_{\mathrm{Wdiv}}(D_1 + D_2).$$

The next examples discuss the variation of the $\mathbb{Q}$-Cartier property in families of divisors. Related positive results are in Section 4.6.

**Example 4.15** Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and $S_C \subset \mathbb{P}^3$ the cone over it. For $p \in C$ let $L_p \subset S_C$ denote the ruling over $p$. Note that $L_p$ is $\mathbb{Q}$-Cartier iff $p$ is a torsion point, that is, $3m[p] \sim \mathscr{O}_C(m)$ for some $m > 0$. The latter is a countable dense subset of the moduli space of the lines $\mathrm{Chow}_{1,1}(S_C) \simeq C$.

In the above example the surface is not $\mathbb{Q}$-factorial and the curve $L_p$ is sometimes $\mathbb{Q}$-Cartier, sometimes not. Next we give a similar example of a flat family of lc surfaces $S \to B$ such that $\{b : S_b \text{ is } \mathbb{Q}\text{-factorial}\} \subset B$ is a countable set of points. Thus being $\mathbb{Q}$-factorial is not a constructible condition.

Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Pick 11 points $P_1, \dots, P_{11} \in C$ and set $P_{12} = -(P_1 + \cdots + P_{11})$. Then there is a quartic curve $D$ such that $C \cap D = P_1 + \cdots + P_{12}$. Thus the linear system $\left|\mathcal{O}_{\mathbb{P}^2}(4)(-P_1 - \cdots - P_{12})\right|$ blows up the points $P_i$ and contracts $C$. Its image is a degree 4 surface $S = S(P_1, \dots, P_{11})$ in $\mathbb{P}^3$ with a single simple elliptic singularity. If $C = (f_3(x, y, z) = 0)$ and $D = (f_4(x, y, z) = 0)$ then

$$S \simeq (f_3(x, y, z)w + f_4(x, y, z) = 0) \subset \mathbb{P}^3.$$

At the point $(x = y = z = 0)$, the singularity of $S$ is analytically isomorphic to the cone $S_C$ and $S$ is smooth elsewhere iff the points $P_1, \dots, P_{12}$ are distinct. If this holds, then the class group of $S$ is generated by the image $L$ of a line in $\mathbb{P}^2$ and the images $E_1, \dots, E_{12}$ of the 12 exceptional curves. They satisfy a single relation $3L = E_1 + \cdots + E_{12}$. Note that $E_i$ is $\mathbb{Q}$-Cartier iff $P_i$ is a torsion point.

If we vary $P_1, \dots, P_{11} \in C$, we get a flat family of lc surfaces parametrized by $\pi : \mathbf{S} \to C^{11} \setminus$ (diagonals), with universal divisors $\mathbf{E}_i \subset \mathbf{S}$. We see that

(4.15.1) $E_i(P_1, \dots, P_{11})$ is $\mathbb{Q}$-Cartier iff $P_i$ is a torsion point and

(4.15.2) $S(P_1, \dots, P_{11})$ is $\mathbb{Q}$-factorial iff $P_i$ is a torsion point for every $i$.

## 4.3 Families of Divisors II

At least three different notions of effective divisors are commonly used in algebraic geometry, but our discussions show that other variants are also necessary.

**4.16** (Five notions of effective divisors) Let $X$ be an arbitrary scheme.

(4.16.1) An effective *Cartier divisor* is a subscheme $D \subset X$ such that, for every $x \in D$, the ideal sheaf $\mathcal{O}_X(-D)$ is locally generated by a non-zero divisor $s_x \in \mathcal{O}_{x,X}$, called a *local equation* of $D$.

(4.16.2) A *divisorial subscheme* is a subscheme $D \subset X$ such that $\mathcal{O}_D$ has no embedded points and $\operatorname{Supp} D$ has pure codimension 1 in $X$.

(4.16.3) A divisorial subscheme $D$ is called an effective, *generically Cartier divisor* if it is Cartier at its generic points. These are called almost Cartier divisors in Hartshorne (1986) and Hartshorne and Polini (2015).

(4.16.4) A divisorial subscheme $D$ is called an effective *Mumford divisor* if $X$ is regular at generic points of $D$. More generally, $D$ is *Mumford along $Z$*, if $X$ and $Z$ are both regular at every generic point of $Z \cap \operatorname{Supp} D$.

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(4.16.5) A *Weil divisor* is a formal, finite linear combination $D = \sum_i m_i D_i$ where $m_i \in \mathbb{Z}$ and the $D_i$ are integral subschemes of codimension 1 in $X$. We say that $D$ is effective if $m_i \geq 0$ for every $i$.

If $A$ is an abelian group then a *Weil A-divisor* is a formal, finite linear combination $D = \sum_i a_i D_i$ where $a_i \in A$. We will only use the cases $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Thus Weil $\mathbb{Z}$-divisor = traditional Weil divisor; we use the terminology "Weil $\mathbb{Z}$-divisor" if the coefficient group is not clear. (A Weil $\mathbb{Z}$-divisor is sometimes called an integral Weil divisor, but the latter could also mean the Weil divisor corresponding to an integral subscheme of codimension 1.)

Note that usually divisorial subschemes and Weil divisors are used only when $X$ is irreducible or at least pure dimensional, but the definition makes sense in general.

If $X$ is smooth then the five variants are equivalent to each other, but in general they are different.

Usually we think of Cartier divisor as the most restrictive notion. If $X$ is $S_2$ then every effective Cartier divisor is a divisorial subscheme. However, if $X$ is not $S_2$, then there are Cartier divisors $D \subset X$ such that $D$ is not a divisorial subscheme, and the natural map from Cartier divisors to divisorial subschemes is not injective; see (4.16.9). These are good to keep in mind, but they will not cause problems for us.

Let $W \subset X$ be a closed subscheme. We can associate to it both a divisorial subscheme and a Weil divisor by the rules

$$\begin{array}{lcl} \operatorname{Div}(W) & := & \operatorname{pure} W := \operatorname{Spec}(\mathscr{O}_W/(\text{torsion})), \quad \text{and} \\ \operatorname{Weil}(W) & := & \sum_i \operatorname{length}_{g_i}(\mathscr{O}_{g_i,W}) \cdot [D_i], \end{array} \tag{4.16.6}$$

where, in the first case, we take the quotient by the subsheaf of those sections whose support has codimension $\geq 2$ in $X$ (see also (10.1)). In the second case, $D_i \subset \operatorname{Supp} W$ are the irreducible components of codimension 1 in $X$ with generic points $g_i \in D_i$. In particular, this associates an effective Weil divisor to any effective Cartier divisor or divisorial subscheme.

Thus, if $X$ is $S_2$, then we have the basic relations among effective divisors

$$\begin{pmatrix} \text{Cartier} \\ \text{divisors} \end{pmatrix} \subset \begin{pmatrix} \text{Mumford} \\ \text{divisors} \end{pmatrix} \subset \begin{pmatrix} \text{generically} \\ \text{Cartier divisors} \end{pmatrix} \subset \begin{pmatrix} \text{divisorial} \\ \text{subschemes} \end{pmatrix}.$$

Assume next that $X$ is regular at a codimension 1 point $g \in X$. Then $\mathscr{O}_{g,X}$ is a DVR, hence its ideals are uniquely determined by their colength. Thus we have the following.

*Claim 4.16.7* If $X$ is a normal scheme then four of the notions agree for effective divisors

$$\begin{pmatrix}\text{Mumford}\\ \text{divisors}\end{pmatrix}=\begin{pmatrix}\text{generically}\\ \text{Cartier divisors}\end{pmatrix}=\begin{pmatrix}\text{divisorial}\\ \text{subschemes}\end{pmatrix}=\begin{pmatrix}\text{Weil}\\ \text{divisors}\end{pmatrix}.$$

We are mainly interested in slc pairs $(X, \Delta)$, thus the underlying schemes $X$ are demi-normal. Fortunately, $X$ is smooth at the generic points of $\Delta$. Thus, for our purposes, we can always imagine that the identifications (4.16.7) hold.

*Convention 4.16.8* Let $X$ be a scheme and $W \subset X$ a subscheme. Assume that $X$ is regular at all generic points of $W$. Then we will frequently identify $\mathrm{Div}(W)$, the divisorial subscheme associated to $W$ and $\mathrm{Weil}(W)$, the Weil divisor associated to $W$. We denote this common object by $[W]$.

We can thus usually harmlessly identify divisorial subschemes and Weil divisors. However – and this is one of the basic difficulties of the theory – it is quite problematic to keep the identification between *families* of divisorial subschemes and *families* of Weil divisors.

*Example 4.16.9* Let $S \subset \mathbb{A}^4$ be the union of the planes $(x_1 = x_2 = 0)$ and $(x_3 = x_4 = 0)$. For $c \neq 0$, consider the Cartier divisors $D_c := (x_1 + cx_3 = 0)$. For any $c$, the corresponding divisorial subscheme is the union of the lines $(x_1 = x_2 = x_3 = 0) \cup (x_1 = x_3 = x_4 = 0)$, hence independent of $c$. However the $D_c$ are different Cartier divisors for different $c \in k$. Indeed, $(x_1+c'x_3)/(x_1+cx_3)$ is a nonregular rational function that is constant $c'/c$ on the first plane and 1 on the second. Note that $S$ is seminormal and the $D_c$ are Mumford.

Corresponding to the five notions of divisors, there are five notions of families. We discuss four of these next, leaving Mumford divisors to Section 4.8.

## Relative Weil divisors

**Definition 4.17** Let $f : X \to S$ be a morphism whose fibers have pure dimension $n$. A Weil divisor $W = \sum m_i W_i$ is called a *relative Weil divisor* if the fibers of $f|_{W_i} : W_i \to f(W_i)$ have pure dimension $n-1$ for every $i$.

We are interested in defining the divisorial fibers of $W \to S$. A typical example is (4.13), where the multiplicity of the scheme-theoretic fiber jumps over the origin. It is, however, quite natural to say that the "correct" fiber is the origin with multiplicity 2; the only problem we have is that scheme theory miscounts the multiplicity. The following theorem, proved in Kollár (1996, 3.17), says that this is indeed frequently the case.

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**Theorem 4.18** *Let $S$ be a normal scheme, $f: X \to S$ a projective morphism, and $Z \subset X$ a closed subscheme such that $f|_Z : Z \to S$ has pure relative dimension $m$. Then there is a section $\sigma_Z : S \to \mathrm{Chow}_m(X/S)$ with the following properties.*

(4.18.1) *Let $g \in S$ be the generic point. Then $\sigma_Z(g) = [Z_g]$, the cycle associated to the generic fiber of $f|_Z : Z \to S$ as in (3.8).*

(4.18.2) $\mathrm{Supp}(\sigma_Z(s)) = \mathrm{Supp}(Z_s)$ *for every $s \in S$.*

(4.18.3) $\sigma_Z(s) = [Z_s]$ *if $f|_Z$ is flat at all generic points of $Z_s$.*

(4.18.4) $s \mapsto (\sigma_Z(s) \cdot L^m)$ *is a locally constant function of $s \in S$, for any line bundle $L$ on $X$.* □

Example (4.10) shows that (4.18) does not hold if $S$ is only seminormal. The notion of well-defined families of algebraic cycles is designed to avoid similar problems, leading to the definition of the Cayley–Chow functor; see Kollár (1996, sec.I.3–4) for details.

## Flat Families of Divisorial Subschemes

Let $X \to S$ be a morphism and $D \subset X$ a subscheme. If Supp $D$ does not contain any irreducible component of a fiber $X_s$, then $\mathscr{O}_{D \cap X_s}/(\text{torsion})$ is (the structure sheaf of) a divisorial subscheme of $X_s$. This notion, however, frequently does not have good continuity properties, as illustrated by (4.13).

We would like to have a notion of flat families of divisorial subschemes, where both the structure sheaf $\mathscr{O}_D$ and the ideal sheaf $\mathscr{O}_X(-D)$ are "well behaved." This seems possible only if $X \to S$ is "well behaved," but then the two aspects turn out to be equivalent.

**Definition–Lemma 4.19** Let $f : X \to S$ be a flat morphism of pure relative dimension $n$ with $S_2$ fibers and $D \subset X$ a closed subscheme of relative dimension $n-1$ over $S$. We say that $f|_D : D \to S$ is a *flat family of divisorial subschemes* if the following equivalent conditions hold.

(4.19.1) $f|_D : D \to S$ is flat with pure fibers of dimension $n-1$ (10.1).

(4.19.2) $\mathscr{O}_X(-D)$ is flat over $S$ with $S_2$ fibers.

If $f$ is projective and pure $D_s$ denotes the largest pure subscheme as in (10.1), these are further equivalent to:

(4.19.3) $s \mapsto \chi(X_s, \mathscr{O}_{\mathrm{pure}\, D_s}(*))$ is locally constant on $S$.

(4.19.4) $s \mapsto \chi(X_s, \mathscr{O}_{X_s}(-\,\mathrm{pure}\, D_s)(*))$ is locally constant on $S$.

*Proof* We have a surjection $\mathscr{O}_X \to \mathscr{O}_D$ and if both of these sheaves are flat then so is the kernel $\mathscr{O}_X(-D)$. If the kernel is flat then $\mathscr{O}_{X_s}(-D_s) \simeq \mathscr{O}_X(-D)|_{X_s}$

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is also the kernel of $\mathscr{O}_{X_s} \to \mathscr{O}_{D_s}$. Since $\mathscr{O}_{X_s}$ is $S_2$, we see that $\mathscr{O}_{X_s}(-D_s)$ is $S_2$ iff $\mathscr{O}_{D_s}$ is pure of dimension $n-1$.

Conversely, assume (2). For any $T \to S$ the pull-back map $q_T^*\mathscr{O}_X(-D) \to q_T^*\mathscr{O}_X$ is an isomorphism over $X_T \setminus D_T$. Since $\mathscr{O}_X(-D)$ is flat with $S_2$ fibers, $q_T^*\mathscr{O}_X(-D)$ does not have any sections supported on $D_T$. Thus the pulled-back sequence

$$0 \to q_T^*\mathscr{O}_X(-D) \to q_T^*\mathscr{O}_X \to q_T^*\mathscr{O}_D \to 0$$

is exact. Therefore, $\operatorname{Tor}_1^S(\mathscr{O}_T, \mathscr{O}_D) = 0$ hence $\mathscr{O}_D$ is flat over $S$ and we already noted that then it has pure fibers of dimension $n-1$.

The last two claims are proved as in (2.75). □

## Relative Cartier Divisors

**Definition–Lemma 4.20** Let $f\colon X \to S$ be a flat morphism with $S_2$ fibers, $x \in X$ a point, and $s := f(x)$. A subscheme $D \subset X$ is a *relative Cartier divisor* at $x \in X$ if the following equivalent conditions hold.

(4.20.1) $D$ is flat over $S$ at $x$ and $D_s := D|_{X_s}$ is a Cartier divisor on $X_s$ at $x$.

(4.20.2) $D$ is a Cartier divisor on $X$ at $x$ and a local equation $g_x \in \mathscr{O}_{x,X}$ of $D$ restricts to a non-zerodivisor on the fiber $X_s$.

(4.20.3) $D$ is a Cartier divisor on $X$ at $x$ and it does not contain any irreducible component of $X_s$ that passes through $x$.

If these hold for all $x \in D$ then $D$ is a *relative Cartier divisor.* If $f\colon X \to S$ is also proper then the functor of relative Cartier divisors is represented by an open subscheme of the Hilbert scheme $\operatorname{CDiv}(X/S) \subset \operatorname{Hilb}(X/S)$; see Kollár (1996, I.1.13) for the easy details.

If (2) holds then $D$ is flat by (4.19). The other nontrivial claim is that (1) implies that $D$ is a Cartier divisor on $X$ at $x$. We may assume that $(x \in X)$ is local. A defining equation $g_s$ of $D_s$ lifts to an equation $g$ of $D$. We have the exact sequence

$$0 \to I_D/(g) \to \mathscr{O}_X/(g) \to \mathscr{O}_D \to 0.$$

Here $\mathscr{O}_X/(g)$ and $\mathscr{O}_D$ are both flat, hence so is $I_D/(g)$. Restricting to $X_s$ we get

$$0 \to (I_D/(g))_s \to \mathscr{O}_{X_s}/(g_s) \xrightarrow{\simeq} \mathscr{O}_{D_s} \to 0.$$

Thus $I_D/(g) = 0$ by Nakayama's lemma and $g$ is a defining equation of $D$. □

Relative Cartier divisors form a very well behaved class, but in applications we frequently have to handle two problems. It is not always easy to see which divisors are Cartier, and we also need to deal with divisors that are not Cartier.

On a smooth variety every divisor is Cartier, thus if $X$ itself is smooth then a divisor $D$ is relatively Cartier iff its support does not contain any of the fibers. In the relative setting, we usually focus on properties of the morphism $f$. Thus we would like to have similar results for smooth morphisms. (See (4.36) and (4.41) for closely related results.)

**Theorem 4.21** *Let $f\colon X \to S$ be a smooth morphism and $W \subset S$ a closed subset such that* $\operatorname{depth}_W S \geq 2$. *Let $D^\circ$ be a Cartier divisor on $X \setminus f^{-1}(W)$ and $D \subset X$ its closure. Assume that* $\operatorname{Supp} D$ *does not contain any irreducible component of any fiber. Then $D$ is Cartier, hence a relative Cartier divisor.*

*Proof* Assume first that $f$ has relative dimension 1. Then $f|_D\colon D \to S$ is quasi-finite, so $f|_D$ is flat by (10.63), so $D$ is a Cartier divisor by (4.20.1).

For the general case, pick a closed point $x \in D$. Since $f$ is smooth, locally it factors through an étale morphism $\tau\colon (x, X) \to ((0, s), \mathbb{A}^n_S)$. Composing with any linear projection we locally factor $f$ as

$$f\colon (x, X) \xrightarrow{g} ((0, s), \mathbb{A}^{n-1}_S) \to S,$$

where $g$ is smooth of relative dimension 1. If $D$ does not contain the fiber of $g$ passing through $x$, then $D$ is a Cartier divisor by the already discussed one-dimensional case.

To find such a $g$, assume first that $k(s)$ is infinite. Let $L \subset \mathbb{A}^n_s$ be a general line through the origin. Then $\pi_s^{-1}(L) \not\subset D_s$. Thus if we choose the projection $\mathbb{A}^n_S \to \mathbb{A}^{n-1}_S$ to have kernel $L$ over $s$, then the argument proves that $D$ is a Cartier divisor at $x$.

If $k(s)$ is finite then consider the trivial lifting $f^{(1)}: X \times \mathbb{A}^1 \to S \times \mathbb{A}^1$. By the previous argument $D \times \mathbb{A}^1$ is a Cartier divisor at the generic point of $\{x\} \times \mathbb{A}^1$, hence $D$ is a Cartier divisor at $x$ by (2.92.1). □

**Examples 4.22** We give two examples showing that in (4.21) we do need depth assumptions on $S$.

Set $S_n := \operatorname{Spec} k[x, y]/(xy)$ and $X_n = \operatorname{Spec} k[x, y, z]/(xy)$. Then $(x, z)$ defines a Weil divisor which is not Cartier.

Set $S_c := \operatorname{Spec} k[x^2, x^3]$ and $X_c = \operatorname{Spec} k[x^2, x^3, y]$. Then $(y^2 - x^2, y^3 - x^3)$ defines a Weil divisor which is not Cartier.

**Lemma 4.23** *Let $X$ be a pure dimensional, $S_2$ scheme, $D \subset X$ a Cartier divisor and $W \subset D$ a subscheme such that* $\operatorname{codim}_D W \geq 2$. *Let $L$ be a rank 1, torsion-free sheaf on $X$ that is locally free along $D \setminus W$. Let $s$ be a section of*

*$L$ such that $s|_{D\setminus W}$ is nowhere zero. Then $L$ is trivial and $s$ is nowhere zero in a neighborhood of $D$.*

*Proof* The section $s$ gives an exact sequence

$$0 \to \mathscr{O}_X \xrightarrow{s} L \to Q \to 0.$$

By (10.7) every associated prime of $Q$ has codimension 1 in $X$. Thus $D \cap \operatorname{Supp} Q$ has codimension 1 in $D$. Therefore, $D$ is disjoint from $\operatorname{Supp} Q$ and $L$ is trivial on $X \setminus \operatorname{Supp} Q$. □

## Relative Generically Cartier Divisors

This is the most important class for moduli purposes.

**Definition 4.24** Let $f: X \to S$ be a morphism. A subscheme $D \subset X$ is a *relative, generically Cartier, effective divisor* or a *family of generically Cartier, effective divisors* over $S$ if there is an open subset $U \subset X$ such that

(4.24.1) $f$ is flat over $U$ with $S_2$ fibers,

(4.24.2) $\operatorname{codim}_{X_s}(X_s \setminus U) \geq 2$ for every $s \in S$,

(4.24.3) $D|_U$ is a relative Cartier divisor (4.20), and

(4.24.4) $D$ is the closure of $D|_U$.

If $U \subset X$ denotes the largest open set with these properties then $Z := X \setminus U$ is the *non-Cartier locus* of $D$.

Thus $\mathscr{O}_X(mD)$ is a mostly flat family of divisorial sheaves on $X$ (3.28) for any $m \in \mathbb{Z}$. Conversely, if $L$ is a mostly flat family of divisorial sheaves on $X$ and $h$ a global section of it that does not vanish on any irreducible component of any fiber, then $(h = 0)$ is a family of generically Cartier, effective divisors over $S$.

**4.25** (Proof of 4.3) All five conditions are local on $S$; the first four are local on $X$. All of them can be checked on a general relative hyperplane section of $X$; see (4.2.6), (4.26) and (10.56).

Thus we may assume that $X \to S$ has relative dimension 1, hence $f$ is smooth along $\operatorname{Supp} D$. We view $D$ as a divisorial subscheme of $X$. After an étale base change we may assume that $D \to S$ is finite.

Applying (3.20) to $F := f_*\mathscr{O}_D$ (with $X = S$) we see that (4.3.5) holds iff $\mathscr{O}_D$ is flat over $S$. By (4.20) the latter holds iff $D$ a relative Cartier divisor. Thus (4.3.3) ⇔ (4.3.4) ⇔ (4.3.5).

As we noted in (4.24), these imply (4.3.1), and (4.3.1) ⇒(4.3.2) is clear. It remains to show that (4.3.2) ⇒(4.3.4).

To see this, fix a point $\tau\colon \{s\} \to S$ and let $T$ be the spectrum of a DVR and $h\colon T \to S$ a morphism that maps the closed point to $\tau(s)$ and the generic point of $T$ to a generic point $g \in S$. Then $h^*_{\rm Wdiv} D$ is flat over $T$ of degree $\deg_{k(g)} \mathscr{O}_{D_g}$. Thus if $\bar\tau : \bar s \to T$ is a lifting of $\tau$ and (4.3.2) holds, then

$$\deg \tau^*_{\rm Wdiv} D = \deg \bar\tau^*_{\rm Wdiv} h^*_{\rm Wdiv} D = \deg_{k(g)} \mathscr{O}_{D_g}.$$

Thus $D \to S$ is flat by (3.20). □

The following Bertini-type results are frequently useful. The first claim is an immediate consequence of (10.56) and the second follows from (10.20).

**Proposition 4.26** *Let* $(0 \in S)$ *be a local scheme,* $X \subset \mathbb{P}^N_S$ *a quasi-projective* $S$*-scheme with fibers of pure dimension* $\geq 2$*, and* $D \subset X$ *a relative divisorial subscheme. Then, for general* $H \in |\mathscr{O}_X(1)|$*,*

(4.26.1) $D$ *is a generically Cartier family of divisors on* $X$ *iff* $D|_H$ *is a generically Cartier family of divisors on* $H$*, and*

(4.26.2) $\mathscr{O}_X(D)|_H \simeq \mathscr{O}_H(D|_H)$. □

## Representability Theorems

**4.27** (Representability of the generically Cartier condition) There are two versions of this question. Let $f\colon X \to S$ be a flat, projective morphism and $D \subset X$ a divisorial subscheme.

The traditional problem is to study those morphisms $q : W \to S$ for which $q^*D$ is a generically Cartier divisor on $X_W$. This gives a representable functor. This will be used during the construction of the moduli of Mumford divisors, so we treat it there (4.77).

From the point of view of Section 4.1, it may seem more natural to study those morphisms $q : W \to S$ for which the Weil-divisor pull-back $q^*_{\rm Wdiv} D$ is a generically Cartier divisor on $X_W$. This, however, does not give a representable functor; see (4.13). This variant is actually not well posed, since the Weil-divisor pull-back is not functorial in general.

Fortunately, it turns out to be relatively easy to ensure the generically Cartier condition. So we focus on studying additional properties of such families.

As a first problem, we start with a family of generically Cartier divisors, and study those morphisms $q\colon W \to S$ for which the generically Cartier pull-back $D_W$ is flat or relatively Cartier.

The first result is a reformulation of (3.29) and (3.30).

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**Theorem 4.28** *Let $S$ be a scheme, $f : X \to S$ a flat, projective morphism with $S_2$ fibers, and $D \subset X$ a family of generically Cartier divisors. Then there is a locally closed decomposition $j^{H\text{-}flat} : S^{H\text{-}flat} \to S$ (resp. a locally closed partial decomposition $j^{car} : S^{car} \to S$) such that, for every morphism $q : W \to S$, the divisorial pull-back $D_W = q^{[*]}D$ is flat (resp. Cartier) iff $q$ factors through $S^{H\text{-}flat}$ (resp. $S^{car}$).* □

This leads to a valuative criterion for Cartier divisors in (4.34).

As we saw in (4.15), the set of fibers where a divisor is $\mathbb{Q}$-Cartier need not be constructible. So the straightforward $\mathbb{Q}$-Cartier version of (4.28) fails. However, this failure of constructibility is the only obstruction.

**Proposition 4.29** *Let $S$ be a reduced scheme, $f : X \to S$ a flat, projective morphism with $S_2$ fibers, and $D$ a family of generically $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) divisors on $X$. Let $S^* \subset S$ be a constructible subset. Assume that $D_s$ is $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) for every point $s \in S^*$.*

*Then there is a locally closed partial decomposition $j^{qcar} : S^{qcar} \to S$ (resp. $j^{rcar} : S^{rcar} \to S$) such that the following holds.*

(4.29.1) *Let $q : W \to S$ be a reduced $S$-scheme such that $q(W) \subset S^*$. Then the divisorial pull-back $D_W \subset X_W$ is $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) iff $q$ factors though $S^{qcar}$ (resp. $S^{rcar}$).*

*Proof* We may assume that $S^*$ is dense in $S$ and start with the $\mathbb{Q}$-Cartier case. By (4.28) there are maximal open subsets $S_1^{\mathrm{car}} \subset S_2^{\mathrm{car}} \subset \cdots$ such that $r! \cdot D$ is Cartier over $S_r^{\mathrm{car}}$. By assumption, $S_r^{\mathrm{car}}$ is dense for $r \gg 1$ and the union of all of them is the open stratum of $S^{\mathrm{qcar}} \to S$. Noetherian induction then gives the other strata.

In the $\mathbb{R}$-Cartier case, we write $D = \sum d_i D^i$ where the $D^i$ are $\mathbb{Q}$-divisors and the $d_i \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$. We already have locally closed partial decompositions $j_i^{\mathrm{qcar}} : S_i^{\mathrm{qcar}} \to S$ using $D^i$, and $j^{\mathrm{rcar}} : S^{\mathrm{rcar}} \to S$ is their fiber product over $S$ using (11.43.2). □

## 4.4 Valuative Criteria

We aim to show that various properties of morphisms can be checked after base change to one-dimensional, regular schemes, equivalently, to spectra of DVRs. We aim to use as few DVRs as possible.

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**Definition 4.30** A morphism $q\colon (x, X) \to (y, Y)$ of local schemes is *local* if $q(x) = y$. A morphism of schemes $q\colon X \to Y$ is *component-wise dominant* if every generic point of $X$ is mapped to a generic point of $Y$. If $X, Y$ are irreducible, then component-wise dominant is the same as dominant.

We are especially interested in local, component-wise dominant morphisms $q\colon (t, T) \to (s, S)$ from the spectrum of a DVR to $S$. To construct these, let $S_1 \subset S$ be an irreducible component and $\pi\colon B_s S_1 \to S_1$ the blow-up of $s$. The exceptional divisor has pure codimension 1. Let $\eta \in \operatorname{Ex}(\pi)$ be a generic point and $\mathscr{O}_\eta$ its local ring. If $S$ is excellent, we can take $T$ to be the normalization of $\operatorname{Spec} \mathscr{O}_\eta$. Then $(\eta, T) \to (s, S_1)$ is essentially of finite type. In general, we need to take $T$ to be one of the irreducible components of the normalization of the completion of $\mathscr{O}_\eta$. Then $T$ is excellent, but $q$ is not essentially of finite type.

**Lemma 4.31** *Let $(s, S)$ be a local scheme and $g\colon S' \to S$ a locally closed partial decomposition (10.83). Then $g$ is an isomorphism iff every local, component-wise dominant morphism $q\colon (t, T) \to (s, S)$ from the spectrum of an excellent DVR to $S$ factors through $g$.*

*Proof* We see that $g$ is proper and dominant by (10.78.1), hence an isomorphism by (10.83.2). □

**Theorem 4.32** (Valuative criterion for divisorial sheaves) *Let $(s, S)$ be a reduced, local scheme and $f\colon X \to S$ a flat morphism of finite type with $S_2$ fibers. Let $L$ be a mostly flat family of divisorial sheaves on $X$ (3.28). Assume that either $f$ is projective or $S$ is excellent. The following are equivalent.*

(4.32.1) *$L$ is flat over $S$ with $S_2$ fibers.*

(4.32.2) *For every local, component-wise dominant morphism $q\colon (t, T) \to (s, S)$ from the spectrum of an excellent DVR to $S$, the hull pull-back (3.27) $L_T^H$ is flat over $T$ with $S_2$ fibers.*

*Proof* It is clear that (1) implies (2). For the converse, we use the theory of hulls and husks from Chapter 9.

Assume first that $f$ is projective. Consider the locally closed decomposition $j\colon \operatorname{Hull}(L) \to S$ given by (9.40). By assumption, every $q\colon (t, T) \to (s, S)$ factors through $j$, so $j$ is an isomorphism by (4.31). Thus $L$ is its own hull, hence it is flat over $S$ with $S_2$ fibers.

This is the main case that we use. The argument in the nonprojective case is similar, but relies on (9.44).

Pick any point $x \in X$ and its image $s := f(x)$. Let $\hat{S}$ denote the completion of $S$ at $s$; it is reduced since $S$ is excellent. Then $L$ is flat over $S$ with $S_2$ fibers at $x$ iff this holds after base change to $\hat{S}$. Thus it is enough to show that (2) ⇒ (1) whenever $s \in S$ is complete, in which case the hull of $L$ is represented by a subscheme $i : S^u \hookrightarrow S$ for local, Artinian $S$-algebras by (9.44).

Let $(R, m)$ be a complete DVR and $q\colon \operatorname{Spec} R \to (s, S)$ a local morphism. By assumption (2), the hull pull-back $L_R^H$ is flat over $R$ with $S_2$ fibers. Thus the same holds for $\operatorname{Spec}(R/m^n)$ for every $n$, hence the restriction of $q$ to $\operatorname{Spec}(R/m^n)$ factors through $i : S^u \hookrightarrow S$. Since this holds for every $n \in \mathbb{N}$, $q$ factors through $i : S^u \hookrightarrow S$. We conclude that $S^u = S$. So, as before, $L$ is its own hull, hence it is flat over $S$ with $S_2$ fibers. □

Putting together (2.79), (2.82) and (4.32) gives the following higher dimensional version.

**Corollary 4.33** *Let $f\colon (X, \Delta) \to S$ be a locally stable morphism to a reduced scheme over a field of characteristic 0. Let $D$ be a relative Mumford $\mathbb{Z}$-divisor (4.68). Assume that either $f$ is projective or $S$ is excellent. Then, in any of the cases (2.79.1–8) and (2.82),*

(4.33.1) *$\mathscr{O}_X(D)$ is flat over $S$ with $S_2$ fibers, and*

(4.33.2) *$\mathscr{O}_X(D)|_{X_s} \simeq \mathscr{O}_{X_s}(D_s)$ for $s \in S$.* □

We can restate (4.32) for Cartier divisors as follows.

**Corollary 4.34** (Valuative criterion for Cartier divisors) *Let $(s, S)$ be a reduced, local scheme, $f\colon X \to S$ a flat morphism of finite type with $S_2$ fibers, and $D$ a relative, generically Cartier divisor on $X$. Assume that either $f$ is projective or $S$ is excellent. Then the following are equivalent.*

(4.34.1) *$D$ is a relative Cartier divisor.*

(4.34.2) *For every local, component-wise dominant morphism $q\colon (t, T) \to (s, S)$ from the spectrum of an excellent DVR to $S$, the divisorial pull-back $D_T \subset X_T$ is a Cartier divisor.* □

Reduction to the Cartier case as in (4.29) gives the following.

**Corollary 4.35** (Valuative criterion for $\mathbb{Q}$- and $\mathbb{R}$-Cartier divisors) *Let $(s, S)$ be a reduced, local scheme, $f\colon X \to S$ a flat morphism of finite type with $S_2$ fibers, and $D$ a family of generically $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) divisors on $X$. Assume that either $f$ is projective or $S$ is excellent. Then the following are equivalent.*

(4.35.1) *$D$ is a $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) divisor.*

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(4.35.2) *For every local, component-wise dominant morphism* $q\colon (t,T) \to (s,S)$ *from the spectrum of an excellent DVR to* $S$, *the divisorial pull-back* $D_T$ *is* $\mathbb{Q}$*-Cartier (resp.* $\mathbb{R}$*-Cartier).* □

The following two consequences of (4.34) are important; see (4.41.1) for a more direct proof of the first one.

**Corollary 4.36** *Let* $S$ *be a reduced scheme,* $f : X \to S$ *a smooth morphism, and* $D$ *a relative, generically Cartier divisor on* $X$*. Assume that either* $f$ *is projective or* $S$ *is excellent. Then* $D$ *is a relative Cartier divisor.*

*Proof* Let $q : T \to S$ be a morphism from the spectrum of a DVR to $S$. Then $X_T$ is regular, hence $D_T$ is Cartier. So $D$ is Cartier by (4.34). □

**Theorem 4.37** *Let* $(s,S)$ *be a reduced, local, excellent scheme,* $f\colon X \to S$ *a flat morphism of finite type with* $S_2$ *fibers, and* $D$ *a relative, generically Cartier divisor on* $X$*. Then* $D$ *is Cartier* $\Leftrightarrow$ $D$ *is* $\mathbb{Q}$*-Cartier,* $D_s$ *is Cartier, and* $D_g$ *is Cartier for all generic points* $g \in S$.

*Proof* The necessity is clear. By (4.34) it is enough to prove the converse after base change to $T$ whenever $q\colon (t,T) \to (s,S)$ is a local, component-wise dominant morphism from the spectrum of an excellent DVR to $S$. The assumptions are preserved.

Let $Z \subset X_t$ be the locus where $D_T$ is not known to be Cartier. After localizing at the generic point of $Z$, we are in the situation of (2.91). Thus $D_T$ is Cartier and so is $D$. □

Another valuative criterion is the following local version of (3.20).

**Theorem 4.38** (Grothendieck, 1960, IV.11.6, IV.11.8) *Let* $(s,S)$ *be a reduced, local scheme,* $f : X \to S$ *a morphism of finite type, and* $F$ *a coherent sheaf on* $X$*. Let* $T$ *be a disjoint union of spectra of DVRs and* $q\colon T \to S$ *a dominant, local morphism. Then* $F$ *is flat over* $S$ *at* $x \in X_s$ *iff* $q_X^*F$ *is flat over* $T$ *along* $q_X^{-1}(x)$. □

## 4.5 Generically $\mathbb{Q}$-Cartier Divisors

In the study of lc and slc pairs, $\mathbb{Q}$-Cartier divisors are more important than Cartier divisors. We have seen many examples of Weil $\mathbb{Z}$-divisors that are $\mathbb{Q}$-Cartier, but not Cartier. By contrast, we show that if a relative Weil $\mathbb{Z}$-divisor is generically $\mathbb{Q}$-Cartier, then it is generically Cartier in characteristic 0.

Let $f : (X, D) \to S$ be a family of pairs and $D$ a relative Weil $\mathbb{Z}$-divisor.

Since we are interested in generic properties, we can focus on a generic point $x$ of $D \cap X_s$. If the assumption (4.2.4) holds then $f$ is smooth at $x$. Thus we may as well assume that $f$ is smooth (but not proper).

If $S$ is normal then $D$ is a Cartier divisor by (4.4), thus here our main interest is in those cases where $S$ is reduced, but not normal. As (4.10) shows, $D$ need not be Cartier in general. However, the next result shows that if some multiple of $D$ is Cartier, then so is $D$, at least in characteristic 0.

Positive characteristic counter examples are given in (4.11) and (4.12).

**Theorem 4.39** *Let $S$ be a reduced scheme, $f : X \to S$ a smooth morphism of relative dimension $\geq 1$, and $D$ a relative Weil $\mathbb{Z}$-divisor on $X$. Assume that $mD$ is Cartier at a point $x \in X$ and* $\operatorname{char} k(x) \nmid m$. *Then $D$ is Cartier at $x$.*

*Proof* By Noetherian induction and shrinking $X$, we may assume that $D$ is Cartier on $X \setminus \{x\}$ and $mD \sim 0$.

By (11.24), $mD \sim 0$ determines a cyclic cover $\tilde{X} \to X$ that is étale over $X \setminus \{x\}$ whenever $\operatorname{char} k(x) \nmid m$. This gives a correspondence between torsion in $\operatorname{Pic}^{\rm loc}(x, X)$ and torsion in the abelian quotient of the fundamental group $\hat{\pi}_1(X \setminus \{x\})$. There are now two ways to finish the proof.

In characteristic 0, we may work over $\mathbb{C}$. After replacing $X$ with a suitable Euclidean neighborhood $x \in U \subset X$, it is enough to prove that $\pi_1(U \setminus \{x\})$ is trivial. We do this in (4.40).

In general, let $X^{\rm wn} \to X$ be the weak normalization (10.74). We prove in (4.41) that $\operatorname{Pic}^{\rm loc}(x^{\rm wn}, X^{\rm wn})$ is free of finite rank. It remains to show that $K^{\rm wn} := \ker[\operatorname{Pic}^{\rm loc}(x, X) \to \operatorname{Pic}^{\rm loc}(x^{\rm wn}, X^{\rm wn})]$ does not contain prime-to-$p$ torsion in characteristic $p > 0$.

Since $X^{\rm wn} \to X$ is finite and purely inseparable, it is a factor of a power $F_q$ of the Frobenius (10.78.2). This gives pull-back maps

$$\operatorname{Pic}^{\rm loc}(x, X) \to \operatorname{Pic}^{\rm loc}(x^{\rm wn}, X^{\rm wn}) \to \operatorname{Pic}^{\rm loc}(x_q, X_q),$$

where the composite is $L \mapsto L^q$. So $K^{\rm wn}$ is $q$-torsion.

Alternatively, one can use Grothendieck (1971, I.11), which implies that $X^{\rm wn} \setminus \{x^{\rm wn}\} \to X \setminus \{x\}$ induces an isomorphism of the fundamental groups. □

**4.40** (Links and smooth morphisms) Let $f : X \to S$ be a smooth morphism of complex spaces of relative dimension $n \geq 1$. We describe the topology of the link of a point $x \in X$ in terms of the topology of the link of $s := f(x) \in S$.

We can write $S \subset \mathbb{C}^N_{\mathbf{z}}$ such that $s$ is the origin and $X \subset S \times \mathbb{C}^n_{\mathbf{t}}$ where $x$ is the origin. Intersecting $S$ with a sphere of radius $\varepsilon$ centered at $s$, we get $L_S$, the

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link of $s \in S$. The intersection of $S$ with the corresponding ball of radius $\varepsilon$ is homeomorphic to the cone $C_S$ over $L_S$.

The link $L_X$ of $x \in X$ can be obtained as the intersection of $X$ with the level set $\max\{\sum |z_i|^2, \sum |t_j|^2\} = \varepsilon^2$. Thus $L_X$ is homeomorphic to the amalgamation of

$$\begin{array}{lcll} L_S \times \mathbb{D}^{2n} & = & \{(\mathbf{z},\mathbf{t}) : \sum |z_i|^2 = \varepsilon^2,\ \sum |t_j|^2 \le \varepsilon^2\} & \text{and of} \\ C_S \times \mathbb{S}^{2n-1} & = & \{(\mathbf{z},\mathbf{t}) : \sum |z_i|^2 \le \varepsilon^2,\ \sum |t_j|^2 = \varepsilon^2\}, & \text{glued along} \\ L_S \times \mathbb{S}^{2n-1} & = & \{(\mathbf{z},\mathbf{t}) : \sum |z_i|^2 = \varepsilon^2,\ \sum |t_j|^2 = \varepsilon^2\}. & \end{array}$$

Let $L_S^i$ be the connected components of $L_S$. Note that $\pi_1(L_S^i \times \mathbb{S}^{2n-1}) \simeq \pi_1(L_S^i) \times \pi_1(\mathbb{S}^{2n-1})$. The first factor gets killed in $\pi_1(C_S \times \mathbb{S}^{2n-1})$; the second is trivial if $n \ge 2$ and gets killed in $\pi_1(L_S^i \times \mathbb{D}^{2n})$ if $n = 1$. Thus $L_X$ is simply connected for $n \ge 1$.

The cohomology of $L_X$ can be computed from the Mayer–Vietoris sequence. Using that $H^i(L_S \times \mathbb{D}^{2n}, \mathbb{Z}) = H^i(L_S, \mathbb{Z})$ and $H^i(C_S \times \mathbb{S}^{2n-1}, \mathbb{Z}) = H^i(\mathbb{S}^{2n-1}, \mathbb{Z})$, for $H^2$ the key pieces are

$$\begin{array}{rcl} & \longrightarrow H^1(L_S,\mathbb{Z}) \oplus H^1(\mathbb{S}^{2n-1},\mathbb{Z}) & \xrightarrow{\sigma^1} H^1(L_S \times \mathbb{S}^{2n-1},\mathbb{Z}) \\ \longrightarrow H^2(L_X,\mathbb{Z}) & \longrightarrow H^2(L_S,\mathbb{Z}) \oplus H^2(\mathbb{S}^{2n-1},\mathbb{Z}) & \xrightarrow{\sigma^2} H^2(L_S \times \mathbb{S}^{2n-1},\mathbb{Z}). \end{array}$$

The Künneth formula gives that the $\sigma^i$ are injections and $\sigma^1$ is an isomorphism if $n \ge 2$. In this case $H^2(L_X, \mathbb{Z}) = 0$. If $n = 1$, then

$$\begin{aligned} H^2(L_X,\mathbb{Z}) &\simeq \operatorname{coker}[H^1(\mathbb{S}^1,\mathbb{Z}) \to H^0(L_S,\mathbb{Z}) \otimes H^1(\mathbb{S}^1,\mathbb{Z})] \\ &\simeq H^0(L_S,\mathbb{Z})/\mathbb{Z}. \end{aligned} \tag{4.40.1}$$

We have thus proved the following.

*Claim 4.40.2* Let $f : X \to S$ be a smooth morphism of complex spaces, $L_X$ the link of a point $x \in X$, and $s := f(x)$. Assume that $\dim_x X > \dim_s S \ge 1$.

Then $L_X$ is simply connected. Furthermore, $H^2(L_X, \mathbb{Z}) = 0$ iff either $n \ge 2$ or the link of $s \in S$ is connected. □

Next we compute the local Picard groups in more detail in the weakly normal case.

**Theorem 4.41** *Let $(s \in S)$ be a local, weakly normal pair (10.74) and $f : X \to S$ a smooth morphism. Let $x \in X_s$ be a point. Then,*

(4.41.1) *if* $\operatorname{codim}(x \in X_s) \ge 2$ *then* $\operatorname{Pic}^{\mathrm{loc}}(x, X) = 0$, *and*

(4.41.2) *if* $\operatorname{codim}(x \in X_s) = 1$ *then* $\operatorname{Pic}^{\mathrm{loc}}(x, X)$ *is free of finite rank.*

*Proof* Set $d = \dim X_s$ and let $\pi\colon X \to \mathbb{A}_S^{d-1}$ be a general projection. Then $\pi$ is generically quasi-finite along the closure of $x$. Let $(w, W)$ be the strict

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Henselization of $\pi(x) \in \mathbb{A}^{d-1}_S$ (2.18). By base change, we have a smooth morphism $\pi' : (x', X') \to (w, W)$ of relative dimension 1, where $x' \in X', w \in W$ are closed points.

By (2.92.1), $\operatorname{Pic}^{\mathrm{loc}}(x, X) \hookrightarrow \operatorname{Pic}^{\mathrm{loc}}(x', X')$, thus it is enough to prove (1–2) for $\operatorname{Pic}^{\mathrm{loc}}(x', X')$.

Every class in $\operatorname{Pic}^{\mathrm{loc}}(x', X')$ can be represented by an effective divisor $D$ that does not contain $X'_w$. Then $\pi'|_D : D \to W$ is finite and flat over $W \setminus \{w\}$.

Let $\{W_i : i \in I\}$ be the connected components of $W \setminus \{w\}$. Then $[D] \mapsto (\operatorname{rank}_{W_i} \pi'_* \mathscr{O}_D : i \in I)$ gives a map

$$\operatorname{Pic}^{\mathrm{loc}}(x', X') \to \mathbb{Z}^{|I|} \to \mathbb{Z}^{|I|}/\mathbb{Z}(1, \dots, 1).$$

We claim that it is an injection. Indeed, if $\pi'_* \mathscr{O}_D$ has constant rank $d$ then $\pi'|_D$ is flat by (10.64), hence $D$ is Cartier by (4.20). This proves (2).

If $\operatorname{codim}(x \in X_s) \geq 2$ then $g(x)$ is not the generic point $\eta_s \in \mathbb{A}^{d-1}_s$. Thus every irreducible component of $\mathbb{A}^{d-1}_S$ contains $\eta_s$, and this continues to hold after strict Henselization. Thus $W \setminus \{w\}$ is connected and we get (1). □

*Complement 4.41.3* The proof shows that in case (2) the rank is bounded by $r - 1$, where $r$ is the maximum number of connected components of $S' \setminus \{s'\}$ for all étale $(s', S') \to (s, S)$. It is ≤ the number of geometric points over $s$ on the normalization of $S$.

## 4.6 Stability Is Representable II

**Assumption.** In this section we work over a field of characteristic 0.

The main result of this section is the following. Eventually we remove the reduced assumption by introducing the notion of K-flatness in Chapter 7.

**Theorem 4.42** *Let $f : (X, \Delta) \to S$ be a projective, well-defined family of pairs. Then the functor of locally stable pull-backs is represented, for reduced schemes, by a locally closed partial decomposition $i^{lst} : S^{lst} \to S$.*

Since ampleness is an open condition for an $\mathbb{R}$-Cartier divisor (11.54.2), (4.42) implies the analogous result for stable morphisms.

**Corollary 4.43** *Let $f : (X, \Delta) \to S$ be a projective, well-defined family of pairs. Then the functor of stable pull-backs is represented, for reduced schemes, by a locally closed partial decomposition $i^{stab} : S^{stab} \to S$.* □

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We start the proof of (4.42), which will be completed in (4.46), with a weaker version.

**Lemma 4.44** *Let $f\colon (X,\Delta) \to S$ be a proper, well-defined family of pairs. Then there is a finite collection of locally closed subschemes $S_i \subset S$ such that*
(4.44.1) *$f_i : (X_{S_i}, \Delta_{S_i}) \to S_i$ is locally stable for every i, and*
(4.44.2) *a fiber $(X_s, \Delta_s)$ is slc iff $s \in \cup_i S_i$.*
*In particular, $\{s : (X_s, \Delta_s)$ is slc$\} \subset S$ is constructible.*

*Proof* Being demi-normal is an open condition by (10.42) and slc implies demi-normal by definition. So we may assume that all fibers are demi-normal and $S$ is irreducible with generic point $g$. Throughout the proof we use $S^\circ \subset S$ to denote a dense open subset which we shrink whenever necessary.

First, we treat morphisms whose generic fiber $X_g$ is normal.
*Case 1:* $(X_g, \Delta_g)$ is lc. Then $K_{X_g} + \Delta_g$ is $\mathbb{R}$-Cartier, hence $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier over an open neighborhood of $g$. Next consider a log resolution $p_g : Y_g \to X_g$. It extends to a simultaneous log resolution $p^\circ : Y^\circ \to X^\circ$ over a suitable $S^\circ \subset S$. Thus, if $E^\circ \subset Y^\circ$ is any exceptional divisor, then $a(E_s, X_s, \Delta_s) = a(E^\circ, X^\circ, \Delta^\circ) = a(E_g, X_g, \Delta_g)$. This shows that all fibers over $S^\circ$ are lc.
*Case 2:* $(X_g, \Delta_g)$ is not lc. Note that the previous argument works if $K_{X_g} + \Delta_g$ is $\mathbb{R}$-Cartier. Indeed, then there is a divisor $E$ with $a(E_g, X_g, \Delta_g) < -1$ and this shows that $a(E_s, X_s, \Delta_s) < -1$ for $s \in S^\circ$. However, if $K_{X_g} + \Delta_g$ is not $\mathbb{R}$-Cartier, then the discrepancy $a(E_g, X_g, \Delta_g)$ is not defined. We could try to prove that $K_{X_s} + \Delta_s$ is not $\mathbb{R}$-Cartier for $s \in S^\circ$, but this is not true in general; see (4.15).

Thus we use the notion of numerically Cartier divisors (4.48) instead. If $K_{X_g} + \Delta_g$ is not numerically Cartier, then, by (4.51), $K_{X_s} + \Delta_s$ is also not numerically Cartier over an open subset $S^\circ \ni g$. Thus $(X_s, \Delta_s)$ is not lc for $s \in S^\circ$.

If $K_{X_g} + \Delta_g$ is numerically Cartier, then the notion of discrepancy makes sense (4.48) and, again using (4.51), the arguments show that if $(X_g, \Delta_g)$ is numerically lc (resp. not numerically lc) then the same holds for $(X_s, \Delta_s)$ for $s$ in a suitable open subset $S^\circ \ni g$. We complete Case 2 by noting that being numerically lc is equivalent to being lc by (4.50).

An alternative approach to the previous case is the following. By (11.30), the log canonical modification (5.15) $\pi_g\colon (Y_g, \Theta_g) \to (X_g, \Delta_g)$ exists and it extends to a simultaneous log canonical modification $\pi : (Y, \Theta) \to (X, \Delta)$ over an open subset $S^\circ \subset S$. By the arguments of Case 1, $(Y_s, \Theta_s)$ is lc for $s \in S^\circ$ and the relative ampleness of the log canonical class is also an open condition. Thus $\pi_s\colon (Y_s, \Theta_s) \to (X_s, \Delta_s)$ is the log canonical modification for $s \in S^\circ$. By assumption, $\pi_g$ is not an isomorphism, so none of the $\pi_s$ are isomorphisms. Therefore, none of the fibers over $S^\circ$ are lc.

If $X_g$ is not normal, the proofs mostly work the same using a simultaneous semi-log-resolution (Kollár, 2013b, Sec.10.4). However, for Case 2 it is more convenient to use the following argument.

Let $\pi_g \colon \bar{X}_g \to X_g$ denote the normalization. Over an open subset $S^\circ \ni g$ it extends to a simultaneous normalization $(\bar{X}, \bar{D}+\bar{\Delta}) \to S$. If $(\bar{X}_g, \bar{D}_g+\bar{\Delta}_g)$ is not lc then $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is not lc for $s \in S^\circ$, hence $(X_s, \Delta_s)$ is not slc, essentially by definition; see Kollár (2013b, 5.10).

Using the already settled normal case, it remains to deal with the situation when $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is lc for every $s \in S^\circ$. By Kollár (2013b, 5.38), $(X_s, \Delta_s)$ is slc iff $\operatorname{Diff}_{\bar{D}^n_s} \bar{\Delta}_s$ is $\tau_s$-invariant. The different can be computed on any log resolution as the intersection of the birational transform of $\bar{D}_s$ with the discrepancy divisor. Thus $\operatorname{Diff}_{\bar{D}^n_s} \bar{\Delta}_s$ is also locally constant over an open set $S^\circ$. Therefore, if $\operatorname{Diff}_{\bar{D}^n_g} \bar{\Delta}_g$ is not $\tau_g$-invariant then $\operatorname{Diff}_{\bar{D}^n_s} \bar{\Delta}_s$ is also not $\tau_s$-invariant for $s \in S^\circ$. Hence $(X_s, \Delta_s)$ is not slc for every $s \in S^\circ$.

In both cases we complete the proof by Noetherian induction. □

The following consequence of (4.44) is quite useful, though it could have been proved before it as in (3.39).

**Corollary 4.45** *Let $f \colon (X, \Delta) \to S$ be a proper, well-defined family of pairs such that $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier. Then $\{s : (X_s, \Delta_s)$ is slc $\} \subset S$ is open.*

*Proof* By (4.44), this set is constructible. A constructible set $U \subset S$ is open iff it is closed under generalization, that is, $x \in U$ and $x \in \bar{y}$ implies that $y \in U$. This follows from (2.3). □

**4.46** (Proof of 4.42) Let $S_i \subset S$ be as in (4.44). We apply (4.29) to the family $f \colon (X, K_{X/S} + \Delta) \to S$ to obtain $S^{\mathrm{rcar}} \to S$ such that, for every reduced $S$-scheme $q : T \to S$ satisfying $q(T) \subset \cup_i S_i$, the pulled-back divisor $K_{X_T/T} + \Delta_T$ is $\mathbb{R}$-Cartier iff $q$ factors as $q : T \to S^{\mathrm{rcar}} \to S$.

Assume now that $f_T \colon (X_T, \Delta_T) \to T$ is slc. Then $K_{X_T/T} + \Delta_T$ is $\mathbb{R}$-Cartier, hence $q$ factors through $S^{\mathrm{rcar}} \to S$. As we observed in (3.23), this implies that $S^{\mathrm{slc}} = (S^{\mathrm{rcar}})^{\mathrm{slc}}$. By definition $K_{X^{\mathrm{rcar}}/S^{\mathrm{rcar}}} + \Delta$ is $\mathbb{R}$-Cartier, thus (4.45) implies that $S^{\mathrm{slc}} = (S^{\mathrm{rcar}})^{\mathrm{slc}}$ is an open subscheme of $S^{\mathrm{rcar}}$. □

We showed in (4.15) that being $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier is not a constructible condition. The next result shows that the situation is better for boundary divisors of lc pairs.

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**Corollary 4.47** *Let $f: (X, \Delta) \to S$ be a proper, flat family of pairs with slc fibers. Let $D$ be an effective divisor on $X$. Assume that*

(4.47.1) *either* Supp $D \subset$ Supp $\Delta$,

(4.47.2) *or* Supp $D$ *does not contain any of the log canonical centers of any of the fibers* $(X_s, \Delta_s)$.

*Then* $\{s : D_s$ *is* $\mathbb{R}$-*Cartier*$\} \subset S$ *is constructible.*

*Proof* Over an open subset, we have a simultaneous log resolution of $(X, D + \Delta)$. Choose $0 < \varepsilon \ll 1$. In the first case, $(X_s, \Delta_s - \varepsilon D_s)$ is slc iff $D_s$ is $\mathbb{R}$-Cartier. In the second case, $(X_s, \Delta_s + \varepsilon D_s)$ is slc iff $D_s$ is $\mathbb{R}$-Cartier. Thus, in both cases, (4.44) implies our claim. □

## Numerically Cartier Divisors

**Definition 4.48** Let $g: Y \to S$ be a proper morphism. An $\mathbb{R}$-Cartier divisor $D$ is called *numerically g-trivial* if $(C \cdot D) = 0$ for every curve $C \subset Y$ that is contracted by $g$.

Let $X$ be a demi-normal scheme. A Mumford $\mathbb{R}$-divisor $D$ on $X$ is called *numerically* $\mathbb{R}$-*Cartier* if there is a proper, birational contraction $p: Y \to X$ and a numerically $p$-trivial $\mathbb{R}$-Cartier Mumford divisor $D_Y$ on $Y$ such that $p_*(D_Y) = D$.

It follows from (11.60) that such a $D_Y$ is unique. If $D$ is a $\mathbb{Q}$-divisor then $D_Y$ is also a $\mathbb{Q}$-divisor since its coefficients are solutions of a linear system of equations. Such a $D$ is called *numerically* $\mathbb{Q}$-*Cartier.*

If $p' : Y' \to X$ is a proper, birational contraction and $Y'$ is $\mathbb{Q}$-factorial, then being numerically $\mathbb{R}$-Cartier can be checked on $Y'$.

Being numerically $\mathbb{R}$-Cartier is preserved by $\mathbb{R}$-linear equivalence, but the exceptional part $D_Y - p_*^{-1}D$ depends on $D \in |D|$.

For $K_X + \Delta$, we can make a canonical choice. Thus we see that $K_X + \Delta$ is numerically $\mathbb{R}$-Cartier iff there is a $p$-exceptional $\mathbb{R}$-divisor $E_{K+\Delta}$ such that $E_{K+\Delta} + K_Y + p_*^{-1}\Delta$ is numerically $p$-trivial.

If $K_X + \Delta$ is numerically $\mathbb{R}$-Cartier, then one can define the *discrepancy* of any divisor $E$ over $X$ by

$$a(E, X, \Delta) := a(E, Y, E_{K+\Delta} + p_*^{-1}\Delta).$$

We can thus define when a demi-normal pair $(X, \Delta)$ is *numerically lc* or *slc.*

If $g: X \to S$ is proper, then a numerically $\mathbb{R}$-Cartier divisor $D$ is called *numerically g-trivial* if $D_Y$ is numerically $(g \circ p)$-trivial on $Y$.

**Examples 4.49** On a normal surface, every divisor is numerically $\mathbb{R}$-Cartier.

The divisor $(x = z = 0)$ is not numerically $\mathbb{R}$-Cartier on the demi-normal surface $(xy = 0) \subset \mathbb{A}^3$.

If $X$ has rational singularities, then a numerically $\mathbb{R}$-Cartier divisor is also $\mathbb{R}$-Cartier by Kollár and Mori (1992, 12.1.4).

Assume that $\dim X \geq 3$ and $D$ is Cartier except at a point $x \in X$. There is a local Picard scheme $\mathbf{Pic}^{\mathrm{loc}}(x, X)$, which is an extension of a finitely generated local Néron–Severi group with a connected algebraic group $\mathbf{Pic}^{\mathrm{loc}-\circ}(x, X)$; see Boutot (1978) or Kollár (2016a) for details. Then $D$ is numerically $\mathbb{R}$-Cartier iff $[D] \in \mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$ where $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)/\,\mathbf{Pic}^{\mathrm{loc}-\circ}(x, X)$ is the torsion subgroup of the local Néron–Severi group.

There are many divisors that are numerically $\mathbb{R}$-Cartier, but not $\mathbb{R}$-Cartier, however, the next result says that the notion of numerically slc pairs does not give anything new.

**Theorem 4.50** (Hacon and Xu, 2016, 1.4) *A numerically slc pair is slc.*

*Outline of the proof* This is surprisingly complicated, using many different ingredients. We start with the normal, numerically $\mathbb{Q}$-Cartier case.

For clarity, let us concentrate on the very special case when $(X, \Delta)$ is dlt, except at a single point $x \in X$. All the key ideas appear in this case, but we avoid a technical inductive argument.

Starting with a thrifty log resolution (Kollár, 2013b, 2.79), the method of (Kollár, 2013b, 1.34) gives a $\mathbb{Q}$-factorial, dlt modification $f : (Y, E + \Delta_Y) \to (X, \Delta)$ such that $K_Y + E + \Delta_Y$ is numerically $f$-trivial, where $E$ is the exceptional divisor dominating $x$ and $\Delta_Y$ is the birational transform of $\Delta$. Let $\Delta_E := \mathrm{Diff}_E\, \Delta_Y$. Then $(E, \Delta_E)$ is a semi-dlt pair such that $K_E + \Delta_E$ is numerically trivial. Next we need a global version of the theorem.

*Claim 4.50.1* Let $(E, \Delta_E)$ be a projective semi-slc pair such that $K_E + \Delta_E$ is $\mathbb{Q}$-Cartier and numerically trivial. Then $K_E + \Delta_E \sim_{\mathbb{Q}} 0$.

The first general proof is in Gongyo (2013), but special cases go back to Kawamata (1985) and Fujino (2000). We discuss a very special case: $E$ is smooth and $\Delta = 0$. The following argument is from Campana et al. (2012) and Kawamata (2013).

We assume that $\mathscr{O}_E(K_E) \in \mathrm{Pic}^\tau(E)$, but after passing to an étale cover of $E$ we have that $\mathscr{O}_E(K_E) \in \mathrm{Pic}^\circ(E)$. Serre duality shows that if $[L] \in \mathrm{Pic}^\tau(E)$ and $h^n(E, L) = 1$, then $L \simeq \mathscr{O}_E(K_E)$.

Next we use a theorem of Simpson (1993) which says that the cohomology groups of line bundles in $\mathrm{Pic}^\circ$ jump precisely along torsion translates of abelian subvarieties. Thus $[K_E]$ is a torsion translate of a trivial abelian subvariety, hence a torsion element of $\mathrm{Pic}^\circ(E)$. □

It remains to lift information from the exceptional divisor $E$ to the dlt model $Y$. To this end consider the exact sequence

$$0 \to \mathscr{O}_Y(m(K_Y + E + \Delta_Y) - E) \to \mathscr{O}_Y(m(K_Y + E + \Delta_Y)) \to \mathscr{O}_E(m(K_E + \Delta_E)) \to 0.$$

Note that $D := m(K_Y + E + \Delta_Y) - E - (K_Y + \Delta_Y) \equiv_f 0$. We apply (Kollár, 2013b, 10.38.1) or the even stronger (Fujino, 2014, 1.10) to conclude that

$$R^1 f_*(\mathscr{O}_Y(m(K_Y + E + \Delta_Y) - E)) = R^1 f_*(\mathscr{O}_Y(D + K_Y + \Delta_Y)) = 0.$$

Hence a nowhere zero global section of $\mathscr{O}_E(m(K_E + \Delta_E))$ lifts back to a global section of $\mathscr{O}_Y(m(K_Y + E + \Delta_Y))$ that is nowhere zero near $E$. Thus $\mathscr{O}_X(m(K_X + \Delta)) \simeq f_*\mathscr{O}_Y(m(K_Y + E + \Delta_Y))$ is free in a neighborhood of $x$. Thus completes the numerically $\mathbb{Q}$-Cartier case.

The $\mathbb{R}$-Cartier case is reduced to the numerically $\mathbb{Q}$-Cartier setting using (11.47) as follows.

Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be a log resolution. Pick curves $C_m$ that span $N_1(Y/X)$ and apply (11.47) to $(Y, \Delta_Y)$. Thus for $n \gg 1$ we get $K_Y + \Delta_Y = \sum_j \lambda_j (K_Y + \Delta_Y^j)$ where the $\Delta_Y^j$ are $\mathbb{Q}$-divisors and $(Y, \Delta_Y^j)$ is lc. Also, since $(C_m \cdot (K_Y + \Delta_Y)) = 0$, (11.47.6.a) shows that $(C_m \cdot (K_Y + \Delta_Y^j)) = 0$. Thus each $(X, f(\Delta_Y^j))$ is a numerically $\mathbb{Q}$-Cartier lc pair. They are thus lc and so is $(X, \Delta)$ by (11.4.4). The demi-normal case now follows using (11.38). □

The advantage of the concept of numerically $\mathbb{R}$-Cartier divisors is that we have better behavior in families.

**Proposition 4.51** *Let $f\colon X \to S$ be a proper morphism with normal fibers over a field of characteristic 0 and $D$ a generically Cartier family of divisors on $X$. Then there is a finite collection of locally closed subschemes $S_i \subset S$ such that*

(4.51.1) *$D_s$ is numerically $\mathbb{R}$-Cartier iff $s \in \cup_i S_i$, and*

(4.51.2) *the pull-back of $D$ to $X \times_S S_i$ is numerically $\mathbb{R}$-Cartier for every $i$.*

*In particular, $\{s \in S : D_s$ is numerically $\mathbb{R}$-Cartier$\} \subset S$ is constructible.*

*Proof* Let $g \in S$ be a generic point. We show that if $D_g$ is numerically $\mathbb{R}$-Cartier (resp. not numerically $\mathbb{R}$-Cartier) then the same holds for all $D_s$ in an open neighborhood $g \in S^\circ \subset S$. Then we finish by Noetherian induction.

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To see our claim, consider a log resolution $p_g\colon Y_g \to X_g$. It extends to a simultaneous log resolution $p^\circ\colon Y^\circ \to X^\circ$ over a suitable open neighborhood $g \in S^\circ \subset S$.

If $D_g$ is numerically $\mathbb{R}$-Cartier then there is a $p_g$-exceptional $\mathbb{R}$-divisor $E_g$ such that $E_g + (p_g)_*^{-1}D_g$ is numerically $p_g$-trivial. This $E_g$ extends to a $p$-exceptional $\mathbb{R}$-divisor $E$ and $E + p_*^{-1}D$ is numerically $p$-trivial over an open neighborhood $g \in S^\circ \subset S$ by (4.52). Thus $D_s$ is numerically $\mathbb{R}$-Cartier for $s \in S^\circ$.

Assume next that $D_g$ is not numerically $\mathbb{R}$-Cartier. Let $E_g^i$ be the $p$-exceptional divisors. Then there are proper curves $C_g^j \subset Y_g$ that are contracted by $p_g$ and such that $(p_g)_*^{-1}D_g$, viewed as a linear function on $\oplus_j\mathbb{R}[C_g^j]$, is linearly independent of the $E_g^i$. Both the divisors $E_g^i$ and the curves $C_g^j$ extend to give divisors $E_s^i$ and curves $C_s^j$ over an open neighborhood $g \in S^\circ \subset S$. Thus $(p_s)_*^{-1}D_s$, viewed as a linear function on $\oplus_j\mathbb{R}[C_s^j]$, is linearly independent of the $E_s^i$, hence $D_s$ is not numerically $\mathbb{R}$-Cartier for $s \in S^\circ$. □

**Lemma 4.52** *Let $p\colon Y \to X$ be a morphism of proper $S$-schemes and $D$ an $\mathbb{R}$-Cartier divisor on $Y$. Then*

$$S^{nt} := \{s \in S : D_s \textit{ is numerically } p_s\textit{-trivial}\}$$

*is an open subset of $S$.*

*Proof* We check Nagata's openness criterion (10.14).

Let us start with the special case when $X = S$. Pick points $s_1 \in \overline{s_2} \subset S$. A curve $C_2 \subset Y_{s_2}$ specializes to $C_1 \subset Y_{s_1}$ and if $(D_{s_1} \cdot C_1) = 0$ then $(D_{s_2} \cdot C_2) = 0$.

Next assume that $D_{s_2}$ is numerically $p_{s_2}$-trivial. By (11.43.2), $D_{s_2} = \sum a_i A_{s_2}^i$ where the $A_{s_2}^i$ are numerically $p_{s_2}$-trivial $\mathbb{Q}$-divisors. Thus each $mA_{s_2}^i$ is algebraically equivalent to 0 for some $m > 0$; see Lazarsfeld (2004, I.4.38). We can spread out this algebraic equivalence to obtain that there is an open subset $U \subset \overline{s_2}$ such that $mD_s$ is algebraically (and hence numerically) equivalent to 0 on all fibers $s \in U$.

Applying this to $Y \to X$ shows that

$$X^{\mathrm{nt}} := \{x \in X : D_x \text{ is numerically trivial on } Y_x\}$$

is an open subset of $X$. Thus $S^{\mathrm{nt}} = S \setminus \pi_X(X \setminus X^{\mathrm{nt}})$ is an open subset of $S$, where $\pi_X : X \to S$ is the structure map. □

**4.53** (Warning on intersection numbers) In general, one cannot define intersection numbers of numerically $\mathbb{R}$-Cartier divisors with curves. This would

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need the stronger property: $(Z \cdot D_Y) = 0$ for every (not necessarily effective) 1-cycle $Z$ on $Y$ such that $p_*[Z] = 0$.

To see that this is indeed a stronger requirement, let $E \subset \mathbb{P}^2$ be a smooth cubic and $S \subset \mathbb{P}^3$ the cone over it. For $x \in E$ let $L_x \subset S$ denote the line over $x$. Set $X := S \times E$ and consider the divisors $D_1$, swept out by the lines $L_{x_0} \times \{x\}$ for some fixed $x_0 \in E$, and $D_2$, swept out by the lines $L_x \times \{x\}$ for $x \in E$. Let $p\colon Y \to X$ be the resolution obtained by blowing up the singular set, with exceptional divisor $F \simeq E \times E$. Then $p_*^{-1}(D_1 - D_2)$ shows that $D_1 - D_2$ is numerically Cartier.

Set $C := F \cap p_*^{-1}(D_1 - D_2)$. It is a section minus the diagonal on $E \times E$. Thus $p_*[C] = 0$, but $(C \cdot p_*^{-1}(D_1 - D_2)) = -2$.

## 4.7 Stable Families over Smooth Base Schemes

All the results of the previous sections apply to families $p\colon (X, \Delta) \to S$ over a smooth base scheme, but the smooth case has other interesting features as well. The following can be viewed as a direct generalization of (2.3).

**Theorem 4.54** *Let* $(0 \in S)$ *be a smooth, local scheme and* $D_1 + \cdots + D_r \subset S$ *an snc divisor such that* $D_1 \cap \cdots \cap D_r = \{0\}$. *Let* $p : (X, \Delta) \to (0 \in S)$ *be a pure dimensional morphism and* $\Delta$ *an* $\mathbb{R}$*-divisor on* $X$ *such that* $\operatorname{Supp} \Delta \cap \operatorname{Sing} X_0$ *has codimension* $\geq 2$ *in* $X_0$. *The following are equivalent:*

(4.54.1) $p : (X, \Delta) \to S$ *is slc.*

(4.54.2) $K_{X/S} + \Delta$ *is* $\mathbb{R}$*-Cartier,* $p$ *is flat and* $(X_0, \Delta_0)$ *is slc.*

(4.54.3) $K_{X/S} + \Delta$ *is* $\mathbb{R}$*-Cartier,* $X$ *is* $S_2$ *and* $(\operatorname{pure}(X_0), \Delta_0)$ *(10.1) is slc.*

(4.54.4) $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ *is slc.*

*Proof* Note that (1) $\Rightarrow$ (2) holds by definition and (2) $\Rightarrow$ (3) since both $S$ and $X_0$ are $S_2$ (10.10). If (3) holds, then (10.72) shows that $p$ is flat and $X_0$ is pure, hence (3) $\Rightarrow$ (2). Next we show that (2) $\Leftrightarrow$ (4) using induction on $r$. Both implications are trivial if $r = 0$.

Assume (4) and pick a point $x \in X_0$. Then $K_X + \Delta + p^*D_1 + \cdots + p^*D_r$ is $\mathbb{R}$-Cartier at $x$ hence so is $K_X + \Delta$. Set $D_Y := p^*D_r$. By (11.17),

$$(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y})$$

is slc at $x$, hence $(X_0, \Delta_0)$ is slc at $x$ by induction. The local equations of the $p^*D_i$ form a regular sequence at $x$ by (4.58), hence $p$ is flat at $x$.

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Conversely, assume that (2) holds. By induction,

$$(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y})$$

is slc at $x$, hence inversion of adjunction (11.17) shows that $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ is slc at $x$. □

**Corollary 4.55** *Let $S$ be a smooth scheme and $p\colon (X,\Delta) \to S$ a morphism. Then $p : (X,\Delta) \to S$ is locally stable iff the pair $(X, \Delta + p^*D)$ is slc for every snc divisor $D \subset S$.* □

**Corollary 4.56** *Let $S$ be a smooth, irreducible scheme and $p : (X,\Delta) \to S$ a locally stable morphism. Then every log center of $(X,\Delta)$ dominates $S$.*

*Proof* Let $E$ be a divisor over $X$ such that $a(E, X, \Delta) < 0$ and let $Z \subset S$ denote the image of $E$ in $S$. If $Z \neq S$ then, possibly after replacing $S$ by an open subset, we may assume that $Z$ is contained in a smooth divisor $D \subset S$. Thus $(X, \Delta + p^*D)$ is slc by (4.55). However, $a(E, X, \Delta + p^*D) \leq a(E, X, \Delta) - 1 < -1$, a contradiction. □

**Corollary 4.57** *Let $S$ be a smooth scheme and $p\colon (X,\Delta) \to S$ a projective, locally stable morphism with normal generic fiber. Let $p^c\colon (X^c, \Delta^c) \to S$ denote the canonical model of $p : (X,\Delta) \to S$ and $p^w : (X^w, \Delta^w) \to S$ a weak canonical model as in Kollár and Mori (1998, 3.50). Then*

(4.57.1) *$p^w : (X^w, \Delta^w) \to S$ is locally stable and*

(4.57.2) *$p^c : (X^c, \Delta^c) \to S$ is stable.*

*Warning 4.57.3* As in (2.47.1), the fibers of $p^c$ are *not* necessarily the canonical models of the fibers of $p$.

*Proof* Let $D \subset S$ be an snc divisor. By (4.55), $(X, \Delta + p^*D)$ is lc and $p^w : (X^w, \Delta^w + (p^*D)^w) \to S$ is also a weak canonical model over $S$ by Kollár (2013b, 1.28). Thus $(X^w, \Delta^w + (p^*D)^w)$ is also slc, where $(p^*D)^w$ is the push-forward of $p^*D$. Next we claim that $(p^*D)^w = (p^w)^*D$. This is clear away from the exceptional set of $(p^w)^{-1}$ which has codimension $\geq 2$ in $X^w$. Thus $(p^*D)^w$ and $(p^w)^*D$ are two divisors that agree outside a codimension $\geq 2$ subset, hence they agree. Now we can use (4.55) again to conclude that $p^w : (X^w, \Delta^w) \to S$ is locally stable.

A weak canonical model is a canonical model iff $K_{X^w/S} + \Delta^w$ is $p^w$-ample and the latter is also what makes a locally stable morphism stable. □

**Lemma 4.58** *Let* $(y \in Y, \Delta + D_1 + \cdots + D_r)$ *be slc. Assume that the* $D_i$ *are Cartier divisors with local equations* $(s_i = 0)$*. Then the* $s_i$ *form a regular sequence.*

*Proof* We use induction on $r$. Since $Y$ is $S_2$, $s_r$ is a non-zerodivisor at $y$. By adjunction $(y \in D_r, \Delta|_{D_r} + D_1|_{D_r} + \cdots + D_{r-1}|_{D_r})$ is slc, hence the restrictions $s_1|_{D_r}, \ldots, s_{r-1}|_{D_r}$ form a regular sequence at $x$. Thus $s_1, \ldots, s_r$ is a regular sequence at $y$. □

The following result of Karu (2000) is a generalization of (2.51) from one-dimensional to higher-dimensional bases.

**Theorem 4.59** *Let* $U$ *be a k-variety and* $f_U\colon (X_U, \Delta_U) \to U$ *a stable morphism. Then there is projective, generically finite, dominant morphism* $\pi : V \to U$ *and a compactification* $V \hookrightarrow \bar{V}$ *such that the pull-back* $(X_U, \Delta_U) \times_U V$ *extends to a stable morphism* $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$.

*Proof* We may assume that $U$ is irreducible with generic point $g$.

Assume first that the generic fiber of $f_U$ is normal and geometrically irreducible. Let $(Y_g, \Delta_g^Y) \to (X_g, \Delta_g)$ be a log resolution. It extends to a simultaneous log resolution $(Y_{U_0}, \Delta_{U_0}^Y) \to (X_{U_0}, \Delta_{U_0})$ over an open subset $U_0 \subset U$. By Abramovich and Karu (2000) (see also Adiprasito et al. (2019)), there is a projective, generically finite, dominant morphism $\pi : V_0 \to U_0$ and a compactification $V_0 \hookrightarrow \bar{V}$ such that the pull-back $(Y_{U_0}, \Delta_{U_0}^Y) \times_{U_0} V_0$ extends to a locally stable morphism $g_{\bar{V}} : (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \to \bar{V}$.

We can harmlessly replace $\bar{V}$ by a resolution of it. Thus we may assume that $\bar{V}$ is smooth and there is an open subset $V \subset \bar{V}$ such that the rational map $\bar{\pi}|_V : V \dashrightarrow U$ is a proper morphism.

Since $g_{\bar{V}}$ is a projective, locally stable morphism, the relative canonical model $f_{\bar{V}}\colon (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$ of $g_{\bar{V}}\colon (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \to \bar{V}$ exists by Hacon and Xu (2013) and it is stable by (4.57.2).

By construction, $(X_{\bar{V}}, \Delta_{\bar{V}})$ and $(X_U, \Delta_U) \times_U V$ are isomorphic over $V_0 \subset V$, but (11.40) implies that in fact they are isomorphic over $V$. This completes the case when the generic fiber of $f_U$ is normal.

In general, we can first pull back everything to the Stein factorization of $X^n \to U$ where $X^n$ is the normalization of $X$. The previous step now gives $f_{\bar{V}}\colon (X_{\bar{V}}^n, \Delta_{\bar{V}}^n) \to \bar{V}$. Finally, (4.56) shows that (11.41) applies and we get $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$. □

**Corollary 4.60** *Let k be a field of characteristic 0 and assume that the coarse moduli space of stable pairs* SP *exists, is separated, and locally of finite type.*

*Then every irreducible component of* SP *is proper over k.*

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*Proof* Let $M$ be an irreducible component of SP with generic point $g_M$. By assumption, there is a field extension $K \supset k(g_M)$ and a stable $K$-variety $(X_K, \Delta_K)$ corresponding to $g_M$.

Since it takes only finitely many equations to define a stable pair, we may also assume that $K/k(g_M)$ is finitely generated, hence so is $K/k$.

By (4.59), there is a smooth, projective $k$-variety $\bar{V}$ and a stable family $\bar{f} : (\bar{Y}, \bar{\Delta}_Y) \to \bar{V}$ such that $k(\bar{V})$ is a finite field extension of $K$ and the generic fiber of $\bar{f}$ is isomorphic to $(X_K, \Delta_K)_{k(\bar{V})}$.

The image of the corresponding moduli morphism $\phi : \bar{Y} \to \mathrm{SP}$ contains $g_M$ and it is proper. It is thus the closure of $g_M$, which is $M$. So $M$ is proper. □

## 4.8 Mumford Divisors

On a normal variety, our basic objects are Weil divisors. On a nonnormal variety, we work with Weil divisors whose irreducible components are not contained in the singular locus. It has been long understood that these give the correct theory of generalized Jacobians of curves; see Serre (1959). Their first appearance in the moduli theory of curves may be Mumford's definition of pointed stable curves given in Knudsen (1983, Def.1.1).

Here we consider the relative version that is compatible with Cayley–Chow forms in a very strong way (4.69). This enables us to construct a universal family of Mumford divisors (4.76), which is a key step in the construction of the moduli space of stable pairs.

We start by recalling the foundational properties of Chow varieties, as treated in Kollár (1996, secs.I.3–4), and then discuss the ideal of Chow equations. We focus on the classical theory, which is over fields. A closer inspection reveals that the theory works for Mumford divisors over arbitrary bases. The end result is that the functor of Mumford divisors (4.69) is representable over reduced bases (4.76).

**Definition 4.61** A *d-cycle* on a scheme $X$ is a finite linear combination $Z := \sum_i m_i[V_i]$, where $m_i \in \mathbb{Z}$ and the $V_i$ are $d$-dimensional irreducible, reduced subschemes. We usually tacitly assume that the $V_i$ are distinct and $m_i \neq 0$. Then the $V_i$ are called the *irreducible components* of $Z$ and the $m_i$ the *multiplicities*. A $d$-cycle is called *effective* if $m_i \geq 0$ for every $i$ and *reduced* if all its multiplicities equal 1.

To a subscheme $W \subset X$ of dimension $\leq d$, we associate a $d$-cycle, called the *fundamental cycle*

$$[W] := \textstyle\sum_i (\mathrm{length}_{w_i} \mathscr{O}_W) \cdot [W_i], \tag{4.61.1}$$

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where $W_i \subset W$ are the $d$-dimensional irreducible components with generic points $w_i \in W_i$. If $W$ is reduced and pure dimensional then $[W]$ determines $W$; we will not always distinguish them clearly. However, if $W$ is nonreduced, then it carries much more information than $[W]$. The only exception is when $W$ is a Mumford divisor.

If $X$ is projective and $L$ is an ample line bundle on $X$, then the *degree* of a $d$-cycle $Z = \sum_i m_i[V_i]$ is defined as $\deg_L Z := \sum_i m_i \deg_L V_i = \sum_i m_i(L^d \cdot V_i)$.

Assume that $X$ is a scheme of finite type over a field $k$ and $K/k$ a field extension. If $V \subset X$ is a $d$-dimensional irreducible, reduced subvariety then $V_K \subset X_K$ is a $d$-dimensional subscheme which may be reducible and, if $\operatorname{char} k > 0$, may be nonreduced. If $Z = \sum m_i V_i$ is a $d$-cycle, we set

$$Z_K := \sum m_i[(V_i)_K]. \tag{4.61.2}$$

$Z$ is called *geometrically reduced* if $Z_{\bar{k}}$ is reduced. If $\operatorname{char} k = 0$ then reduced is the same as geometrically reduced.

Given an embedding $X \hookrightarrow \mathbb{P}^n$, every $d$-cycle on $X$ is also a $d$-cycle on $\mathbb{P}^n$. Thus Cayley–Chow theory focuses primarily on cycles in $\mathbb{P}^n$

**4.62** (Cayley–Chow correspondence over fields I) Fix a projective space $\mathbb{P}^n$ over a field $k$ with dual projective space $\check{\mathbb{P}}^n$. Points in $\check{\mathbb{P}}^n$ are hyperplanes in $\mathbb{P}^n$.

For $d \le n-1$ we have an incidence correspondence

$$\mathbf{I}^{(n,d)} := \{(p, H_0, \dots, H_d) : p \in H_0 \cap \cdots \cap H_d\} \subset \mathbb{P}^n \times (\check{\mathbb{P}}^n)^{d+1}, \tag{4.62.1}$$

which comes with the coordinate projections

$$\mathbb{P}^n \xleftarrow{\pi_1} \mathbf{I}^{(n,d)} \xrightarrow{\pi_2} (\check{\mathbb{P}}^n)^{d+1} \xrightarrow{\sigma_i} (\check{\mathbb{P}}^n)^d, \tag{4.62.2}$$

where $\pi_1$ is a $(\check{\mathbb{P}}^{n-1})^{d+1}$-bundle and $\sigma_i$ deletes the $i$th component. The projection $\pi_2$ is a $\mathbb{P}^{n-d-1}$-bundle over a dense open subset. For a closed subscheme $Y \subset \mathbb{P}^n$ set $\mathbf{I}_Y^{(n,d)} := \pi_1^{-1}(Y)$.

Let $Z \subset \mathbb{P}^n$ be an irreducible, geometrically reduced, closed subvariety of dimension $d$. Its *Cayley–Chow hypersurface* is defined as

$$\begin{aligned} \operatorname{Ch}(Z) &:= \pi_2(\mathbf{I}_Z^{(n,d)}) \\ &= \{(H_0, \dots, H_d) \in (\check{\mathbb{P}}^n)^{d+1} : Z \cap H_0 \cap \cdots \cap H_d \neq \emptyset\}. \end{aligned} \tag{4.62.3}$$

An equation of $\operatorname{Ch}(Z)$ is called a *Cayley–Chow form.* Next note that

$$\mathbf{I}_Z^{(n,d)} \cap \pi_2^{-1}(H_0, \dots, H_d) = Z \cap H_0 \cap \cdots \cap H_d. \tag{4.62.4}$$

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In particular, a general $H_0 \cap \cdots \cap H_d$ is disjoint from $Z$ and a general $H_0 \cap \cdots \cap H_d$ containing a smooth point $p \in Z$ meets $Z$ only at $p$ (scheme theoretically). Thus we see the following.

*Claim 4.62.5* Let $Z$ be a geometrically reduced $d$-cycle. Then $\pi_2 \colon \mathbf{I}_Z^{(n,d)} \to \operatorname{Ch}(Z)$ is birational and $\operatorname{Ch}(Z)$ is a hypersurface in $(\check{\mathbb{P}}^n)^{d+1}$. □

For any $H_0, \dots, H_{d-1}$ the fiber of the coordinate projection $\sigma_d \colon \operatorname{Ch}(Z) \to (\check{\mathbb{P}}^n)^d$ is $\check{\mathbb{P}}^n$ if $\dim(Z \cap H_0 \cap \cdots \cap H_{d-1}) \geq 1$; otherwise it is the set of hyperplanes that contain one of the points of $Z \cap H_0 \cap \cdots \cap H_{d-1}$. Similarly for all the other $\sigma_i$. Thus we proved the following.

*Claim 4.62.6* Let $Z$ be a geometrically reduced $d$-cycle of degree $r$. Then a general geometric fiber of any of the projections $\sigma_i : \operatorname{Ch}(Z) \to (\check{\mathbb{P}}^n)^d$ is the union of $r$ distinct hyperplanes in $\check{\mathbb{P}}^n$. In particular, the projections are geometrically reduced and $\operatorname{Ch}(Z)$ has multidegree $(r, \dots, r)$. □

For $p \in \mathbb{P}^n$, let $\check{p}$ denote the set of hyperplanes passing through $p$. Then $p \in Z$ iff $\check{p} \times \cdots \times \check{p} \subset \operatorname{Ch}(Z)$. This leads us to the definition of the inverse of the map $Z \mapsto \operatorname{Ch}(Z)$. Let $D \subset (\check{\mathbb{P}}^n)^{d+1}$ be a geometrically reduced subscheme. (In practice, $D$ will always be a hypersurface.) Define

$$\operatorname{Ch}^{-1}_{\text{set}}(D) := \{p : \check{p} \times \cdots \times \check{p} \subset D\} \subset \mathbb{P}^n. \tag{4.62.7}$$

For now we will view $\operatorname{Ch}^{-1}_{\text{set}}(D)$ as a reduced subscheme; scheme-theoretic versions will be discussed in (4.71).

It is easy to see that $\dim \operatorname{Ch}^{-1}_{\text{set}}(D) \leq d$ and an irreducible hypersurface $D$ is of *Cayley–Chow type* if $\dim \operatorname{Ch}^{-1}_{\text{set}}(D) = d$. An arbitrary hypersurface $D$ is of *Cayley–Chow type* if all of its irreducible components are. The basic correspondence of Cayley–Chow theory is the following; see Kollár (1996, I.3.24.5).

*Claim 4.62.8* Fix $n, d, r$, and a base field $k$. Then the maps Ch and $\operatorname{Ch}^{-1}_{\text{set}}$ provide a one-to-one correspondence between

$$\left\{ \begin{array}{c} \text{geometrically reduced} \\ d\text{-cycles of degree } r \text{ in } \mathbb{P}^n \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{geometrically reduced} \\ \text{Cayley–Chow type hypersurfaces of} \\ \text{degree } (r, \dots, r) \text{ in } (\check{\mathbb{P}}^n)^{d+1} \end{array} \right\}.$$

*Proof* We already saw the ⇒ part. To see the converse, observe the inclusion $\operatorname{Ch}(\operatorname{Ch}^{-1}_{\text{set}}(D)) \subset D$. Thus if $Z \subset \operatorname{Ch}^{-1}_{\text{set}}(D)$ is any subvariety of dimension $d$, then $\operatorname{Ch}(Z) \subset D$, hence $\operatorname{Ch}(Z)$ is an irreducible component of $D$. Thus $D = \operatorname{Ch}(\operatorname{Ch}^{-1}_{\text{set}}(D))$. □

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Let $Z \subset \mathbb{P}^n$ be a pure dimensional subscheme or a cycle. The Chow equations are the "most obvious" equations of $Z$. They generate a homogeneous ideal (or an ideal sheaf), which was studied in various forms in Catanese (1992), Dalbec and Sturmfels (1995), and Kollár (1999). Its relationship with the scheme-theoretic $\mathrm{Ch}^{-1}_{\mathrm{sch}}$ will be given in (4.73).

**4.63** (Element-wise power) Let $R$ be a ring, $I \subset R$ an ideal, and $m \in \mathbb{N}$. Set

$$I^{[m]} := (r^m : r \in I).$$

These ideals have been studied mostly when $\mathrm{char}\, k = p > 0$ and $q$ is a power of $p$; one of the early occurrences is in Kunz (1976). In these cases, $I^{[q]}$ is called a *Frobenius power* of $I$. Other values of the exponent are also interesting. Of the following properties, (1) is clear and (4.63.2–3) are implied by (4.63.4–5). We assume for simplicity that $R$ is a $k$-algebra.

(4.63.1) If $I$ is principal then $I^{[m]} = I^m$.

(4.63.2) If $\mathrm{char}\, k = 0$ then $I^{[m]} = I^m$.

(4.63.3) If $m < \mathrm{char}\, k$ then $I^{[m]} = I^m$.

(4.63.4) If $k$ is infinite then $(r_1, \dots, r_n)^{[m]} = ((\sum c_i r_i)^m : c_i \in k)$.

Note that (3) is close to being optimal. For example, if $I = (x, y) \subset k[x, y]$ and $\mathrm{char}\, k = p \geq 3$ then $(x, y)^{[p+1]} = (x^{p+1}, x^p y, xy^p, y^{p+1}) \subsetneq (x, y)^{p+1}$.

*Claim 4.63.5* Let $k$ be an infinite field. Then

$$\langle (c_1 x_1 + \cdots + c_n x_n)^m : c_i \in k \rangle = \langle x_1^{i_1} \cdots x_n^{i_n} : \tbinom{m}{i_1 \dots i_n} \neq 0 \rangle.$$

Here $\binom{m}{i_1 \dots i_n}$ denotes the coefficient of $x_1^{i_1} \cdots x_n^{i_n}$ in $(x_1 + \cdots + x_n)^m$.

*Proof* The containment $\subset$ is clear. If the two sides are not equal then the left-hand side is contained in some hyperplane of the form $\sum \lambda_I x^I = 0$, but this would give a nontrivial polynomial identity $\sum \binom{m}{i_1 \dots i_n} \lambda_I c^I = 0$ for the $c_i$. □

**4.64** (Ideal of Chow equations) Let $Z$ be a $d$-cycle of degree $r$ in $\mathbb{P}^n$. Let $\varrho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ be a projection that is defined along $Z$. Then $\varrho_*(Z)$ is a $d$-cycle in $\mathbb{P}^{d+1}$, thus it can be identified with a hypersurface; hence with a homogeneous polynomial $\phi(Z, \varrho)$ of degree $r$. Its pull-back to $\mathbb{P}^n$ is a homogeneous polynomial $\Phi(Z, \varrho)$ of degree $r$. Together they generate the *ideal sheaf of Chow equations* $I^{\mathrm{ch}}(Z) \subset \mathscr{O}_{\mathbb{P}^n}$.

Over a finite field $k$ there may not be any projections defined along $Z$. The definition gives $I^{\mathrm{ch}}(Z)$ over $\bar{k}$ and it is clearly defined over $k$.

$^0*$ This is not related to the symbolic power, frequently denoted by $I^{(m)}$.

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The embedded primes of $I^{\mathrm{ch}}(Z)$ are quite hard to understand, so frequently we focus on the *Chow hull* of the cycle $Z$:

$$\mathrm{CHull}(Z) := \mathrm{pure}(\mathrm{Spec}\, \mathscr{O}_{\mathbb{P}^n}/I^{\mathrm{ch}}(Z)).$$

Any Zariski dense set of projections generate $I^{\mathrm{ch}}(Z)$. That is, if $P \subset \mathrm{Gr}(n-d, n+1)$ is Zariski dense then $I^{\mathrm{ch}}(Z) = (\Phi(Z,\varrho) : \varrho \in P)$. It is enough to show that this holds in every Artinian quotient $\sigma : \mathscr{O}_{\mathbb{P}^n} \twoheadrightarrow A$. Let $B \subset A$ be the ideal generated by $\sigma(\Phi(Z,\varrho) : \varrho \in P)$. All the $\sigma(\Phi(Z,\varrho))$ are points of an irreducible subvariety $G \subset A$ obtained as an image of $\mathrm{Gr}(n-d, n+1)$. By assumption, $G \cap B$ contains the points $\sigma(\Phi(Z,\varrho) : \varrho \in P)$, hence it is dense in $G$. So $G \subset B$, since $B$ is Zariski closed, if we think of $A$ as a $k$-vectorspace.

*Claim 4.64.1* Let $Z$ be a geometrically reduced cycle. Then $I^{\mathrm{ch}}(Z) \subset I_Z$ and the two agree along the smooth locus of $Z$.

*Proof* Let $p \in Z$ be a smooth point and $v \in T_p\mathbb{P}^n \setminus T_pZ$. A general projection $\varrho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ maps $\langle T_pZ, v\rangle$ isomorphically onto $T_{\varrho(p)}\mathbb{P}^{d+1}$. Then $d\Phi(Z,\varrho)$ is nonzero on $v$. Thus the $\Phi(Z,\varrho)$ generate $I_Z$ in a neighborhood of $p$. □

For the nonreduced case, we need a definition.

**Definition–Lemma 4.65** Let $Z \subset \mathbb{P}^n$ be an irreducible, $d$-dimensional subscheme such that $\mathrm{red}\, Z$ is geometrically reduced. Its *width* is defined in the following equivalent ways.

(4.65.1) The *projection width* of $Z$ is the generic multiplicity of $\pi(Z)$ for a general projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$.

(4.65.2) The *power width* of $Z$ is the smallest $m$ such that $I_{\mathrm{red}\, Z}^{[m]} \cdot \mathscr{O}_Z$ is generically 0 along $Z$.

In general, we first take a purely inseparable field extension $K/k$ such that $\mathrm{red}(Z_K)$ is geometrically reduced and define the width of $Z$ as the width of $Z_K$.

For example, it is easy to see that the width of $\mathrm{Spec}\, k[x,y]/(x,y)^m$ is $m$ and the width of $\mathrm{Spec}\, k[x,y]/(x^m, y^m)$ is $2m-1$.

*Proof* For a general projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ let $\phi_\pi$ be an equation of $\pi(\mathrm{red}\, Z)$ and $\Phi_\pi$ its pull-back to $\mathbb{P}^n$. Then $Z$ has projection width $m$ iff $\Phi_\pi^m \cdot \mathscr{O}_Z$ is generically 0 for every $\pi$, and $m$ is the smallest such. Since the $\Phi_\pi$ generically generate $I_{\mathrm{red}\, Z}$, this holds iff $I_{\mathrm{red}\, Z}^{[m]} \cdot \mathscr{O}_Z$ is generically 0 and $m$ is the smallest. Thus the projection width equals the power width. □

**Proposition 4.66** *Let $Z_i \subset \mathbb{P}^n$ be distinct, geometrically irreducible cycles of the same dimension. Then* $\mathrm{CHull}(\sum m_i Z_i) = \mathrm{pure}(\mathrm{Spec}\, \mathscr{O}_{\mathbb{P}^n}/ \cap_i I(Z_i)^{[m_i]})$.

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*Proof* The equations of the projections $\phi(\sum Z_i, \varrho)$ (as in (4.64)) generate $I_{\sum Z}$ at its smooth points. So if $p \in Z_i$ is a smooth point of $\sum Z$, then $I(Z_i)^{[m_i]}$ agrees with $I^{\mathrm{ch}}(\sum m_i Z_i)$ at $p$ by (4.63.4). □

The following consequence of (4.66) is key to our study of Mumford divisors.

**Corollary 4.67** *Let $k$ be an infinite field, $X \subset \mathbb{P}^n_k$ a reduced subscheme of pure dimension $d+1$ and $D \subset X$ a Mumford divisor, viewed as a divisorial subscheme. Then* $\mathrm{pure}(X \cap \mathrm{CHull}(D)) = D$.

*Proof* The containment ⊃ is clear, hence equality can be checked after a field extension. Write $D = \sum m_i D_i$ where the $D_i$ are geometrically irreducible and reduced. Then $\mathrm{CHull}(D) = \mathrm{pure}(\mathrm{Spec}\, \mathscr{O}_{\mathbb{P}^n_k} / \cap_i I(D_i)^{[m_i]})$ by (4.66). Let $g_i \in D_i$ be the generic point and $R_i$ its local ring in $\mathbb{P}^n_k$. Let $J_i \subset R_i$ be the ideal defining $X$ and $(J_i, h_i)$ the ideal defining $D_i$. The ideal defining the left-hand side of (4.67.1) is then $(J_i + (J_i, h_i)^{[m_i]})/J_i$. This is the same as $(h_i)^{[m_i]}$, as an ideal in $R_i/J_i$, which equals $(h_i^{m_i})$ by (4.63.1). □

## Relative Mumford Divisors

**Definition 4.68** Let $S$ be a scheme and $f\colon X \to S$ a morphism of pure relative dimension $n$ that is mostly flat (3.26). A *relative Mumford divisor* on $X$ is a relative, generically Cartier divisor $D$ (4.24) such that, for every $s \in S$, the fiber $X_s$ is smooth at all generic points of $D_s$.

Let $S'$ be another scheme and $h : S' \to S$ a morphism. Then the pull-back $h^{[*]}D$ is again a relative Mumford divisor on $X \times_S S' \to S'$. This gives the *functor of Mumford divisors,* denoted by

$$\mathcal{MD}iv(X/S)(*)\colon \{S\text{-schemes}\} \to \{\text{sets}\}. \tag{4.68.1}$$

We prove in (4.76) that if $f$ is projective, then the functor of effective Mumford divisors is represented by an $S$-scheme

$$\mathrm{Univ}^{\mathrm{md}}(X/S) \to \mathrm{MDiv}(X/S), \tag{4.68.2}$$

whose connected components are quasi-projective over $S$.

We will see that relative, effective Mumford divisors form the right class for moduli purposes over a reduced base, but not in general. Fixing this problem leads to the notion of K-flatness in Chapter 7.

The following result – whose proof will be given after (4.76.5) – turns a relative, effective Mumford divisor into a flat family of Cartier divisors on another morphism, leading to the existence of $\mathrm{MDiv}(X/S)$ in (4.76).

**Theorem 4.69** *Let $S$ be a reduced scheme, $f\colon X \to S$ a projective morphism that is mostly flat (3.26), and $j : X \hookrightarrow \mathbf{P}_S$ an embedding into a $\mathbb{P}^N$-bundle. Then the maps* Ch *and* $\mathrm{Ch}_X^{-1}$ *– to be defined in (4.70) and (4.75.2) – provide a one-to-one correspondence*

$$\left\{ \begin{matrix} \textit{relative Mumford} \\ \textit{divisors on } X \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{flat Cayley–Chow forms of} \\ \textit{Mumford divisors on } X \end{matrix} \right\}. \tag{4.69.1}$$

*Comments 4.69.2* There are two remarkable aspects of this equivalence. First, the left-hand side depends only on $X \to S$, while the right-hand side is defined in terms of an embedding $j : X \hookrightarrow \mathbf{P}_S$.

Second, on the left we have families that are usually not flat, on the right families of hypersurfaces in a product of projective spaces; these are the simplest possible flat families.

The correspondence (4.69.1) fails very badly over nonreduced bases. We see in (7.14) that, in an analogous local setting, the left-hand side is locally infinite dimensional for $S = \operatorname{Spec} \mathbb{C}[\varepsilon]$, but the right-hand side is locally finite dimensional. Nonetheless, we will be guided by (4.69.1). The rough plan is that we declare the right-hand side to give the correct answer and then work backwards to see what additional conditions this imposes on relative Mumford divisors. This leads us to the notion of C-flatness (7.37). Independence of the embedding $j : X \hookrightarrow \mathbf{P}_S$ then becomes a major issue in Chapter 7.

**4.70** (Definition of Ch) In order to construct $\mathrm{Chow}_{d,r}(\mathbb{P}^n_S)$, the Chow variety of degree $r$ cycles of dimension $d$ in $\mathbb{P}^n_S$, we start with the incidence correspondence as in (4.62)

$$\begin{array}{ccc} & \mathrm{Inc}_S(\text{point}, (\check{\mathbb{P}}^n)^{d+1}) & \\ {\scriptstyle\sigma} \swarrow & & \searrow {\scriptstyle\tau} \\ \mathbb{P}^n_S & & (\check{\mathbb{P}}^n)^{d+1}_S. \end{array} \tag{4.70.1}$$

Note that here $\sigma = \sigma_{n,d,r}$ is a $(\check{\mathbb{P}}^{n-1})^{d+1}$-bundle. The fibers of $\tau = \tau_{n,d,r}$ are linear spaces of dimension $\geq n - d - 1$ and $\tau$ is a $\mathbb{P}^{n-d-1}$-bundle over a dense open subset.

Let now $D \subset \mathbb{P}^n_S$ be a generically flat family of $d$-dimensional subschemes (3.26). Assume also that the generic embedding dimension of $D_s$ is $\leq d + 1$ for every $s \in S$. (This is satisfied iff each $D_s$ is a Mumford divisor on some $X \subset \mathbb{P}^n_s$; a more general definition is in (7.46).) Set $\mathrm{Ch}(D) := \tau_*(\sigma^{-1}(D))$.

*Claim 4.70.2* The map, $\tau\colon \sigma^{-1}(D) \to \mathrm{Ch}(D)$ is a local isomorphism on the preimage of a dense open subset $U \subset D$ such that $U \cap D_s$ is dense in $D_s$ for every $s \in S$.

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*Proof* Pick $p \in D_s$ such that $T_{D_s}$ has dimension $d+1$ at $p$. If $L_s \supset p$ is a general linear subspace of dimension $n-d-1$, then $L_s \cap D_s = \{p\}$, scheme theoretically. This is exactly the fiber of $\tau : \sigma^{-1}(D) \to \mathrm{Ch}(D)$ over any $(H_0, \dots, H_d)$ for which $L_s = H_0 \cap \cdots \cap H_n$. □

*Corollary 4.70.3* Ch($D$) is a generically flat family of Cartier divisors. If $S$ is reduced, then Ch($D$) is flat over $S$.

*Proof* By assumption, $D$ is a generically flat family, hence so is $\sigma^{-1}(D)$ since $\sigma$ is smooth. The first part is now immediate from (4.70.2). The second claim then follows from (4.36). □

**4.71** (Definition of $\mathrm{Ch}^{-1}_{\mathrm{sch}}$) Although Ch($D$) is not a flat family of Cartier divisor in general, we decide that from now on we are only interested in the cases when it is flat. Thus let $H^{\mathrm{cc}} \subset (\check{\mathbb{P}}^n_S)^{d+1}$ be a relative hypersurface of multidegree $(r, \dots, r)$. We first define its *scheme-theoretic Cayley–Chow inverse,* denoted by $\mathrm{Ch}^{-1}_{\mathrm{sch}}(H^{\mathrm{cc}})$. It is a first approximation of the "correct" Cayley–Chow inverse.

Working with (4.70.1) consider the restriction of the left-hand projection

$$\sigma^{\mathrm{cc}} : (\mathrm{Inc}_S \cap \tau^{-1}(H^{\mathrm{cc}})) \to \mathbb{P}^n_S. \tag{4.71.1}$$

Fix $s \in S$ and a point $p_s \in \mathbb{P}^n_s$. Note that the preimage of $p_s$ consists of all $(d+1)$-tuples $(H_0, \dots, H_d)$ such that $p_s \in H_i$ for every $i$ and $(H_0, \dots, H_d) \in H^{\mathrm{cc}}_s$. In particular, if $Z$ is a $d$-cycle of degree $r$ on $\mathbb{P}^n_S$ and $H^{\mathrm{cc}} = \mathrm{Ch}(Z)$ is its Cayley–Chow hypersurface, then $\sigma^{\mathrm{cc}}$ is a $(\check{\mathbb{P}}^{n-1}_S)^{d+1}$-bundle over Supp $Z$.

The key observation is that this property alone is enough to define $\mathrm{Ch}^{-1}_{\mathrm{sch}}$ and to construct the Chow variety. So we define $\mathrm{Ch}^{-1}_{\mathrm{sch}}(H^{\mathrm{cc}}) \subset \mathbb{P}^n_S$ as the unique, largest, closed subscheme over which $\sigma^{\mathrm{cc}}$ is a $(\check{\mathbb{P}}^{n-1})^{d+1}$-bundle. (Its existence is a special case of (3.19), but we derive its equations in (4.72.2).)

The set-theoretic behavior of the projection $\varrho : \mathrm{Ch}^{-1}_{\mathrm{sch}}(H^{\mathrm{cc}}) \to S$ is described in (4.62). The fibers have dimension $\leq d$ and $Z_s \subset \mathbb{P}^n_s$ is a $d$-dimensional irreducible component iff Ch($Z_s$) is an irreducible component of $H^{\mathrm{cc}}_s$. It is not hard to see that there is a maximal closed subset $S(H^{\mathrm{cc}}) \subset S$ over which $H^{\mathrm{cc}}$ is the Cayley–Chow hypersurface of a family of $d$-cycles; see Kollár (1996, I.3.25.1).

However, we do not yet have the "correct" scheme structure on $S(H^{\mathrm{cc}})$, since the scheme structure of the fibers of $\varrho$: $\mathrm{Ch}^{-1}_{\mathrm{sch}}(H^{\mathrm{cc}}) \to S$ is not the "correct" one. Before we move ahead, we need to understand this scheme structure.

**4.72** (Scheme structure of $\mathrm{Ch}^{-1}_{\mathrm{sch}}(H^{\mathrm{cc}})$) Let $S$ be a scheme and $H^{\mathrm{cc}} := (F = 0) \subset (\check{\mathbb{P}}^n)^{d+1}_S$ a hypersurface of multidegree $(r, \dots, r)$. We aim to write down equations for $\mathrm{Ch}^{-1}_{\mathrm{sch}}(F = 0)$.

Choose coordinates $(x_0{:}\cdots{:}x_n)$ on $\mathbb{P}^n_S$ and dual coordinates $(\check{x}_{0j}{:}\cdots{:}\check{x}_{nj})$ on the $j$th copy of $\check{\mathbb{P}}^n_S$ for $j = 0, \dots, d$. So $F = F(\check{x}_{ij})$ is a homogeneous polynomial of multidegree $(r, \dots, r)$. For notational simplicity we compute in the affine chart $\mathbb{A}^n_S = \mathbb{P}^n_S \setminus (x_0 = 0)$.

For $(x_1, \dots, x_n) \in \mathbb{A}^n_S$, the hyperplanes $H$ in the $j$th copy of $\check{\mathbb{P}}^n_S$ that pass through $(x_1, \dots, x_n)$ are all written as $(-\sum_{i=1}^n x_i \check{x}_{ij} : \check{x}_{1j} : \cdots : \check{x}_{nj})$.

Let $M(\check{x}_{ij})$ be all the monomials in the $\check{x}_{ij}$ and write

$$\begin{aligned} F(-\textstyle\sum_{i=1}^n x_i\check{x}_{i0} : \check{x}_{10} : \cdots : \check{x}_{n0}; \cdots ; -\sum_{i=1}^n x_i\check{x}_{id} : \check{x}_{1d} : \cdots : \check{x}_{nd}) \\ =: \textstyle\sum_M F_M(x_1, \dots, x_n) M(\check{x}_{ij}). \end{aligned} \tag{4.72.1}$$

Since the monomials $M(\check{x}_{ij})$ are linearly independent, this vanishes for all $\check{x}_{ij}$ iff $F_M = 0$ for every $M$. Equivalently:

*Claim* 4.72.2 The subscheme $\mathrm{Ch}^{-1}_{\mathrm{sch}}(F = 0) \cap \mathbb{A}^n_S$ is given by the equations $F_M(x_1, \dots, x_n) = 0$ for all monomials $M$, with $F_M$ as in (4.72.1). □

Assume that $(F = 0) = \mathrm{Ch}(Y)$. If we fix $\check{x}_{ij} = c_{ij}$, then these give the matrix of a linear projection $\pi_{\mathbf{c}} : \mathbb{A}^n_S \to \mathbb{A}^{d+1}_S$. The corresponding Chow equation of $Y$ is $\sum_M F_M(x_1, \dots, x_n) M(c_{ij}) = 0$. Thus we proved the following.

**Theorem 4.73** *Let $Z \subset \mathbb{P}^n_k$ be a d-cycle of degree r. Then* $\mathrm{Ch}^{-1}_{\mathrm{sch}}(\mathrm{Ch}(Z)) \subset \mathbb{P}^n_k$ *is the subscheme defined by the ideal of Chow equations $I^{ch}(Z)$.* □

Note that we proved a little more. If the residue field of $S$ is infinite, then $I^{\mathrm{ch}}(Y)|_{\mathbb{A}^n_S}$ is generated by the Chow equations of the linear projections $\pi_{\mathbf{c}} : \mathbb{A}^n_S \to \mathbb{A}^{n+1}_S$. A priori we would need to use the more general projections (7.34.4), but this is just a matter of choosing the hyperplane at infinity.

Combining (4.73) and (4.67) gives the following.

**Corollary 4.74** *Let k be a field, $X \subset \mathbb{P}^n_k$ a subscheme of pure dimension $d + 1$, and $D \subset X$ a Mumford divisor. Then* $\mathrm{pure}(X \cap \mathrm{Ch}^{-1}_{\mathrm{sch}}(\mathrm{Ch}(D))) = D$. □

**4.75** (Construction of $\mathrm{MDiv}(X/S)$) As we noted in (4.69.2), we construct $\mathrm{MDiv}(X/S)$ by starting on the right-hand side of (4.69.1)

Let $S$ be a scheme, $f : X \to S$ a mostly flat, projective morphism of pure dimension $d$, and $j : X \hookrightarrow \mathbb{P}^n_S$ an embedding.

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We fix the intended degree to be $r$ and let $\mathbf{P}_{n,d,r} = |\mathscr{O}_{(\check{\mathbb{P}}^n)^{d+1}}(r,\dots,r)|$ be the linear system of hypersurfaces of multidegree $(r,\dots,r)$ in $(\check{\mathbb{P}}^n)^{d+1}$, with universal hypersurface $\mathbf{H}^{\mathrm{cc}}_{n,d,r} \subset (\check{\mathbb{P}}^n)^{d+1} \times \mathbf{P}_{n,d,r}$. Thus (4.70.1) extends to

$$\begin{array}{ccc} & \mathrm{Inc}_S(\mathrm{point}, (\check{\mathbb{P}}^n)^{d+1}) \times_S \mathbf{P}_{n,d,r} & \\ \swarrow^{\sigma_{n,d,r}} & & \searrow^{\tau_{n,d,r}} \\ \mathbb{P}^n_S \times_S \mathbf{P}_{n,d,r} & & (\check{\mathbb{P}}^n)^{d+1}_S \times_S \mathbf{P}_{n,d,r}. \end{array} \tag{4.75.1}$$

As in (4.71), we get $\mathrm{Ch}^{-1}_{\mathrm{sch}}(\mathbf{H}^{\mathrm{cc}}_{n,d,r}) \subset \mathbb{P}^n_S \times_S \mathbf{P}_{n,d,r}$. We are interested in $d$-cycles that lie on $X$, so we should take

$$\mathrm{Ch}^{-1}_X(\mathbf{H}^{\mathrm{cc}}_{n,d,r}) := \mathrm{Ch}^{-1}_{\mathrm{sch}}(\mathbf{H}^{\mathrm{cc}}_{n,d,r}) \cap (X \times_S \mathbf{P}_{n,d,r}) \subset \mathbb{P}^n_S \times_S \mathbf{P}_{n,d,r}. \tag{4.75.2}$$

By (4.74), if $D_s \subset X_s$ is a Mumford divisor of degree $r$ then the fiber of the coordinate projection $\varrho_{n,d,r} : \mathrm{Ch}^{-1}_X(\mathbf{H}^{\mathrm{cc}}_{n,d,r}) \to \mathbf{P}_{n,d,r}$ over $[\mathrm{Ch}(D_s)]$ is $D_s$ (aside from possible embedded points).

This leads us to the following. Recall the difference between mostly flat (in codimension $\leq 1$) and generically flat (in codimension 0) as in (3.26).

**Theorem 4.76** *Let $S$ be a scheme, $f : X \to S$ a mostly flat, projective morphism of pure relative dimension $d+1$, and $j : X \hookrightarrow \mathbb{P}^n_S$ an embedding. Then the functor of generically flat families of degree $r$ Mumford divisors on $X$ is represented by a locally closed subscheme* $\mathrm{MDiv}_r(X/S)$ *of* $\mathbf{P}_{n,d,r}$ *(4.75). Over* $\mathrm{MDiv}_r(X)$ *we have*

(4.76.1) $\mathrm{Univ}^{md}_r(X/S) \subset X \times_S \mathrm{MDiv}_r(X/S)$*, a universal, generically flat family of degree $r$ Mumford divisors on $X$, and*

(4.76.2) $\mathbf{H}^{cc}_r \subset (\check{\mathbb{P}}^n)^{d+1} \times_S \mathrm{MDiv}_r(X/S)$*, a flat family of multidegree* $(r,\dots,r)$ *hypersurfaces,*

*that correspond to each other under* Ch *and* $\mathrm{Ch}^{-1}_X$.

*Proof* As we noted in (4.62), every fiber of $\varrho_{n,d,r}$ has dimension $\leq d$. So

$$\{H^{\mathrm{cc}}_s : \dim(\mathrm{Sing}\, X_s \cap \mathrm{Supp}\, \mathrm{Ch}^{-1}_X(H^{\mathrm{cc}}_s)) \leq d-1\}$$

defines a closed subset of $\mathbf{P}_{n,d,r}$; let $\mathbf{P}^{\circ}_{n,d,r}$ denote its complement. Thus $[H^{\mathrm{cc}}_s] \in \mathbf{P}^{\circ}_{n,d,r}$ iff the divisorial part of $\mathrm{Ch}^{-1}_X(H^{\mathrm{cc}}_s)$ satisfies the Mumford condition.

Now apply (4.77) to $\mathrm{Ch}^{-1}_X(H^{\mathrm{cc}})$ over $\mathbf{P}^{\circ}_{n,d,r}$ to get a locally closed decomposition $j^{\mathrm{flat}} : \mathbf{P}^{\mathrm{flat}}_{n,d,r} \to \mathbf{P}^{\circ}_{n,d,r}$, representing the functor of generically flat pull-backs of $\mathrm{Ch}^{-1}_X(H^{\mathrm{cc}})$ as in (4.77). Over each connected component of $\mathbf{P}^{\mathrm{flat}}_{n,d,r}$, the degree of the $d$-dimensional part is locally constant. The union of those connected components where this degree equals $r$ is $\mathrm{MDiv}_r(X/S)$. □

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*Warning 4.76.3* In the nonreduced case the resulting $\mathrm{MDiv}(X)$ a priori depends on the projective embedding $j\colon X \hookrightarrow \mathbb{P}^n_S$. We write $\mathrm{MDiv}(X \subset \mathbb{P}^n_S)$ if we want to emphasize this. In Chapter 7 we construct a subscheme $\mathrm{KDiv}(X) \subset \mathrm{MDiv}(X \subset \mathbb{P}^n_S)$, that does not depend on the embedding. The two have the same underlying reduced structure and a positive answer to Question 7.42 would imply that in fact $\mathrm{MDiv}(X \subset \mathbb{P}^n_S) = \mathrm{KDiv}(X)$.

We have used the following variant of (3.19).

**Proposition 4.77** *Let $f\colon X \to S$ be a projective morphisms and $F$ a coherent sheaf on $X$ such that* $\operatorname{Supp} F \to S$ *has fiber dimension $\leq d$. Then there is a locally closed decomposition $j^{flat}_F\colon S^{flat}_F \to S$ such that $F_W$ is flat at $d$-dimensional points of the fibers iff $W \to S$ factors through $j^{flat}_F$.*

*Proof* We may replace $X$ by the scheme-theoretic support $\operatorname{SSupp} F$. The question is local on $S$. By (10.46.1), we may assume that there is a finite morphism $\pi\colon X \to \mathbb{P}^d_S$. Note that $F_W$ is flat at $d$-dimensional points iff the same holds for $(\pi_W)_* F_W$. We may thus assume that $X = \mathbb{P}^d_S$; the important property is that now $f\colon X \to S$ is flat with integral geometric fibers. By (3.19.1) we get a decomposition $\amalg_i X_i \to X$, where $F|_{X_i}$ is locally free of rank $i$.

Let $Z \subset X$ be a closed subscheme. Applying (3.19) to the projection $\mathscr{O}_Z$, we see that there is a unique largest subscheme $S(Z) \subset S$ such that $f^{-1}(S(Z)) \subset Z$, scheme theoretically. For a locally closed subscheme $Z \subset X$ set $S(Z) = S(\bar{Z}) \setminus S(\bar{Z} \setminus Z)$, where $\bar{Z}$ denotes the closure of $Z$. Note that $S(Z)$ is the largest subscheme $T \subset S$ with the following property:

(4.77.1) There is an open subscheme $X^\circ_T \subset X_T$ that contains the generic point of $X_t$ for every $t \in T$ and $X^\circ_T \subset Z$, scheme theoretically.

We claim that $S^{\mathrm{flat}}_F = \amalg_i S(X_i)$. One direction is clear. $F|_{X_i}$ is locally free of rank $i$, so the restriction of $F$ to $S(X_i) \times_S X$ is locally free, hence flat, at the generic point of every fiber.

Conversely, let $W$ be a connected scheme and $q : W \to S$ a morphism such that $F_W$ is generically flat over $W$ the fiber dimension of $\operatorname{Supp} F_W \to S$ is $n$. Since $X_w$ is integral, $F_w$ is generically free for every $w \in W$, so $F_W$ is locally free at the generic point of every fiber. Let $X^\circ_W \subset X_W$ be the open set where $F_W$ is locally free.

By assumption, $X^\circ_W$ contains the generic point of every fiber $X_w$, so $X^\circ_W$ is connected. Thus $F_W$ has constant rank, say $i$, on $X^\circ_W$. Therefore, the restriction of $q_X : X_W \to X$ to $X^\circ_W$ lifts to $q^\circ_X : X^\circ_W \to X_i$. By (4.77.1), this means that $q$ factors as $q : W \to S(X_i) \to S$. □

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