Forum

A Mathematical Method of obtaining an Astronomical Vessel Position

Arturo Chiesa and Raffaele Chiesa

A method is presented which enables a vessel's position to be obtained directly as the intersection point of the position circles of the celestial bodies observed with a sextant. This method offers an important advantage: it does not require the introduction of an 'estimated position' to start the calculation, as is necessary with the usual method based on position lines. The mathematical procedure is rather complex, but it lends itself to solution with programmable calculators, even of small dimensions and cost, such as minicalculators. Furthermore, the procedure is integrated in such a way that it is sufficient to enter the series of sextant readings into the calculator to obtain the vessel position immediately. Any type of celestial body may be used. The method also allows the 'running fix' procedure to be adopted. Some offshore sail navigators have already made extensive use of the procedure, programmed on BASIC minicalculators.

1. THE MATHEMATICAL SOLUTION OF THE ASTRONOMICAL POSITION. The calculation of a vessel's position as the intersection point of the position circles of observed celestial bodies is a mathematically defined problem; therefore a direct analytical solution is possible. The relative formulae are rather complex, but the present availability of programmable calculators and their widespread distribution provides the possibility of solving the mathematical procedure in a simple and rigorous way. Even calculators of limited dimensions and cost, such as minicalculators, have sufficient capacity.

The advantages offered by the mathematical solution of the problem are essentially two.

(i) A vessel's position is obtained according to its exact definition, as the intersection point of the position circles of the observed celestial bodies.

(ii) No introduction of an estimated position is required to start the calculation as is necessary with procedures based on lines of position.

2. THE USUAL METHOD OF LINES OF POSITION. According to this method, a vessel's position is obtained as the intersection point of the tangents to the position circles of the various observed celestial bodies; more exactly, as the intersection point of straight lines drawn on a Mercator chart. Simple geometrical considerations demonstrate that the intersection of such straight lines does not exactly coincide with the intersection point of the position circles lying on the Earth's surface, which is the exact vessel position.

The reason why the method of lines of position is commonly used in astronavigation is that it lends itself to the application of tabular methods, such as HO 214, HO 249, or HO 229.

However, if a programmable calculator is available, there is no reason to use the position line method instead of the analytical one. Even though the differences between the vessel positions obtained with the two methods are usually very small, the analytical method is to be preferred for the two reasons already stated: because it gives the exact position and, above all, because it does not require the introduction of an estimated position. Simply by entering the sextant readings into the calculator, the coordinates of the vessel position are immediately obtained.

3. THE METHOD OF THE INTERSECTION OF CIRCLES OF POSITION. The following initial data are always available for any observed celestial body: (i) the equatorial celestial coordinates (hour angle, GHA, and declination, Dec, of the celestial body) at the instant of observation; and (ii) the true altitude, H, of the body deduced from the observed sextant altitude. These data enable the instantaneous position circles of the celestial bodies to be drawn on the Earth's surface: the geographical positions of the celestial bodies are the centres of the circles and the zenith distances ($ZD = 90° - H$) are their radii. It may be worthwhile to point out that the drawing of the position circles of the celestial bodies on the Earth's surface is always possible, and consequently there is no need to assume an estimated position for the vessel.

Let us first consider the case in which there are only two observed bodies. Two position circles are then obtained which intersect at two points. *Both points are valid*; that is, two vessel positions are possible.

The existence of two possible positions is a physical reality, independent of the mathematical method used. In fact, if two circles drawn on the Earth's surface intersect, they always intersect at two points. Consequently, if two celestial bodies S_1 and S_2 are observed at the same instant with altitudes H_1 and H_2 respectively, two points P_1 and P_2 always exist on the Earth's surface, where the two bodies S_1 and S_2 are seen with the respective altitudes H_1 and H_2. However, the two points are generally so distant from one another that in practice no doubt can arise in choosing the one corresponding to the actual position of the vessel.

Example. During an Atlantic passage, on 15 September 1988, a little earlier than sunrise, namely at 8.58.00 GMT, a planet and a star are observed: Venus with true altitude 34° 54·5′ and Sirius with true altitude 22° 05·0′. The calculations carried out according to the procedure described in this paper provide the two vessel positions: P_1 (46° 33·6′ N., 55° 18·8′ W.); P_2 (18° 58·7′ S., 43° 56·7′ E.). The first point is on an Atlantic route, about 60 miles south of Newfoundland. The second point lies in the Indian Ocean, near the west coast of Madagascar, more than 6500 miles from the first point.

With the usual altitude formula it can be easily verified that, at both points P_1 and P_2, the two celestial bodies are seen with their altitudes, Venus 34° 54·5′ and Sirius 22° 05·0′.

Let us now describe the mathematical procedure. With reference to Figure 1, the known elements are: (i) the coordinates on the Earth's surface of the geographical positions S_1 and S_2 of the celestial bodies: $\phi_{1,2} = \mathrm{Dec}_{1,2}$, $\lambda_{1,2} = \mathrm{GHA}_{1,2}$; (ii) the arcs $S_1P_1 = S_1P_2$ and $S_2P_1 = S_2P_2$ correspond to the zenith distances $ZD_{1,2} = 90° - H_{1,2}$.

The various steps of the mathematical procedure are as follows.

(1) Determination of the orthodromic distance D and of the initial orthodromic course angle $R = NS_1S_2$ between the two geographical points S_1 and S_2. Various formulae are available. The most common are:

$$\tan R = \frac{\sin(\lambda_2 - \lambda_1)}{\tan\phi_2 \cos\phi_1 - \sin\phi_1 \cos(\lambda_2 - \lambda_1)}$$

$$\cos D = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_2 - \lambda_1)$$

where ϕ_1 and ϕ_2 correspond to the declinations, Dec_1 and Dec_2, and λ_1 and λ_2 correspond to the hour angles GHA_1 and GHA_2 of the celestial bodies.

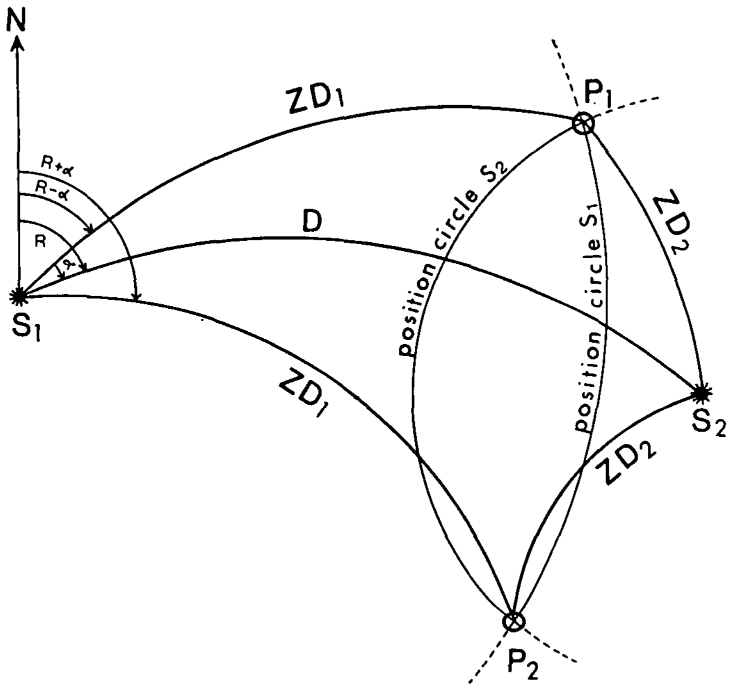

Fig. 1. Intersection of astronomical position circles. S_1, S_2 are the geographical positions of the two observed celestial bodies. ZD_1, ZD_2 are the zenith distances: $ZD_{1,2} = 90° - H_{1,2}$, where H_1 and H_2 are the two true altitudes. P_1, P_2 are the two possible vessel positions

(2) Determination of the angle α of the spherical triangle $S_1P_1S_2$, of which the three arcs ZD_1, ZD_2 and D are known. With simple manipulations from the so-called half-angle formulae, one obtains:

$$\sin\frac{\alpha}{2} = \sqrt{\left(\frac{\cos[(D+H_1+H_2)/2] \quad \sin[(D+H_1+H_2)/2]}{\sin D \quad \cos H_1}\right)}$$

(3) Determination of the angles $R_1 = R - \alpha$ and $R_2 = R + \alpha$, i.e. the initial orthodromic course angles of the known arcs S_1P_1 and S_1P_2.

(4) Lastly, determination of the coordinates of the two points P_1 and P_2 (the two possible vessel positions) given: (i) the coordinates of the departure point S_1 (ϕ_1, λ,); (ii) the initial orthodromic course angles R_1 and R_2; (iii) the distance $ZD_1 = 90° - H_1$.

The following formulae of spherical trigonometry are used:

$$\lambda_{p1,2} = \lambda_A + \arctan\frac{\cos[(ZD_1 - 90 + \phi_1)/2]}{\tan(R_{1,2}/2) \quad \cos[(ZD_1 + 90 - \phi_1)/2]}$$

$$+ \arctan\frac{\sin[(ZD_1 - 90 + \phi_1)/2]}{\tan(R_{1,2}/2) \quad \sin[(ZD_1 + 90 - \phi_1)/2]}$$

$$\phi_{p1,2} = \arccos\frac{\sin R_{1,2} \quad \sin ZD_1}{\sin(\lambda_{p1,2} - \lambda_1)}$$

4. VESSEL POSITION FROM THE ALTITUDES OF THREE CELESTIAL BODIES (OR THREE OBSERVATIONS). If more than two observations are made, the procedure becomes more complicated. The position circles of the observed bodies intersect two by two, but among all possible intersection points only one exists of 'multiple coincidence', where all circles intersect. This is precisely the position point; other points, where circles intersect only two by two, are not significant.

From here on, the mathematical procedure should simply consist of choosing, among the various intersection points, the one common to all circles. However, when applying the procedure in practical cases, a complication arises. In practice, the sextant readings, even at best, are always affected by a certain degree of inaccuracy. Suffice it to consider the variability of atmospheric refraction.

As a result, instead of a unique coincident point, a group of points, very close to but distinct from one another, is always obtained. Thus the need arises to find an algorithm allowing discrimination of those points which contribute to determining the 'multiple point' from those, usually far more distant, where the circles intersect only two by two.

The algorithm devised has a cyclic form. In order to avoid too complicated iterations, only the case of three celestial bodies has been considered; thus the intersection points are six in all. Once the intersection points are obtained, a procedure of successive elimination is used: among the eight possible triplets, one set is chosen whose points are distant from one another less than a preselected value. The coordinates of the vessel position are finally obtained as average values of the coordinates of the points of the selected triplet.

The observation of three celestial bodies is in general sufficient to obtain a good astronomical vessel position. If, however, it is desired to use more than three observations, the procedure can be repeated in groups of three celestial bodies each time. The whole procedure lends itself to be performed by a calculator, hence the repetition of the calculation takes a very short time.

The algorithm has been further improved to allow for excessive observational errors. If the calculator is unable to find a group of points which are close to each other within a preset tolerance, instead of the result the message 'ERROR' appears on the screen. This must be considered as a warning that one or more observations are affected by too large an error. When, on the other hand, the position coordinates appear on the screen, this means that the observations are to be considered valid within the preset tolerance. This is a further advantage of the method presented.

5. RUNNING FIX PROCEDURE. One of the most common procedures in astronavigation is the so-called 'running fix' procedure. The celestial body, usually the Sun, or possibly the Moon during its daylight periods, is observed at various times throughout the day and all sights are then transferred to the last one, according to the courses and distances sailed in the meantime. The method here described allows this procedure to be carried out very easily. It is simply a matter of transferring the geographical positions of the observed bodies before proceeding to the calculation of the intersections.

Lastly, it may be worthwhile to underline that the method presented here does not impose any restriction as far as the times of observation are concerned. In particular, there is no need to take either a meridian sight or sights at symmetric hours around the meridian passage, as is often done with other methods. The only obvious condition is that the azimuths of the observed bodies should be sufficiently separated from one another (usually not less than 30°).

6. INTEGRATION OF THE PROCEDURE. As already said, the method described here may easily be adopted using a programmable calculator. In a recently published

manual,[1] an actual program is described, according to which the vessel position is obtained simply by entering into the calculator the series of sextant readings (altitudes vs. times) of the various observations. Furthermore, once the vessel position is obtained, the calculator also supplies course, distance and passage time for navigation to a possible destination point.

Different types of celestial bodies can be used in the same group of observations, such as, for instance, two sights of the Sun and one of a star or a planet or the Moon. In fact, at every observation, the calculator asks what type of celestial body is being observed. The number of sextant readings of each observation can range from two to seven.

The authors are pleased to state that the navigators of three sailing yachts have made extensive use of the method described, programmed on small minicalculators, during their Atlantic passages.

REFERENCE

[1] Chiesa, A. and Chiesa, R. (1988) *Navigazione piana radiogoniometrica astronomica con programmi di calcolo in BASIC*. C.L.U.P. Publishers, Milan, Italy.

Key Words

1. Astro. 2. Computers.

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