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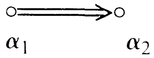
λ-MAPPINGS BETWEEN REPRESENTATION RINGS OF LIE ALGEBRAS

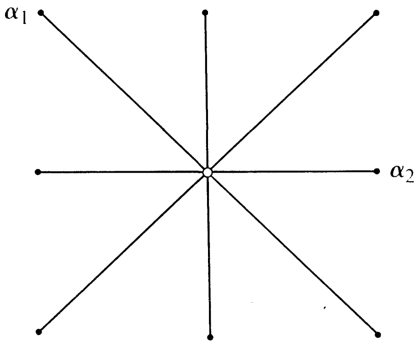
R. V. MOODY AND A. PIANZOLA

Introduction. In **[10]** Patera and Sharp conceived a new relation, subjoining, between semisimple Lie algebras. Our objective in this paper is twofold. Firstly, to lay down a mathematical formalization of this concept for arbitrary Lie algebras. Secondly, to give a complete classification of all maximal subjoinings between Lie algebras of the same rank, of which many examples were already known to the above authors.

The notion of subjoining is a generalization of the subalgebra relation between Lie algebras. To give an intuitive idea of what is involved we take a simple example. Suppose $\mathfrak{g}$ is a complex simple Lie algebra of type B_2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Δ the corresponding root system. We have the standard root diagram



B_2



Inside B_2 there lies the subalgebra $A_1 \times A_1$ which can be identified with the sum of $\mathfrak{h}$ and the root spaces corresponding to the long roots of B_2. If ρ is a representation of B_2 then its restriction to $A_1 \times A_1$ is a representation for the subalgebra and this leads to a homomorphism between their representation rings:

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$$f:R(B_2) \to R(A_1 \times A_1)$$

or more conveniently between their character rings. For example the character of the 14-dimensional representation of B_2 of highest weight $2\omega_1 + 2\omega_2$ branches into

$$\chi_{2,2} + \chi_{1,1} + \chi_{0,0}$$

of dimensions $9 + 4 + 1 = 14$ where $\chi_{i,j}$ is the character of the $A_1 \times A_1$ representation of highest weight (i, j).

Now the subalgebra $A_1 \times A_1$ can be said to exist in B_2 precisely because the root system of $A_1 \times A_1$ can be embedded as a closed subroot system of the same type in Δ.

The subroot system of type $A_1 \times A_1$ consisting of short roots does not correspond to a subalgebra of B_2. Nonetheless there is a homomorphism

$$f':R(B_2) \to R(A_1 \times A_1).$$

For comparison the character of $2\omega_1 + 2\omega_2$ now "branches" into

$$\chi_{0,4} + \chi_{4,0} + \chi_{2,2} - \chi_{0,2} - \chi_{2,0} + \chi_{0,0}$$

of dimensions $5 + 5 + 9 - 3 - 3 + 1 = 14$. Although $A_1 \times A_1$ is not a subalgebra in this case, it is evidently tied to B_2 in an extremely close way. Both cases are examples of subjoining of $A_1 \times A_1$ to B_2 but the second is obviously something new.

In [**10**] subjoining was perceived as a process of moving between weight spaces by means of certain transition matrices. For the purposes of building a mathematical formalization of this we found it more convenient to work with homomorphisms between the representation rings.

The representation ring of a complex semisimple Lie algebra of rank l is well known to be isomorphic to the polynomial ring in l-variables over $\mathbf{Z}$. Obviously most homomorphisms between two such rings are irrelevant to the Lie algebras underlying them. However the Lie algebra confers an additional structure on its representation ring; namely the structure of λ-ring. Not surprisingly, the subalgebra relation between two Lie algebras determines a λ-homomorphism between their representation rings. But not every λ-homomorphism is determined by some subalgebra relation, and it is among these others that we find the new subjoinings. Since the representation ring of an arbitrary Lie algebra $\mathfrak{g}$ depends only on $\mathfrak{g}$ factored by its nil-radical most of our discussion will deal with reductive Lie algebras.

The first part of the paper is devoted to determining a fundamental relation between λ-homomorphisms between the representation rings of reductive Lie algebras on the one hand and certain mappings between their weight systems and Weyl groups on the other. This is described in Theorem 1 and proved in Sections 2, 3, 4.

The proof is based on ideas from the theory of the arithmetic of Galois extensions and seems to indicate that there is still more to be gleaned from this situation. In Section 5 we use our results to give several equivalent definitions of subjoinings.

The second part of the paper is a classification of maximal subjoinings between equal rank reductive Lie algebras. This depends ultimately on the classification of maximal equal rank subalgebras of the simple Lie algebras which was carried out over 40 years ago by Borel and de Siebenthal.

Some of the results of this paper were announced in [**9**].

After we had completed this paper Prof. J. F. Adams kindly pointed out his work (partly coauthored with Z. Mahmud) on maps between classifying spaces of compact simple Lie groups [**13**, **14**]. It is very interesting that these maps are closely related to λ-maps between the corresponding representation rings. In the course of their work they essentially establish part of our Theorem 2.1 [**13**, Theorem 1.7, Corollary 1.13, Theorem 2.21] and give some examples of what we called subjoining. (See also [**15**].)

It is a pleasure to thank J. Patera for initiating our interest in this problem and for much encouragement in bringing this paper to fruition. We also wish to thank the Centre de Recherche de Mathématiques Appliquées at the Université de Montréal for its hospitality while this work was being completed.

1. Preliminary concepts and conventions. Throughout this work k denotes an algebraically closed field of characteristic zero.

Z denotes either the integer numbers or the ring generated by the multiplicative unit of k. Thus **Q**, the field of quotients of **Z**, is used to denote either the rational numbers or the prime field of k.

If V is a k-space and A is a subset of V the symbol $\{A\}_k$ will be used to denote the k-span of A in V.

a) *On Lie algebras*. By a Lie algebra we mean a finite dimensional Lie algebra over k.

Recall that a Lie algebra is said to be *reductive* if it can be written as the direct product of a semisimple and a commutative Lie algebra. Henceforth

$$\mathfrak{g} = \mathfrak{s} \times \mathfrak{a} \quad \text{and} \quad \mathfrak{g}' = \mathfrak{s}' \times \mathfrak{a}'$$

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will denote two such algebras.

Choose Cartan subalgebras $\mathfrak{h}_{\mathfrak{s}}$, $\mathfrak{h}'_{\mathfrak{s}}$ for $\mathfrak{s}$ and $\mathfrak{s}'$ respectively thereby fixing Cartan subalgebras

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{s}} \times \mathfrak{a} \quad \text{and} \quad \mathfrak{h}' \times \mathfrak{h}'_{\mathfrak{s}} \times \mathfrak{a}'$$

for $\mathfrak{g}$ and $\mathfrak{g}'$. We assume the following standard notation.

Lie algebra	$\mathfrak{s}$	$\mathfrak{s}'$
Root system	Δ	Δ'
Base for the system	$\{\alpha_1, \ldots, \alpha_l\}$	$\{\alpha'_1, \ldots, \alpha'_{l'}\}$
Weyl group	W	W'
Weight lattice	P	P'
Fundamental weights	$\{\omega_1, \ldots, \omega_l\}$	$\{\omega'_1, \ldots, \omega'_{l'}\}$
Coroot system	Σ	Σ'
Coroot lattice	$\mathfrak{h}_{\mathbf{Z}}$	$\mathfrak{h}'_{\mathbf{Z}}$

In dealing with reductive Lie algebras we will find the following natural extension of the concept of root system useful.

Definition. (Borel-Tits) Let V be a k-vector space, M a subvector space of V. A *root system* for the pair (V, M) is a subset Δ of V such that

i) The vector space U spanned by Δ is supplementary to M

ii) Δ is a root system (in the usual sense) in U.

Let $\alpha_1, \ldots, \alpha_l$ be the fundamental roots of Δ (viewed as a root system in U). The Weyl group W is the group generated by the reflections $r_{\alpha_1}, \ldots, r_{\alpha_l}$ where the r_α's are now viewed as automorphisms of V acting trivially on M.

Since $V = U \oplus M$ we may write $V^* = U^* \oplus M^*$ where

$$\langle U, M^* \rangle = \{0\} = \langle M, U^* \rangle.$$

Thus, the dual root system Σ is a root system for the pair (V^*, Z^*).

One defines the group of weights to be

$$P(\Delta\ Z) = \{\lambda \in V^* | \langle \alpha, \lambda \rangle \in M \text{ for all } \alpha \in \Delta\}.$$

Notice that if $x \in M^*$ then $\langle \Delta, x \rangle = \{0\}$ and therefore $x \in P(\Delta, M)$.

Remark. Let $\mathfrak{g} = \mathfrak{s} \times \mathfrak{a}$ as above. Then

$$\mathfrak{h} = \{\mathfrak{h}_Z\}_k \oplus \mathfrak{a} \quad \text{and} \quad \mathfrak{h}^* = \{P\}_k \oplus \mathfrak{a}^*.$$

After having identified $\mathfrak{h}_{\mathfrak{s}}$ and $\mathfrak{h}^*_{\mathfrak{s}}$ via the Killing form $(.\,,.)$ on $\mathfrak{s}$, we have a non-degenerate bilinear form

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$$\langle .\,,.\rangle:\mathfrak{h}^* \times \mathfrak{h} \to k$$

given by

$$\langle \mu + \varphi, h + a\rangle = (\mu, h) + \varphi(a)$$

for all $\mu \in P$, $\varphi \in \mathfrak{a}^*$, $h \in \mathfrak{h}$ and $a \in \mathfrak{a}$.

Σ is a root system for the pair $(\mathfrak{h}, \mathfrak{a})$ and the group of weights $P(\Sigma, \mathfrak{a}) \subset \mathfrak{h}^*$ is easily seen to be $P \times \mathfrak{a}^*$.

b) *On* λ-*structures and representation rings*. (For references and details about this material the reader is referred to the Appendix.)

Let A be a commutative ring with identity and denote by $A[[t]]$ the ring of formal power series in one variable t with coefficients in A. We define $1 + A[[t]]^+ \subset A[[t]]$ by

$$1 + A[[t]]^+ := \left\{1 + \sum_{i \in \mathbf{Z}_{>0}} a_i t^i | a_i \in A\right\}.$$

The multiplication on $A[[t]]$ gives $1 + A[[t]]^+$ the structure of an albelian group. Let

$$\lambda_t: A \to 1 + A[[t]]^+$$

be a mapping and write its action by

$$\lambda_t: a \to \lambda^0(a) + \lambda^1(a)t + \lambda^2(a)t^2 + \ldots, \quad \text{for all } a \in A.$$

The pair (A, λ_t) is said to be a *pre* λ-*ring* if

$\lambda^1(a) = a$ for all $a \in A$ and

λ_t is a group homomorphism from the additive group of A into the (multiplicative) group of $1 + A[[t]]^+$.

If (A, λ_t) is a pre λ-ring we then have for all $a, b \in A$:

I. $\lambda^0(a) = 1$

II. $\lambda^1(a) = a$

III. $\lambda^n(a + b) = \sum_{i=0}^{n} \lambda^i(a)\lambda^{n-i}(b)$ for all $n \in \mathbf{Z}_{\geqq 0}$.

Conversely, given a family of mappings $\{\lambda^n\}_{n \in \mathbf{Z}_{\geqq 0}}: A \to A$ satisfying these three properties then λ_t as above exists and (A, λ_t) is a pre λ-ring. We say that $a \in A$ has λ-*degree* n, for some $n \in \mathbf{Z}_{\geqq 0}$, if $\lambda^n(a) \neq 0$ and $m > n \Rightarrow \lambda^m(a) = 0$. If this is the case we write

$$\deg_{\lambda_t}(a) = n.$$

A λ-ring is a pre λ-ring where certain relations for elements of the form $\lambda^m(\lambda^n(ab))$ are imposed. For the most part it is not necessary to know the explicit form of these relations, and for smoothness of exposition we have deferred their discussion to the Appendix. In the sequel we use the term λ-*structure* to refer either to pre λ-rings or λ-rings.

Let A be a λ-structure. An ideal J of A is said to be a λ-*ideal* if $\lambda_t(J) \subset 1 + J[[t]]^+$. The quotient ring A/J has a natural λ-structure by defining

$$\lambda^n(a + J) = \lambda^n(a) + J$$

for all $n \in \mathbf{Z}_{\geqq 0}$ and $a \in A$.

If (A, λ_t) and (R, Λ_t) are two λ-structures then a ring morphism $f:A \to R$ is said to be a λ-*morphism* if

$$f_t\lambda_t(a) = \Lambda_t f(a) \quad \text{for all } a \in A$$

where f_t denotes the natural extension of f to $1 + A[[t]]^+$.

If $f:A \to R$ is a λ-morphism then $J = \ker f$ is a λ-ideal of A and conversely, if J is a λ-ideal of A then $f:A \to A/J$ is a surjective λ-morphism with kernel J.

Let $\mathfrak{g}$ be a Lie algebra. In the (additive) free abelian group on the set of isomorphism classes $[V]$ of simple $\mathfrak{g}$-modules V we can define a multiplication using the tensor product of $\mathfrak{g}$-modules to obtain a commutative ring with identity $R(\mathfrak{g})$ called the *representation ring* of $\mathfrak{g}$ (see Appendix 2). The mapping

$$\Lambda_t:R(\mathfrak{g}) \to 1 + R(\mathfrak{g})[[t]]^+$$

defined (in the free basis) by

$$\Lambda_t:[V] \mapsto 1 + [V]t + \Lambda^2[V]t^2 + \dots$$

where $\Lambda^n[V]$ stands for the isomorphism class of the n^{th}-exterior power of the $\mathfrak{g}$-module V, gives $R(\mathfrak{g})$ a pre λ-ring structure.

If $\mathfrak{g} = \mathfrak{s} \times \mathfrak{a}$ is reductive we denote by $Z[P]$ and $Z[\mathfrak{a}^*]$ the group algebras on P and $\mathfrak{a}^*$ over $\mathbf{Z}$ respectively. Thus if $x \in Z[P]$ and $y \in Z[\mathfrak{a}^*]$ we have

$$x = \sum_{\mu \in P} n_\mu e(\mu), \quad n_\mu \in Z$$

and

$$y = \sum_{\boldsymbol{\varphi} \in \mathfrak{a}^*} n_{\boldsymbol{\varphi}} e(\boldsymbol{\varphi}), \quad n_{\boldsymbol{\varphi}} \in Z$$

where both sums have finite support. P and $\mathfrak{a}^*$ are treated multiplicatively in $Z[P]$ and $Z[\mathfrak{a}^*]$. There their elements are denoted in the form $e(\mu)$, $e(\varphi)$ etc. to avoid confusion.

Let $Z[P]^W$ be the subring of $Z[P]$ consisting entirely of elements that are invariant under the action of the Weyl group W. It is well known [**4**, Chapter 8] that

$$R(\mathfrak{s}) \simeq Z[P]^W.$$

Clearly $R(\mathfrak{a}) \simeq Z[\mathfrak{a}^*]$ and moreover

$$R(\mathfrak{s} \times \mathfrak{a}) \simeq R(\mathfrak{s}) \otimes_Z R(\mathfrak{a}) \simeq Z[P]^W \otimes_Z Z[\mathfrak{a}^*].$$

The mapping

$$\lambda_{\mathfrak{s}}: Z[P] \to 1 + Z[P][[t]]^+$$

defined by

$$\lambda_{\mathfrak{s}}: \sum_{\mu \in P} n_\mu e(\mu) \mapsto \prod (1 + e(\mu)t)^{n_\mu}$$

gives $Z[P]$ a pre λ-ring structure that can be carried to $Z[P]^W$ since

$$\lambda_{\mathfrak{s}}(Z[P]^W) \subset 1 + Z[P]^W[[t]]^+.$$

Similarly for $Z[\mathfrak{a}^*]$.

$\mathbf{Z}[a^*]$ and $\mathbf{Z}[P]$ (hence $\mathbf{Z}[P]^W$) are in fact λ-rings since the elements $e(\varphi)$ and $e(\mu)$ are of λ-degree 1 (see Appendix 1, Proposition 1). The tensor products $\mathbf{Z}[P]^W \otimes \mathbf{Z}[\mathfrak{a}^*]$ are thus also λ-rings with $\lambda_t = (\lambda_{\mathfrak{s}} \otimes \lambda_{\mathfrak{a}})$ and the elements $e(\mu) \otimes e(\varphi)$ of λ-degree 1.

Not surprisingly the λ-mapping Λ_t of $R(\mathfrak{s} \times \mathfrak{a})$ defined upon exterior powers of $\mathfrak{s} \times \mathfrak{a}$-modules is none other than that induced from $\lambda_t = (\lambda_{\mathfrak{s}} \otimes \lambda_{\mathfrak{a}})$ via the isomorphism

$$R(\mathfrak{s} \times \mathfrak{a}) \simeq Z[P]^W \otimes_Z Z[\mathfrak{a}^*].$$

Throughout the rest of this paper $\otimes$ means $\otimes_Z$.

2. Statement of the main theorem on λ-ring homomorphisms. (Notation as in Section 1.)

THEOREM 2.1. (A) *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be reductive Lie algebras, $R(\mathfrak{g})$, $R(\mathfrak{g}')$ their respective representation rings. Let*

$$f: R(\mathfrak{g}) \to R(\mathfrak{g}')$$

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be a λ*-ring morphism. Then*

(1) *There exists a group morphism*

$$f_\circ : P \times \mathfrak{a}^* \to P' \times \mathfrak{a}'^*$$

which induces f *via the natural extension of* $f_\circ$ *to a ring morphism*

$$Z[P] \otimes Z[\mathfrak{a}^*] \to Z[P'] \otimes Z[\mathfrak{a}'^*]$$

and the canonical identifications of $R(\mathfrak{g})$ *and* $R(\mathfrak{g}')$ *with* $Z[P]^W \otimes Z[\mathfrak{a}^*]$ *and* $Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]$ *respectively. Furthermore, let* $\mathfrak{j} = \ker f_\circ$ *and let*

$$W_D = \{w \in W | w\mathfrak{j} \subset \mathfrak{j}\}$$

$$W_I = \{w \in W | wx \equiv x \bmod \mathfrak{j}: \forall x \in P \times \mathfrak{a}^*\}.$$

Then $W_I \lhd W_D$ *and, if we denote by* $^- : W_D \to W_D/W_I$ *the canonical homomorphism, then*

(2) *There exists a group morphism*

$$\psi : W' \to W_D/W_I$$

such that whenever $w \in W_D$ *and* $\psi(w') = \bar{w}$ *for some* $w' \in W'$ *the following diagram commutes*

$$\begin{tikzcd} P \times \mathfrak{a}^* \arrow[r, "f_\circ"] \arrow[d, "w"'] & P' \times \mathfrak{a}'^* \arrow[d, "w'"] \\ P \times \mathfrak{a}^* \arrow[r, "f_\circ"'] & P' \times \mathfrak{a}'^* \end{tikzcd} \qquad \text{(D)}$$

The pair $(f_\circ, \psi)$ *is determined up to conjugation by* W.

(B) *Conversely suppose that* $f_\circ : P \times \mathfrak{a}^* \to P' \times \mathfrak{a}'^*$, $\mathfrak{j}$, W_D, W_I *and*

$$\psi : W' \to W_D/W_I$$

as above exist and satisfy (D1). *Then there exists a* λ*-ring morphism*

$$f : R(\mathfrak{g}) \to R(\mathfrak{g}')$$

which intrinsically defines $f_\circ$, $\mathfrak{j}$, W_D, W_I *and* ψ *up to conjugation by* W.

Note. The groups W_D and W_I are called the *decomposition* and the *inertia* groups of the mapping f. The meaning of "up to conjugation by W" will be made precise later.

For $(\mu, \varphi) \in P \times \mathfrak{a}^*$ we define $e(\mu, \varphi) := e(\mu) \otimes e(\varphi)$.

Notice that for all $w \in W$

$$we(\mu, \varphi) = we(\mu) \otimes e(\varphi) = e(w\mu) \otimes e(\varphi) = e(w\mu, \varphi).$$

Proof of (B). Let $f_\circ : P \times \mathfrak{a}^* \to P' \times \mathfrak{a}'^*$, $\psi : W' \to W_D/W_I$ be as stated. Let

$$f: Z[P] \otimes Z[\mathfrak{a}^*] \to Z[P'] \otimes Z[\mathfrak{a}'^*]$$

be the ring morphism obtained by extending $f_\circ$ in the natural way. Both $Z[P] \otimes Z[\mathfrak{a}^*]$ and $Z[P'] \otimes Z[\mathfrak{a}'^*]$ are constructable (they both have free bases consisting entirely of elements of λ-degree one). Since f maps elements of $\deg_\lambda = 1$ into elements of $\deg_\lambda = 1$, f is a λ-ring morphism [Appendix 1, Proposition 2].

At this point, it will suffice to show that

$$f: Z[P]^W \otimes Z[\mathfrak{a}^*] \to Z[P']^{W'} \otimes Z[\mathfrak{a}'^*].$$

Let

$$x = \sum n_{\mu,\varphi} e(\mu) \otimes e(\varphi) \in Z[P]^W \otimes Z[\mathfrak{a}^*].$$

Let $w' \in W'$ and choose $w \in W_D$ such that $\overline{w} = \psi(w')$. Then

$$w'f(x) = \sum w' n_{\mu,\varphi} e(f_\circ(\mu, \varphi)) = \sum n_{\mu,\varphi} f_\circ(we(\mu, \varphi)) = fw(x)$$
$$= f(x).$$

3. The construction of $f_\circ$. In this section we show how a λ-mapping

$$f: Z[P]^W \otimes Z[\mathfrak{a}^*] \to Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]$$

determines a group morphism

$$f_\circ : P \times \mathfrak{a}^* \to P' \times \mathfrak{a}'^*.$$

Once this is proved, we prove part (1) of Theorem 2.1 (A).

LEMMA. *Let A be a ring and Φ an* (*additive*) *commutative torsion free group. Let $A[\Phi]$ denote the group algebra of Φ over A. Then if A is entire* (*resp. integrally closed*) *so is $A[\Phi]$.*

This is proved in [**2**], Chapter 5, Section 1, Example 24.

PROPOSITION 3.1. (1) $Z[P] \otimes Z[\mathfrak{a}^*]$ *is integrally closed.*

(2) $Z[P]^W \otimes [\mathfrak{a}^*]$ *is entire.*

If $l = \operatorname{rank}(\mathfrak{s})$ *and* $\{Z[\mathfrak{a}^*]\}_{\text{field}}$, $\hat{L}$, *and* $\hat{K}$ *denote the fields of quotients of* $Z[\mathfrak{a}^*]$, $Z[P] \otimes Z[\mathfrak{a}^*]$, *and* $Z[P]^W \otimes Z[\mathfrak{a}^*]$ *respectively then*

(3) $Z[P]^W \otimes Z[\mathfrak{a}^*] \simeq Z[\mathfrak{a}^*][\chi_1, \ldots, \chi_l]$ *and*

$$\hat{K} \simeq \{Z[\mathfrak{a}^*]\}_{\text{field}}(\chi_1, \dots, \chi_l)$$

(4) $Z[P] \otimes Z[\mathfrak{a}^*] \simeq Z[\mathfrak{a}^*](X_1, \dots, X_l, X_1^{-1}, \dots, X_l^{-1}]$ *and*

$$\hat{L} \simeq \{Z[\mathfrak{a}^*]\}_{\text{field}}(X_1, \dots, X_l)$$

where $\chi_1, \dots, \chi_l$; $X_1, \dots, X_l$ *are indeterminates over* $Z[\mathfrak{a}^*]$.

Proof. Let $\Phi = (P \times \mathfrak{a}^*, +)$. Then $Z[P] \otimes Z[\mathfrak{a}^*] \simeq Z[\Phi]$ via

$$\sum_{\mu \in P} n_\mu e(\mu) \otimes \sum_{\phi \in \mathfrak{a}^*} n_\phi e(\phi) \mapsto \sum_{(\mu,\phi) \in P \times \mathfrak{a}^*} n_\mu n_\phi e(\,(\mu, \phi)\,).$$

by which we may identify them (thus $e(\mu, \phi) = e(\,(\mu, \phi)\,)$). Thus (1) follows by our last lemma while (2) follows from (1). Finally (3) and (4) follow from the well known isomorphisms ([**4**], Chapter 8)

$$Z[x_1, \dots, x_l] \simeq Z[P]^W$$

via

$$\chi_i \mapsto \text{ch } M(\omega_i)$$

where $M(\omega_i)$ is the irreducible $\mathfrak{s}$-module of highest weight ω_i and ch is the character map (see notation in Section 1), and

$$Z[X_1, \dots, X_l, X_1^{-1}, \dots, X_l^{-1}] \simeq Z[P]$$

via

$$X_1^{n_1} \dots X_l^{n_l} \to e(n_1\omega_1 + \dots + n_l\omega_l)$$

where $n_1, \dots, n_l \in Z$

PROPOSITION 3.2. *$Z[P] \otimes Z[\mathfrak{a}^*]$ is the integral closure of $Z[P]^W \otimes Z[\mathfrak{a}^*]$ in $\hat{L}$ (= field of quotients of $Z[P] \otimes Z[\mathfrak{a}^*]$).*

Proof. Each $q \in Z[P] \otimes Z[\mathfrak{a}^*]$ is a root of

$$\prod_{w \in W} (X - w(q)) \in (Z[P]^W \otimes Z[\mathfrak{a}^*])[X]$$

which shows that q is integral over $Z[P]^W \otimes Z[\mathfrak{a}^*]$. Since $Z[P] \otimes Z[\mathfrak{a}^*]$ is integrally closed the proposition follows.

Let $\mathfrak{p}_0$ be the kernel of $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$. Then $\mathfrak{p}_0$ is a λ-ideal and a prime ideal of $Z[P]^W \otimes Z[\mathfrak{a}^*]$. Let

$$\pi:Z[P]^W \otimes Z[\mathfrak{a}^*] \to Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0$$

be the quotient mapping. We have the following commutative diagram

$$\begin{array}{ccc} Z[P]^W \otimes Z[\mathfrak{a}^*] & \xrightarrow{\pi} & Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0 \\ \downarrow{\scriptstyle \lambda_t} & & \downarrow{\scriptstyle \bar{\lambda}_t} \\ 1 + (Z[P]^W \otimes Z[\mathfrak{a}^*])[[t]]^+ & \xrightarrow[\pi_t]{} & 1 + (Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0)[[t]]^+ \end{array}$$

where $\bar{\lambda}_t$ is the quotient λ-mapping.

Let $\mathfrak{p}$ be any prime ideal of $Z[P] \otimes Z[\mathfrak{a}^*]$ lying over $\mathfrak{p}_0$ [**8**, pg. 9]. We introduce the following fields of quotients.

$$\begin{array}{ll} K & \text{for } Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0 \\ K' & \text{for } Z[P']^{W'} \otimes Z[\mathfrak{a}'^*] \\ L & \text{for } Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p} \\ L' & \text{for } Z[P'] \otimes Z[\mathfrak{a}'^*] \\ \hat{K} & \text{for } Z[P]^W \otimes Z[\mathfrak{a}^*] \\ \hat{L} & \text{for } Z[P] \otimes Z[\mathfrak{a}^*] \end{array} \left.\vphantom{\begin{array}{l} a \\ b \end{array}}\right\} \text{ as before.}$$

We may consider L as an extension of K in a natural way, and likewise L' over K' and $\hat{L}$ over $\hat{K}$.

Let

$$\bar{f}: Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0 \rightarrow Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]$$

be the injective reduction of f, i.e., $f = \bar{f} \circ \pi$. Thus $\bar{f}$ is a λ-monomorphism and $\bar{f}$ extends to an embedding $\bar{f}: K \rightarrow K'$

$$\begin{array}{ccc} L & & L' \\ | & & | \\ K & \xrightarrow{\bar{f}} & K' \\ | & & | \\ Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0 & \xrightarrow{\bar{f}} & Z[P']^{W'} \otimes Z[\mathfrak{a}^*] \end{array}$$

Let $^{-}: Z[P] \otimes Z[\mathfrak{a}^*] \rightarrow Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p}$ be the quotient map (thus $^{-}$ is an extension of π).

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PROPOSITION 3.3. (a) *L/K is a finite Galois extension.*

(b) *$\hat{L}/\hat{K}$ and L'/K' are finite Galois extensions with Galois groups W and W' respectively.*

Proof. It is clear that

$$L = K(\overline{e(\omega_1) \otimes e(0)}, \dots, \overline{e(\omega_l) \otimes e(0)}\,).$$

Let

$$\xi_i = \sum_{\mu \in W\omega_i} e(\mu) \otimes e(0) \in Z[P]^W \otimes Z[\mathfrak{a}^*]$$

$$\begin{aligned} \bar{\lambda}_t \bar{\xi}_i &= \bar{\lambda}_t \pi \xi_i = \pi_t\, \lambda_t\, \xi_t \\ &= \pi_t \prod_{\mu \in W\omega_i} (1 + e(\mu) \otimes e(0)t) \\ &= \prod_{\mu \in W\omega_i} (1 + \overline{e(\mu) \otimes e(0)}t) \end{aligned}$$

which lies in $(Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_0)[t] \subset K[t]$ and has roots $-e(\mu)^{-1} \otimes e(0)$ (in particular $-e(\omega_i)^{-1} \otimes e(0)$) in L. Thus L is the splitting field of $\prod_{i=1}^{l} \bar{\lambda}_t \bar{\xi}_i$. On the other hand L is of characteristic 0; hence L/K is Galois. Similarly for $\hat{L}/\hat{K}$ and L'/K'.

It is obvious that W acts on $\hat{L}$ as automorphisms of $\hat{L}$ over $\hat{K}$. The resulting homomorphism $W \to \mathrm{Gal}\,(\hat{L}/\hat{K})$ is faithful since any $w \in W$ is entirely determined by its action on the set

$$e(\omega_1) \otimes e(0), \dots, e(\omega_l) \otimes e(0).$$

Consider the fixed field $\hat{L}^W$ of $\hat{L}$ under W. If $a, b \in Z[P] \otimes Z[\mathfrak{a}^*]$ and $a/b \in \hat{L}^W$ then from

$$a/b = \prod_{w \in W} w(a)/b \prod_{w \neq 1} w(a)$$

we see that the left side and the numerator, hence also the denominator, are W-invariant and thus belong to $Z[P]^W \otimes Z[\mathfrak{a}^*]$. Therefore $a/b \in \hat{K}$. Similarly for L'/K'.

Let $\tilde{K}'$ be an algebraic closure of K' containing L' and let $\tilde{f}$ be an extension of $\bar{f}$ to an embedding of L into $\tilde{K}'$. Let $\tilde{f}_t$, $\bar{f}_t$ be the corresponding mappings for the formal power series

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$$\begin{tikzcd}
 & \widetilde{K}' \\
L \arrow[ur, "\widetilde{f}"] & L' \arrow[u, no head] \\
K \arrow[u, no head] \arrow[r, "\bar{f}"] & K' \arrow[u, no head]
\end{tikzcd}$$

We claim that L falls into L' under $\widetilde{f}$. More precisely:

LEMMA. *For all* $\mu \in P$ *and* $\varphi \in \mathfrak{a}^*$ *there exist* $\mu' \in P'$ *and* $\varphi' \in \mathfrak{a}'^*$ *such that*

$$\widetilde{f}\ \overline{e(\mu) \otimes e(\varphi)} = e(\mu') \otimes e(\varphi').$$

Proof. Recall that $\lambda_t = (\lambda_{\mathfrak{g}} \otimes \lambda_{\mathfrak{a}})_t$, $\lambda'_t = (\lambda_{\mathfrak{g}'} \otimes \lambda_{\mathfrak{a}'})_t$, $\widetilde{f}|_K = \bar{f}$ and $\lambda'_t \bar{f} = \bar{f}_t \lambda_t$.

Let $\mu \in P$, $\varphi \in \mathfrak{a}^*$. Define

$$\xi = \sum_{\alpha \in W\mu} e(\alpha) \otimes e(\varphi) \in Z[P]^W \otimes Z[\mathfrak{a}^*].$$

First

$$(1) \quad \widetilde{f}_t \lambda_t \xi = \bar{f}_t \prod_{\alpha \in W\mu} (1 + \overline{e(\alpha) \otimes e(\varphi)}\, t) = \prod_{\alpha \in W\mu} (1 + \widetilde{f}(\overline{e(\alpha) \otimes e(\varphi)}\,)t).$$

Next since $\bar{f}\xi \in Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]$ we may write

$$\bar{f}\xi = \sum n_{\mu',\varphi'}\, e(\mu') \otimes e(\varphi').$$

Thus

$$(2) \quad \lambda'_t \bar{f} \xi = \prod_{n_{\mu,\varphi}} (1 + e(\mu') \otimes e(\varphi')t)^{n_{\mu',\varphi'}}.$$

By equating the roots of the polynomials (1) and (2) (which are equal) we obtain

$$\widetilde{f}\ \overline{e(\alpha) \otimes e(\varphi)} = e(\mu') \otimes e(\varphi')$$

for some $\mu' \in P'$ and $\varphi' \in \mathfrak{a}'^*$ (depending on α and φ). In particular

$$\widetilde{f}\ \overline{e(\mu) \otimes e(\varphi)} = e(\mu') \otimes e(\varphi').$$

The mapping

$$f_{\circ}:(\mu, \varphi) \to \overline{e(\mu) \otimes e(\varphi)} \to \tilde{f}\,\overline{e(\mu) \otimes e(\varphi)} \to e(\mu') \otimes e(\varphi') \to (\mu', \varphi')$$

is a group morphism from $P \times \mathfrak{a}^*$ into $P' \times \mathfrak{a}'^*$.

Recall that

$$e(\mu', \varphi') = e(\mu') \otimes e(\varphi') \quad \text{for all } (\mu', \varphi') \in P' \times \mathfrak{a}'^*.$$

The mapping

$$Z[P] \otimes Z[\mathfrak{a}^*] \to Z[P'] \otimes Z[\mathfrak{a}'^*]$$

induced by $f_{\circ}$ is $\tilde{f} \circ {}^{-}$ since

$$\begin{aligned} &\Sigma\, n_{\mu,\varphi} e(\mu) \otimes e(\varphi) \to \Sigma\, n_{\mu,\varphi} e(f_{\circ}(\mu, \varphi)) \\ &= \Sigma\, n_{\mu,\varphi}\, \tilde{f}\,\overline{e(\mu) \otimes e(\varphi)} \\ &= \tilde{f}\, \Sigma\, n_{\mu,\varphi}\, \overline{e(\mu) \otimes e(\varphi)}. \end{aligned}$$

Its restriction to $Z[P]^W \otimes Z[\mathfrak{a}^*]$ is $\bar{f} \circ \pi = f$. This establishes part (1) of Theorem 2.1 (A).

4. The decomposition and inertia groups. In this section we will finish the proof of Theorem 2.1 (A). Let us start by recalling the following result from the theory of fields [**8**, p. 15].

THEOREM 4.1. *Let A be an integrally closed commutative integral domain and K its field of quotients. Let L/K be a finite Galois extension and let B be the integral closure of A in L. Let $\mathfrak{p}$ be a maximal ideal of A and $\mathcal{P}$ a prime ideal of B lying over $\mathfrak{p}$. The set of such $\mathcal{P}$ form one orbit under the Galois group $G(L/K)$ of L over K. Let*

$$G_D = \{\sigma \in \mathrm{Gal}(L/K) | \sigma\mathcal{P} = \mathcal{P}\}$$

$$G_I = \{\sigma \in \mathrm{Gal}(L/K) | \sigma(x) \equiv x \bmod \mathcal{P} \textit{ for all } x \in B\}.$$

Then $\mathcal{P}$ is a maximal ideal of B and under the natural inclusion

$$A/\mathfrak{p} \hookrightarrow B/\mathcal{P}$$

$B/\mathcal{P}$ is a normal extension of $A/\mathfrak{p}$. Furthermore every $\sigma \in G_D$ stabilizes B and thereby induces an automorphism σ of $B/\mathcal{P}$ which fixes $A/\mathfrak{p}$. The resulting mapping

$$^{-}:G_D \to \mathrm{Gal}(B/\mathcal{P} \;/\; A/\mathfrak{p})$$

is surjective with kernel G_I. Thus

$$G_D/G_I \simeq \mathrm{Gal}(B/\mathcal{P} \;/\; A/\mathfrak{p}).$$

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Let us return to the situation at which we arrived at the end of Section 3.

$$\begin{tikzcd}
L \arrow[r, "\tilde{f}"] \arrow[d, dash] & L' \arrow[d, dash] \\
K \arrow[r, "\bar{f}"] \arrow[d, dash] & K' \arrow[d, dash] \\
Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_o \arrow[r, "\bar{f}"] & Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]
\end{tikzcd}$$

Recall that

$$\mathfrak{p}_o = \ker f: Z[P]^W \otimes Z[\mathfrak{a}^*] \rightarrow Z[P']^{W'} \otimes Z[\mathfrak{a}'^*]$$

and $\mathfrak{p}$ is a prime ideal of $Z[P] \otimes Z[\mathfrak{a}^*]$ lying over $\mathfrak{p}_o$. We localize at $\mathfrak{p}_o$. Let

$$S_o := (Z[P]^W \otimes Z[\mathfrak{a}^*]) - \mathfrak{p}_o$$

$$A := S_o^{-1}(Z[P]^W \otimes Z[\mathfrak{a}^*]) \subset \hat{K}$$

$$B := S_o^{-1}(Z[P] \otimes Z[\mathfrak{a}^*]) \subset \hat{L}.$$

The following is a consequence of Proposition 3.2.

PROPOSITION 4.2. (1) *A is integrally closed in $\hat{K}$ and B is the integral closure of A in $\hat{L}$;*

(2) *$S_o^{-1}\mathfrak{p}_o$ is maximal in A and $S_o^{-1}\mathfrak{p}$ is a prime ideal of B lying over A.*

Let $\tilde{\mathfrak{p}}_o := S_o^{-1}\mathfrak{p}_o$ and $\tilde{\mathfrak{p}} := S_o^{-1}\mathfrak{p}$ and apply the last theorem to

$$\hat{L} \supset B \supset \tilde{\mathfrak{p}}$$

$$\hat{K} \supset A \supset \tilde{\mathfrak{p}}_o$$

to see that $B/\tilde{\mathfrak{p}}$ is Galois over $A/\tilde{\mathfrak{p}}_o$ with

$$\mathrm{Gal}(B/\tilde{\mathfrak{p}} \,/\, A/\tilde{\mathfrak{p}}_o) \simeq W_D/W_I \quad \text{where}$$

$$W_D = \{w \in W | w\tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}\} \quad \text{and}$$

$$W_I = \{w \in W | wx \equiv x \bmod \tilde{\mathfrak{p}} \text{ for all } x \in B\}.$$

Now the kernel of the composition

$$Z[P] \otimes Z[\mathfrak{a}^*] \rightarrow B \rightarrow B/\tilde{\mathfrak{p}}$$

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is $Z[P] \otimes Z[\mathfrak{a}^*] \cap \tilde{\mathfrak{p}} = \mathfrak{p}$ so we have an injection of L = field of quotients of $Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p} \hookrightarrow B/\tilde{\mathfrak{p}}$ and this is in fact an isomorphism of fields since

$$B = S_0^{-1} (Z[P] \otimes Z[\mathfrak{a}^*]).$$

A similar thing occurs with K and $A/\tilde{\mathfrak{p}}_0$. We conclude that

$$\mathrm{Gal}(L/K) = W_D/W_I$$

and, as we already know

$$\mathrm{Gal}(L'/K') = W'.$$

L is the splitting field of a certain polynomial over K. Hence $\tilde{f}(L)$ is the splitting field of the corresponding polynomial over $\tilde{f}(K) \subset K'$. Thus $\mathrm{Gal}(L'/K')$ induces automorphisms of L over K. Let

$$\psi: W' \to W_D/W_I$$

be the resulting homomorphism of Galois groups.

LEMMA 4.3. *Let $w' \in W'$ and let $w \in W_D$ be such that $\psi(w') = \bar{w} := wW_I$. Then the diagram* (D1) *of Theorem* 2.1 (A) (2) *is commutative.*

Proof. Let $(\mu, \varphi) \in P \times \mathfrak{a}^*$. We have to show that

$$w'f_0(\mu, \varphi) = f_0w(\mu, \varphi).$$

Now

$$e(w'f_0(\mu, \varphi)) = w'\tilde{f}\,\overline{e(\mu) \otimes e(\varphi)} = \tilde{f}\psi(w')\overline{e(\mu) \otimes e(\varphi)}$$
$$= \tilde{f}\,\bar{w}\,\overline{e(\mu) \otimes e(\varphi)} = \tilde{f}\,\overline{e(w\mu) \otimes e(\varphi)} = e(f_0w(\mu, \varphi)).$$

Up to now the definition of W_D and W_I depend upon $\tilde{\mathfrak{p}}$. We now relate them to $\mathfrak{j} = \ker f_0 \subset P \times \mathfrak{a}^*$ as prescribed by Theorem 2.1 (A)(2).

PROPOSITION 4.4. $1 - e(\mathfrak{j})$ *generates* $\mathfrak{p}$ *as an ideal of* $Z[P] \otimes Z[\mathfrak{a}^*]$.

Proof. Let $\langle 1 - e(\mathfrak{j}) \rangle$ be the ideal of $Z[P] \otimes Z[\mathfrak{a}^*]$ generated by $\{1 - e(\mathfrak{j})\}$.

Recall that $\mathfrak{p}$ is the kernel of the composite map

$$Z[P] \otimes Z[\mathfrak{a}^*] \twoheadrightarrow Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p} \xrightarrow{\tilde{f}} Z[P'] \otimes Z[\mathfrak{a}'^*]$$

and

$$\tilde{f}(\overline{1 - e(\xi)}) = 1 - e(f_0(\xi)) \quad \text{for all } \xi \in P \times \mathfrak{a}^*.$$

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Thus $\langle 1 - e(\mathfrak{j}) \rangle \subset \mathfrak{p}$.

If $\mathfrak{p} = \{0\}$ then $\mathfrak{j} = \ker f_{\circ} = \{0\}$ and we are done. Suppose $\mathfrak{p} \neq \{0\}$. Let

$$0 \neq x = \sum_{I} n_{\xi} e(\xi) \in \mathfrak{p}; \quad n_{\xi} \neq 0 \text{ for all } \xi \in I \subset P \times \mathfrak{a}^*.$$

Choose $\xi_{\circ} \in I$ and let $\xi'_{\circ} \in P' \times \mathfrak{a}'^*$ be such that

$$f_{\circ}(\xi_{\circ}) = \xi'_{\circ}$$

Let $I_{\circ} \subset I$ be defined by

$$I_{\circ} := \{\xi \in I | f_{\circ}(\xi) = \xi'_{\circ}\}.$$

Since the coefficient of $e(\xi'_{\circ})$ in $0 = \tilde{f} \sum_{I} n_{\xi} e(\xi)$ is $\sum_{I_{\circ}} n_{\xi}$ we see that $I_{\circ}$ has at least two elements.

Let $y = \sum_{I_{\circ}} n_{\xi} e(\xi)$. Then

$$\begin{aligned} y &= e(\xi_{\circ}) \sum n_{\xi} e(\xi - \xi_{\circ}) \\ &= e(\xi_{\circ}) \sum - n_{\xi}(1 - e(\xi - \xi_{\circ})) \in \langle 1 - e(\mathfrak{j}) \rangle \end{aligned}$$

and

$$x = y + \sum_{I \setminus I_{\circ}} n_{\xi} e(\xi).$$

The result now follows by induction on Card (I).

PROPOSITION 4.5.

$$W_D = \{w \in W | w(\mathfrak{j}) = \mathfrak{j}\}$$

$$W_I = \{w \in W | wx \equiv x \bmod \mathfrak{j} \quad \textit{for all } x \in P \times \mathfrak{a}^*\}.$$

Proof. Let $w \in W_D$. If $(\mu, \varphi) \in P \times \mathfrak{a}^*$ then

$$(\mu, \varphi) \in \mathfrak{j} \Leftrightarrow 1 - e(\mu, \varphi) \in \mathfrak{p}.$$

Thus

$$\begin{aligned} (\mu, \varphi) \in \mathfrak{j} \Rightarrow w(1 - e(\mu, \varphi)) &= 1 - e(w\mu, \varphi) \\ &\in \tilde{\mathfrak{p}} \cap Z[P] \otimes Z[\mathfrak{a}^*] = \mathfrak{p} \\ \Rightarrow (w\mu, \varphi) = w(\mu, \varphi) \in \mathfrak{j}. \end{aligned}$$

Conversely if $w \in W$ and $w(\mathfrak{j}) = \mathfrak{j}$ then $w(1 - e(\mathfrak{j})) = 1 - e(\mathfrak{j})$ and by our last proposition $w\mathfrak{p} = \mathfrak{p}$.

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By definition $wS_\circ = S_\circ$, hence $w\tilde{\mathfrak{p}} = wS_\circ^{-1}\mathfrak{p} = \tilde{\mathfrak{p}}$ and $w \in W_D$. Likewise, if $\xi \in P \times \mathfrak{a}^*$ then

$$w \in W_I \Rightarrow we(\xi) \equiv e(\xi) \bmod \tilde{\mathfrak{p}}$$
$$\Rightarrow 1 - e(w\xi - \xi) \in \tilde{\mathfrak{p}} \cap Z[P] \otimes Z[\mathfrak{a}^*] = \mathfrak{p}$$
$$\Rightarrow w\xi - \xi \in \mathfrak{j}$$

and conversely, if $we(\xi) \equiv e(\xi) (\bmod \mathfrak{j})$ for all $\xi \in P \times \mathfrak{a}^*$ then

$$1 - e(w\xi - \xi) \in 1 - e(\mathfrak{j})$$

and hence $we(\xi) \equiv e(\xi) \bmod \mathfrak{p}$ for all $\xi \in P \times \mathfrak{a}^*$. Therefore

$$wx \equiv x \bmod \tilde{\mathfrak{p}} \quad \text{for all } x \in B$$

and thus $w \in W_I$.

The construction of $f_\circ$ (hence $\mathfrak{j}$, W_D, W_I and ψ) depends only upon the choice of $\mathfrak{p}$ in $Z[P] \otimes Z[\mathfrak{a}^*]$ lying over $\mathfrak{p}_\circ$ and the lifting of $\bar{f}$ to $\tilde{f}$. But

$$\mathfrak{p} = \tilde{\mathfrak{p}} \cap Z[P] \otimes Z[\mathfrak{a}^*]$$

where $\tilde{\mathfrak{p}}$ is chosen in B to lie over $\tilde{\mathfrak{p}}_\circ = S_\circ^{-1}\mathfrak{p}_\circ$ in A. According to the general theorem at the head of this paragraph $\tilde{\mathfrak{p}}$, hence also $\mathfrak{p}$, is unique up to the action of $\mathrm{Gal}(\hat{L}/\hat{K}) = W$.

Suppose that $\mathfrak{q}$ lies over $\mathfrak{p}_\circ$ and $\tilde{g}$ is a lifting of $\bar{f}$ to an embedding.

$$\tilde{g}: Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{q} \to L'.$$

Let $\mathfrak{q} = w_1\mathfrak{p}$, $w_1 \in W$ and let

$$[w_1]: Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p} \to Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{q}$$

be the corresponding isomorphism of rings. Then $\tilde{g} \circ [w_1]$ is an embedding of $Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p}$ into L' extending $\bar{f}$. Since the fields of quotients L and $L_\mathfrak{q}$, say, of $Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{p}$ and $Z[P] \otimes Z[\mathfrak{a}^*]/\mathfrak{q}$ respectively are the splitting fields of the same polynomial over K (= field of quotients of $Z[P]^W \otimes Z[\mathfrak{a}^*]/\mathfrak{p}_\circ$) their images under $\tilde{f}$ and $\tilde{g} \circ [w_1]$ are the same. Thus

$$\tilde{f}^{-1} \circ \tilde{g} \circ [w_1] \in \mathrm{Gal}(L/K)$$

which is $W_D/W_I \subset W$.

The net effect on $f_\circ$ is to replace it by $f_\circ \circ w^{-1}$ for some $w \in W$, $\mathfrak{j}$ by $w\mathfrak{j}$, W_D by wW_Dw^{-1}, W_I by wW_Iw^{-1} and ψ by $i_w \circ \psi$ where

$$i_w: W_D/W_I \to wW_Dw^{-1}/wW_Iw^{-1}$$

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is derived from the inner automorphism of w on W.

The pair $(f_\circ \circ w^{-1}, i_w \circ \psi)$ is said to be the *conjugate of* $(f_\circ, \psi)$ *by* w. Our argument then makes it clear that the construction of $(f_\circ, \psi)$ is unique up to conjugation by W. This concludes the proof of Theorem 2.1.

PROPOSITION 4.6. *Let* $\mathfrak{g}$ *be a Lie algebra and* $R(\mathfrak{g})$ *its representation ring. Then there exists a unique* λ*-ring morphism*

$$\dim : R(\mathfrak{g}) \rightarrow Z$$

such that dim *maps each class* $[V]$ *into* $\dim_k(V)$ *for every* $\mathfrak{g}$*-module* V. *In particular if* $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ *is a morphism of* λ*-rings then the following diagram commutes*:

$$\begin{tikzcd} R(\mathfrak{g}) \arrow[rr, "f"] \arrow[dr, "\dim"'] & & R(\mathfrak{g}') \arrow[dl, "\dim'"] \\ & Z & \end{tikzcd}$$

Proof. (See Appendix for details.) Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{g}$. Then

$$R(\mathfrak{g}) \overset{\lambda}{\cong} R(\mathfrak{g}/\mathfrak{n}) \overset{\lambda}{\cong} Z[P]^W \otimes Z[\mathfrak{a}^*].$$

Theorem 2.1 applied to $\mathfrak{g}/\mathfrak{n}$ and the trivial Lie algebra $\{0\}$ shows the existence of unique λ-mapping

$$\dim : Z[P] \times Z[\mathfrak{a}^*] \rightarrow Z$$

with

$$\dim_\circ : P \times \mathfrak{a}^* \rightarrow P' = \{0\}.$$

(Conjugation does not affect this map.)

If V is an irreducible $\mathfrak{g}$-module, hence an irreducible $\mathfrak{g}/\mathfrak{n}$-module then

$$[V] \text{ corresponds to } \sum n_\mu e(\mu) \otimes e(\varphi),$$

the character of $\mathfrak{g}/\mathfrak{n}$ on V, and

$$\widetilde{\dim} : \left(\sum n_\mu e(\mu) \otimes e(\varphi)\right) \mapsto \sum n_\mu = \dim_k(V)$$

shows that $\dim := \widetilde{\dim}|_{R(\mathfrak{g})} : R(\mathfrak{g}) \rightarrow Z$ has the required property. In view of the uniqueness, the second assertion is obvious.

5. The definition of subjoining. To motivate the material in this section we start by examining the subalgebra relation.

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Let $\mathfrak{g} = \mathfrak{s} \times \mathfrak{a}$, and $\mathfrak{g}' = \mathfrak{s}' \times \mathfrak{a}'$ be reductive Lie algebras with $\mathfrak{g}'$ a subalgebra of $\mathfrak{g}$ embedded by the mapping i. Choose Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ for $\mathfrak{g}$ and $\mathfrak{g}'$ respectively with $\mathfrak{h}' \subset \mathfrak{h}$. Obviously

(1) $i:\mathfrak{h}' \rightarrow \mathfrak{h}$ is injective.

Write $\mathfrak{h} = \mathfrak{h}_{\mathfrak{s}} \times \mathfrak{a}$ where $\mathfrak{h}_{\mathfrak{s}} = \mathfrak{s} \cap \mathfrak{h}$ and similarly for $\mathfrak{h}'$. Notice that

$$\mathfrak{s}' = [\mathfrak{g}', \mathfrak{g}'] \xrightarrow{i} [\mathfrak{g}, \mathfrak{g}] = \mathfrak{s};$$

in particular $\mathfrak{h}'_{\mathfrak{s}} \subset \mathfrak{h}_{\mathfrak{s}}$.

Let $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ be the λ-ring morphism induced by restriction (see Appendix). By Theorem 2.1 this determines a group morphism

$$f_{\circ}:P \times \mathfrak{a}^* \rightarrow P' \times \mathfrak{a}'^*$$

which is none other than the transpose i^* of i restricted to $P \times \mathfrak{a}^*(\subset \mathfrak{h}^*)$. It is easy to see that $f_{\circ}(\mathfrak{a}^*) \subset \mathfrak{a}'^*$ and moreover

(2) $f_{\circ,\mathfrak{a}} := f_{\circ}|_{\{0\} \times \mathfrak{a}^*}$ is k-linear.

Furthermore

(3) $f_{\circ}(P \times \{0\}) \cap (P' \times \{0\})$ is of finite index in $P' \times \{0\}$.

Were this not so then there would exist an element $0 \neq h' \in \mathfrak{h}'_{\mathfrak{s}}$ such that $0 \neq i(h') \in \mathfrak{h}_{\mathfrak{s}}$ would vanish under the action of $P \times \{0\}$.

Conditions (1), (2), and (3) form the basis of our definition of subjoining.

Let us return to the general situation where $\mathfrak{g}$ and $\mathfrak{g}'$ are reductive and f is a λ-ring morphism from $R(\mathfrak{g})$ into $R(\mathfrak{g}')$. We retain the notation of Sections 1, 2, 3, and 4. We know the existence of a group morphism:

$$f_{\circ}:P \times \mathfrak{a}^* \rightarrow P' \times \mathfrak{a}'^*.$$

If $(\mu, \varphi) \in P \times \mathfrak{a}^*$, write

$$f_{\circ}(\mu, \varphi) = f_{\circ,\mathfrak{s}}(\mu) + f_{\circ,\mathfrak{a}}(\varphi)$$

where $f_{\circ,\mathfrak{s}}(\mu) := f_{\circ}(\mu, 0)$ and $f_{\circ,\mathfrak{a}}(\varphi) := f_{\circ}(0, \varphi)$. In general

$$f_{\circ,\mathfrak{s}}(\mu) = (\mu', \varphi') \in P' \times \mathfrak{a}'^*$$

but notice that since $1 \otimes e(\varphi) \in R(\mathfrak{g})$ and f is compatible with $f_{\circ}$ we must have (after identifying $\mathfrak{a}'^*$ inside $P' \times \mathfrak{a}'^*$)

$$f_{\circ,\mathfrak{a}}(\varphi) \in \mathfrak{a}'^* \quad \text{for all } \varphi \in \mathfrak{a}^*$$

by an argument on λ-degrees (see Appendix, 3, Proposition 7).

$\mathfrak{a}^*$ has the structure of a k-linear space but unfortunately it does not follow that $f_{\circ,\mathfrak{a}}$ is k-linear. However, since in the subalgebra relationship this is indeed the case it is natural for us to make the following

Assumption. $f_{\circ,\mathfrak{a}}:\mathfrak{a}^* \to \mathfrak{a}'^*$ is k-linear.

Notice that this is expressible at the level of representation rings since $\mathfrak{a}^*$ is canonically identified with the elements of λ-degree 1 in $R(\mathfrak{g})$ (Appendix 3, Proposition 7). We will say that f is k-linear if $f_{\circ,\mathfrak{a}}$ is k-linear. Recall that

$$\mathfrak{h}^* = \{P\}_k \oplus \mathfrak{a}^* \quad \text{and} \quad \mathfrak{h}'^* = \{P'\}_k \oplus \mathfrak{a}'^*.$$

Thus $f_\circ$ extends uniquely to a k-linear mapping

$$f_k:\mathfrak{h}^* \to \mathfrak{h}'^*$$

by

$$f_k:k\mu + \varphi \mapsto k\mu' + (k\varphi'_\mu + \varphi'_\mathfrak{a})$$

for all $(\mu, \varphi) \in P \times \mathfrak{a}^*$ and $k \in k$ where $f_{\circ,\mathfrak{s}}(\mu) = (\mu', \varphi'_\mu)$ and $f_{\circ,\mathfrak{a}}(\varphi) = \varphi'_\mathfrak{a}$. Notice that (D) has the k-linear extension

$$\begin{array}{ccc} \mathfrak{h}^* & \xrightarrow{f_\mathfrak{k}} & \mathfrak{h}'^* \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w'} \\ \mathfrak{h}^* & \xrightarrow[f_\mathfrak{k}]{} & \mathfrak{h}'^* \end{array} \qquad (\mathrm{D})_\mathfrak{k}$$

whenever $w \in \psi(w')$.

The transpose of f_k, which for simplicity we denote as f^*, is a mapping

$$f^*:\mathfrak{h}' \to \mathfrak{h}.$$

This mapping is extremely important for everything that follows. In the case when subjoining is the subalgebra relation, f^* is simply the inclusion map. Loosely speaking to the extent that f^* differs from the identity, so much does the subjoining differ from a subalgebra inclusion.

Let us recall that for a root system of pairs (Section 1) we have

$$\langle \mathfrak{h}'^*_\mathfrak{s}, \mathfrak{a}' \rangle' = \{0\} = \langle \mathfrak{a}'^*, \mathfrak{h}'_\mathfrak{s} \rangle'$$

where $\mathfrak{h}' = \mathfrak{h}'_\mathfrak{s} \times \mathfrak{a}'$.

As before let $\mathfrak{h}'_Z \subset \mathfrak{h}'_\mathfrak{s}$ be the coroot lattice of $\mathfrak{s}'$. One can see that for x'

$\in \mathfrak{h}'$ to belong to $\mathfrak{h}'_Z$ it is necessary and sufficient that

(1) $\langle P', x' \rangle' \subset Z$ and

(2) $\langle \mathfrak{a}'^*, x' \rangle' = \{0\}$.

It follows that $f^*(\mathfrak{h}'_Z) \subset \mathfrak{h}_Z$. For suppose $x' \in \mathfrak{h}'_Z$ and let $\mu \in P$ with $f_\circ(\mu, 0) = (\mu', \varphi'_\mu)$. Then

(1) $\langle \mu, f^*(x') \rangle = \langle \mu' + \varphi'_\mu, x' \rangle' = \langle \mu', x' \rangle' \in Z$

(2) $\langle \mathfrak{a}^*, f^*(x') \rangle \subset \langle f_\circ(\mathfrak{a}^*), x' \rangle' \subset \langle \mathfrak{a}'^*, x' \rangle' = \{0\}$

whence our claim. With $f^*_Z := f^*|_{\mathfrak{h}'_Z}$ we have

$$f^*_Z : \mathfrak{h}'_Z \rightarrow \mathfrak{h}_Z.$$

The two group morphisms f^*_Z and ψ are the essential tools in the study of subjoinings.

Definition 5.1. A $\mathfrak{k}$-linear λ-ring morphism $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ is said to be a *pre-subjoining* if the induced group morphism $f_\circ : P \times \mathfrak{a}^* \rightarrow P' \times \mathfrak{a}'^*$ satisfies the following condition:

$$\text{PSJ}:[P' \times \{0\} : f_\circ(P \times \{0\}) \cap (P' \times \{0\})] < \infty.$$

Remark. f given, $f_\circ$ is unique up to conjugation by W. For any $w \in W$ it is clear that $f_\circ$ satisfies PSJ if and only if $f_\circ \circ w^{-1}$ satisfies PSJ (see Section 4).

Let $\mathfrak{a}'^*_{im} \subset \mathfrak{a}'^*$ be the k-span of the elements $\varphi' \in \mathfrak{a}'^*$ such that $\varphi' = f_\circ(\xi)$ for some $\xi \in P \times \mathfrak{a}^*$. Let L_f be the smallest field containing $e(\mathfrak{a}'^*_{im})$ and $f(Z[P] \otimes Z[\mathfrak{a}^*])$ and K_f the smallest field containing $e(\mathfrak{a}'^*_{im})$ and $f(R(\mathfrak{g}))$. Recall $L(\mathfrak{g}')$ and $K(\mathfrak{g}')$, the field of quotients of $Z[P'] \otimes Z[\mathfrak{a}'^*]$ and $R(\mathfrak{g}')$ respectively.

PROPOSITION 5.2. *For a pre-subjoining* $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ *the following are equivalent*:

(1) $L(\mathfrak{g}')$ *is a finite extension of* L_f

(2) $\mathfrak{a}'^*_{im} = \mathfrak{a}'^*$

(3) $f_k : \mathfrak{h}^* \rightarrow \mathfrak{h}'^*$ *is surjective*

(4) $f^* : \mathfrak{h}' \rightarrow \mathfrak{h}$ *is injective*

(5) $K(\mathfrak{g}')$ *is a finite extension of* K_f.

We start by proving the following

LEMMA. *Let* $k_\circ$ *be an extension of degree* $\geqq 1 +$ rank P *of* $\mathbf{Q}$ *in* k *and let* $\mathfrak{b}'^*$ *be a* $k_\circ$*-subspace of* $\mathfrak{a}'^*$ *such that*

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$$\mathfrak{a}'^*_{im} \subset \mathfrak{b}'^* \subsetneq \mathfrak{a}'^*.$$

Let F be the subfield of $L(\mathfrak{g}')$ generated by $f(Z[P] \otimes Z[\mathfrak{a}^*])$ and $e(\mathfrak{b}'^*)$. Then $L(\mathfrak{g}') \neq F$.

Proof (of the lemma). F is the field of quotients of the ring generated by $e(\mathfrak{b}'^*)$ and $e(f_{\mathrm{o}}(P \times \mathfrak{a}^*))$. The typical element in this ring is

$$\sum n_i e(\lambda_i, \psi_i + b_i) \quad (*)$$

where $n_i \in Z$, $(\lambda_i, \psi_i) \in f_{\mathrm{o}}(P, 0)$ and $b_i \in \mathfrak{b}'^*$.

Let $\boldsymbol{\varphi}' \in \mathfrak{a}'^* \backslash \mathfrak{b}'^*$ and suppose that $1 \otimes e(\boldsymbol{\varphi}') \in F$. Then for some elements of the form (*) we have

$$\sum n_i e(\lambda_i, \psi_i + b_i + \boldsymbol{\varphi}') = \sum n_i' e(\lambda_i', \psi_i' + b_i')$$

and comparing terms we conclude that

$$(\lambda_i, \psi_i + b_i + \boldsymbol{\varphi}') = (\lambda_k', \psi_k' + b_k') \quad \text{for some } i \text{ and } k$$

and thus that

$$\boldsymbol{\varphi}' \in f_{\mathrm{o}}(P, 0) \bmod \mathfrak{b}'^*.$$

Let $l = \operatorname{rank} P$. Choose $k_1, \ldots, k_{l+1} \in k_0$ linearly independent over $\mathbf{Q}$. By the above argument applied to $k_j\boldsymbol{\varphi}'$, $j = 1, \ldots, l + 1$ we find

$$k_j\boldsymbol{\varphi}' \equiv f_{\mathrm{o}}(\mu_j, 0) \bmod \mathfrak{b}'^* \quad \text{for some } \mu_j \in P.$$

Since $f_{\mathrm{o}}(\mu_1, 0), \ldots, f_{\mathrm{o}}(\mu_{l+1}, 0)$ are linearly dependent over $\mathbf{Q}$ we have

$$\left(\sum q_j k_j\right)\boldsymbol{\varphi}' \in \mathfrak{b}'^* \text{ for some } q_j \in \mathbf{Q}$$

whence $\boldsymbol{\varphi}' \in \mathfrak{b}'^*$ which is not the case. We conclude that $1 \otimes e(\boldsymbol{\varphi}') \notin F$. Hence $L(\mathfrak{g}') \notin F$. This concludes the proof of the lemma.

Proof (of Proposition 5.2). "not (2)" $\Rightarrow$ "not (1)": Choose $\mathfrak{k}_{\mathrm{o}}$ as in the lemma. By assumption $\mathfrak{a}'^*_{im} \subsetneq \mathfrak{a}'^*$. Apply the above lemma to $\mathfrak{b}'^* = \mathfrak{a}'^*_{im}$ to find $\boldsymbol{\varphi}_1' \notin \mathfrak{a}'^*_{im}$ such that $1 \otimes e(\boldsymbol{\varphi}_1') \notin L_f$. Let $\mathfrak{b}'^*_1 = k_{\mathrm{o}}\boldsymbol{\varphi}_1' + \mathfrak{a}'^*_{im}$ and F_1 be the compositum of L_f and the field generated by $e(\mathfrak{b}'^*_1)$; find $\boldsymbol{\varphi}_2' \notin \mathfrak{b}_1'^*$ such that $1 \otimes e(\boldsymbol{\varphi}_2') \notin F_1$. Let $\mathfrak{b}_2'^* = k_{\mathrm{o}}\boldsymbol{\varphi}_2' + \mathfrak{b}_1'^*$ and so on. This produces an infinite tower of proper extensions

$$L_f \subset F_1 \subset F_2 \subset \ldots \subset L(\mathfrak{g}')$$

whence $L(\mathfrak{g}')$ is not a finite extension of L_f.

(2) $\Rightarrow$ (1): By assumption $\mathfrak{a}'^*_{im} = \mathfrak{a}'^*$ hence

$$L_f \supset \{Z[\mathfrak{a}'^*]\}_{\text{field}}.$$

By Proposition 3.1, $L(\mathfrak{g}')$ is generated over $\{Z[\mathfrak{a}'^*]\}_{\text{field}}$ by l'-elements $X_1, \ldots, X_{l'}$, that are identified with a free basis of P' over $\mathbf{Z}$. By PSJ all these elements are algebraic over L_f.

(2) $\Rightarrow$ (3). Let $k\mu' + \varphi' \in \mathfrak{h}'^*$. By PSJ for a certain $N \in \mathbf{N}$ there exists $\mu \in P$ such that $f_O(\mu, 0) = (N\mu', 0)$. Use (2) to find $x \in \mathfrak{h}^*$ such that $f_k(x) = \varphi'$. Then

$$f_k\left(\frac{k}{N}\mu + x\right) = k\mu' + \varphi'.$$

(3) $\Rightarrow$ (2). Let $\varphi' \in \mathfrak{a}'^*$. By assumption there exists a $k\mu + \varphi \in \mathfrak{h}^*$ such that $f_{\mathfrak{t}}(k\mu + \varphi) = \varphi'$. Let

$$(\mu', \varphi'_\mu) := f_{O,\mathfrak{s}}(\mu) \quad \text{and} \quad \varphi'_{\mathfrak{a}} := f_{O,\mathfrak{a}}(\varphi).$$

Then

$$\varphi' = k\mu' + (k\varphi'_\mu + \varphi'_{\mathfrak{a}}),$$

so that $\mu' = 0$. We conclude that $\varphi'_\mu \in \mathfrak{a}'^*_{im}$ and therefore

$$\varphi' = k\varphi'_\mu + \varphi'_{\mathfrak{a}} \in \mathfrak{a}'^*_{im}.$$

(3) $\Leftrightarrow$ (4). This is clear.

(5) $\Leftrightarrow$ (1). This is immediate from the following diagram

$$\begin{tikzcd} K(\mathfrak{g}') \arrow[r, dash, "\text{finite}"] \arrow[d, dash] & L(\mathfrak{g}') \arrow[d, dash] \\ K_f \arrow[r, dash, "\text{finite}"'] & L_f \end{tikzcd}$$

The top extension is finite by Proposition 3.3. As for the bottom, L_f (respectively K_f) is the smallest subfield of $L(\mathfrak{g}')$ (respectively $K(\mathfrak{g}')$) containing $f(Z[P] \otimes Z[\mathfrak{a}^*])$ and $e(\mathfrak{a}'^*_{im})$ (respectively $f(R(\mathfrak{g}))$ and $e(\mathfrak{a}'^*_{im})$). However, $Z[P] \otimes Z[\mathfrak{a}^*]$ is integral over $Z[P]^W \otimes Z[\mathfrak{a}^*] \simeq R(\mathfrak{g})$ by Proposition 3.2 and the result follows.

Definition 5.3. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two reductive Lie algebras. A pre-subjoining $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ is said to be a *subjoining* if f satisfies any of the equivalent conditions of Proposition 5.2.

Remark. For $\mathfrak{g}'$ semi-simple we can see that the notions of pre-subjoining and subjoining coincide. In fact, let $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ be a pre-subjoining and suppose that $\mathfrak{g}'$ is semisimple. Then (2) in Proposition

5.2 obviously holds and f is a subjoining. Conversely suppose that PSJ is not the case. Choose a basis $\lambda'_1, \ldots, \lambda'_{l'}$ for P' such that $d_1\lambda'_1, \ldots, d_k\lambda'_k$ (with $d_1, \ldots, d_k \in \boldsymbol{N}$) is a basis for $f_0(P \times \{0\}) \cap P'$. By assumption $k < l'$. Let $\mu' = \lambda'_{l'}$ and imagine an algebraic relation over L_f, say,

$$e(\mu')^N + a_1 e(\mu')^{N-1} + \ldots + a_{N-1} e(\mu') + a_N = 0.$$

After clearing denominators we see that since $R(\mathfrak{g}')$ has no non-trivial elements of λ-degree one we can assume that $a_i \in f(Z[P] \otimes 1)$ for all $1 \leqq i \leqq N$. We conclude that $e(\mu') = e(\lambda'_{l'})$ is algebraic over $Z[e(\lambda'_1), \ldots, e(\lambda'_k)]$, a contradiction. Thus (1) = PSJ and we see that if $\mathfrak{g}'$ is semisimple then PSJ $\Leftrightarrow$ (1) $\Leftrightarrow$ (5). This allows us to characterize the concept of subjoining at the level of representation rings;

PROPOSITION 5.2′. ($\mathfrak{g}'$ *semi-simple*). *For a λ-ring morphism $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ to be a subjoining it is necessary and sufficient that the field of quotients of $R(\mathfrak{g}')$ be a finite extension of the field generated by $f(R(\mathfrak{g}))$.*

Let us end this remark by pointing out that if $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ is a subjoining then rank $\mathfrak{g} \geqq$ rank $\mathfrak{g}'$. This follows from Proposition 5.2 (3).

PROPOSITION 5.4. *Let $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ be a subjoining. Then*

$$\psi: W' \rightarrow W_D/W_I$$

is injective.

Proof. Suppose that $1 \in \psi(w')$ for some $w' \in W'$. Dualizing diagram $(\mathrm{D})_f^*$ we obtain the following commutative diagram

$$\begin{tikzcd} \mathfrak{h} \arrow[d, "1"'] & \mathfrak{h}' \arrow[l, "f^*"'] \arrow[d, "w'"] \\ \mathfrak{h} & \mathfrak{h}' \arrow[l, "f^*"] \end{tikzcd}$$

from which $w' = 1$ since f^* is injective.

PROPOSITION 5.5. *For a subjoining $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ the following are equivalent*:

(1) rank $\mathfrak{g}$ = rank $\mathfrak{g}'$

(2) *f_k is injective*

(3) *f^* is surjective*

(4) *$f_0:P \times \mathfrak{a}^* \rightarrow P' \times \mathfrak{a}'^*$ is injective.*

Proof. If f is a subjoining then f_k is surjective and f^* is injective. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Finally, since f_k is a k-linear extension of $f_\circ$ we have (2) $\Leftrightarrow$ (4).

Definition 5.6. A subjoining $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ is said to be an *equal rank subjoining* if it satisfies any of the equivalent conditions of Proposition 5.5.

Remark. If f is equal rank then condition (4) of the above proposition tells us that $\psi: W' \to W$ since from $\mathfrak{j} = \ker f_\circ = \{0\}$ we have $W_D = W$ and $W_I = \{1\}$.

The study of the composition of subjoinings in general is fairly intricate. For our purpose it is sufficient to examine the equal rank subjoining case and that is quite straightforward.

A pair of subjoinings

$$f:R(\mathfrak{g}) \to R(\mathfrak{g}')$$
$$g:R(\mathfrak{g}') \to R(\mathfrak{g}'')$$

of equal rank reductive Lie algebras leads to the following picture

$$(1)\quad \begin{tikzcd}
P \times \mathfrak{a}^* \arrow[r, "f_\circ"] \arrow[d, dash] & P' \times \mathfrak{a}'^* \arrow[r, "g_\circ"] \arrow[d, dash] & P'' \times \mathfrak{a}''^* \arrow[d, dash] \\
L \arrow[r, "\tilde{f}"] \arrow[d, dash] & L' \arrow[r, "\tilde{g}"] \arrow[d, dash] & L'' \arrow[d, dash] \\
K \arrow[r, "\bar{f}"] \arrow[d, dash] & K' \arrow[r, "\bar{g}"] \arrow[d, dash] & K'' \arrow[d, dash] \\
R(\mathfrak{g}) \arrow[r, "f"] & R(\mathfrak{g}') \arrow[r, "g"] & R(\mathfrak{g}'') \\
W & W' \arrow[l, "\psi_f"'] & W'' \arrow[l, "\psi_g"']
\end{tikzcd}$$

The composition $h:= g \circ f$ is compatible with $\bar{h}:= \bar{g} \circ \bar{f}$, $\tilde{h}:= \tilde{g} \circ \tilde{f}$, $h_\circ = g_\circ \circ f_\circ$, and $\psi_h:= \psi_f \circ \psi_g$, and is a subjoining. We have $h_Z^* = f_Z^* \circ g_Z^*$

$$(2)\quad h_Z'' \xrightarrow{g_Z^*} h_Z' \xrightarrow{f_Z^*} h_Z.$$

From (1) and (2) we see that if h is an isomorphism then f_Z^*, g_Z^*, ψ_f, and ψ_g

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are isomorphisms. In Proposition 6.3 we will show that this implies that $\mathfrak{g} \simeq \mathfrak{g}' \simeq \mathfrak{g}''$ and f and g are isomorphisms. In particular an isomorphism cannot be split non-trivially.

Definition 5.7. A subjoining is *proper* if it is not an isomorphism.

Again, in the equal rank case, from (1) and (2) and using the fact that f^*_Z, g^*_Z, ψ_f, and ψ_g are injective we see that only finitely many non-equivalent splittings $h = g \circ f$ can exist for a given equal rank subjoining h.

Definition 5.8. (equal rank) A subjoining is *maximal* if it cannot be split as a product of two proper subjoinings.

Definition 5.9. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras and $\mathfrak{n}$ and $\mathfrak{n}'$ their respective nil-radicals. Then there exist canonical λ-ring isomorphisms (Appendix 2, Proposition 6),

$$\psi:R(\mathfrak{g}) \to R(\mathfrak{g}/\mathfrak{n}) \quad \text{and} \quad \psi':R(\mathfrak{g}') \to R(\mathfrak{g}'/\mathfrak{n}').$$

Thus, given a λ-ring morphism $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ there exists a unique λ-ring morphism $f_R:R(\mathfrak{g}/\mathfrak{n}) \to R(\mathfrak{g}'/\mathfrak{n})$ that makes the following diagram commutative

$$\begin{tikzcd} R(\mathfrak{g}) \arrow[r, "f"] \arrow[d, "\psi"'] & R(\mathfrak{g}') \arrow[d, "\psi'"] \\ R(\mathfrak{g}/\mathfrak{n}) \arrow[r, "f_R"'] & R(\mathfrak{g}'/\mathfrak{n}') \end{tikzcd}$$

Conversely, given $f_R:R(\mathfrak{g}/\mathfrak{n}) \to R(\mathfrak{g}'/\mathfrak{n}')$ then f as above exists and is unique.

f as above is said to be a *subjoining* if f_R is a subjoining in the sense of Definition 5.3. As before we say that f is proper if it is not an isomorphism, maximal if it is not the composition of two proper subjoinings, and equal rank if rank $\mathfrak{g}$ = rank $\mathfrak{g}'$. (Recall that rank $\mathfrak{g} :=$ rank $(\mathfrak{g}/\mathfrak{n})$). It can be easily shown that f is either proper, maximal, or equal rank if and only if f_R is.

6. Equal rank subjoining. In this and subsequent sections we apply the theory hitherto constructed to the special situation of subjoinings between equal rank reductive Lie algebras. The important point here is that we have embeddings $\mathfrak{h}'_Z \to \mathfrak{h}_Z$ and $W' \to W$ which allow us to reduce the problem to the level of the coroot systems Σ' and Σ of $\mathfrak{g}'$ and $\mathfrak{g}$ in $\mathfrak{h}'_Z$ and

$\mathfrak{h}_Z$ respectively. The reduction is worked out in Section 6. In Section 8 we apply this to the case of maximal equal rank subjoinings, finally achieving the classification in Section 9.

Let $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ be an equal rank subjoining between two reductive Lie algebras. By Proposition 5.5 (2)

$$f_k:\mathfrak{h}^* \to \mathfrak{h}'^*$$

is a vector space isomorphism. Let $V' \subset \mathfrak{h}'^*$ be defined by

$$V' := f_k(\{P\}_k).$$

Then $\dim_k V' = \operatorname{rank} \mathfrak{s}$. By PSJ we see that $\{P'\}_k \subset V'$. Now $\{P'\}_k$ is a nonsingular space (relative to $(\, . \, , \, . \,)'$) and its orthogonal complement $\mathfrak{d}'^*$ in V' is thus orthogonal to all roots of Δ'. Thus $\mathfrak{d}'^*$ is in $\mathfrak{h}'^{*\perp} = \mathfrak{a}'^*$ and it follows that

$$\mathfrak{d}'^* = \{\varphi \in \mathfrak{a}'^* | \varphi = f_\circ(\mu, 0) \text{ for some } \mu \in P\}_k.$$

In other words we see that $f_{\circ,\mathfrak{s}}:P \to P' \times \mathfrak{d}'^*$ together with $\psi:W' \to W$ induces an equal rank subjoining (Theorem 2.1 (B)) $f_{\mathfrak{s}}:R(\mathfrak{s}) \to R(\mathfrak{s}' \times \mathfrak{d}')$ such that $f_{\mathfrak{s}} := f|_{R(\mathfrak{s})\otimes 1}$.

According to this, from now on we make the assumption that $\mathfrak{g}$ is semisimple since $\mathfrak{a}^*$ and its isomorphic copy $f_{\mathfrak{k}}(\mathfrak{a}^*) \subset \mathfrak{a}'^*$ can be removed from the picture for the purpose of classification. Recall that since $f_\circ$ is injective

$$\psi:W' \to W$$

and ψ is injective.

Recall the following commutative diagram ($(\mathrm{D})_f^*$)

$$\begin{array}{ccc} W' \times \mathfrak{h}'_Z & \longrightarrow & \mathfrak{h}'_Z \\ \Big\downarrow {\scriptstyle \psi \times f_Z^*} & & \Big\downarrow {\scriptstyle f_Z^*} \\ W \times \mathfrak{h}_Z & \longrightarrow & \mathfrak{h}_Z \end{array}$$

Let Σ' and Σ be the coroot systems of $\mathfrak{g}'$ and $\mathfrak{g}$ in $\mathfrak{h}'_Z$ and $\mathfrak{h}_Z$ respectively. Thus Σ' is the set of $\alpha'^\vee$, $\alpha' \in \Delta'$ where $\alpha'^\vee \in \mathfrak{h}'_Z$ is defined by the equation for the reflection $r_{\alpha'}$ on $\mathfrak{h}'$. Thus, for all $x = k\mu'^\vee + a' \in \mathfrak{h}' = \mathfrak{h}'_{\mathfrak{s}} \times \mathfrak{a}'$

$$r_{\alpha'}:x \mapsto x - \langle \alpha', x\rangle' \alpha'^\vee = x - \langle \alpha', k\mu'^\vee\rangle'_{\mathfrak{s}}\, \alpha'^\vee = kr_{\alpha'}(\mu'^\vee) + a'.$$

Similarly for Σ, for which $\mathfrak{h} = \mathfrak{h}'_{\mathfrak{s}}$ by our last assumption.

Let $\alpha' \in \Sigma'$. Let $H_{\alpha'} \subset \mathfrak{h}'$ be the hyperplane corresponding to $r_{\alpha'}$. Under the action of ψ, $r_{\alpha'}$ is mapped into a symmetry on $\mathfrak{h}$, that is, $\psi(r_{\alpha'})$ reverses

$f^*(\alpha')$ and pointwise fixes the hyperplane $f^*(H_{\alpha'})$ of $\mathfrak{h}$ (see diagram $(\mathrm{D})_{\mathrm{f}}^*$ above). Since $\psi(r_{\alpha'}) \in W$, $\psi(r_{\alpha'})$ is a reflection and corresponds to some root $\alpha \in \Sigma$. We have

$$f^*:\alpha' \mapsto k_{\alpha'}\alpha.$$

Since $f^*(\mathfrak{h}'_Z) \subset \mathfrak{h}_Z$ and α is indivisible in $\mathfrak{h}_Z$, it follows that $k_{\alpha'} \in Z$ and that we can certainly arrange things so that $k_{\alpha'} > 0$; then α is uniquely determined by α' and we may write k_α instead of $k_{\alpha'}$.

PROPOSITION 6.1. (Notation as above). *Let $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ be an equal rank subjoining. Then there exists a unique mapping*

$$\Sigma' \to \Sigma; \ \alpha' \mapsto \alpha$$

and unique positive integers k_α $(= k_{\alpha'})$ such that

(i) $f_Z^{*\prime}(\alpha') = k_\alpha \alpha$

(ii) $\psi(r_{\alpha'}) = r_\alpha$.

Furthermore if $\alpha' \mapsto \alpha$, $\beta' \mapsto \beta$ then $\alpha' \perp \beta' \Leftrightarrow \alpha \perp \beta$.

Proof. It remains to prove the last statement. However

$$\begin{aligned} \alpha' \perp \beta' &\Leftrightarrow r_{\alpha'}\beta' = \beta' \\ &\Leftrightarrow \psi(r_{\alpha'})f_Z^*(\beta') = f_Z^*(\beta') \\ &\Leftrightarrow r_\alpha k_\beta \beta = k_\beta \beta \Leftrightarrow \alpha \perp \beta. \end{aligned}$$

Remark. Suppose Σ is decomposable, say

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_s \quad \text{with } \Sigma_i \perp \Sigma_j \text{ whenever } i \neq j.$$

We write

$$\Sigma = \Sigma_1 \vee \ldots \vee \Sigma_s$$

for this situation.

Let $\mathfrak{h}_i$ be the k-span of Σ_i and $\mathfrak{h}'_i = f^{*-1}(\mathfrak{h}_i) \subset \mathfrak{h}'$. Thus

$$\mathfrak{h} = \mathop{\perp}_{i=1}^{s} \mathfrak{h}_i \quad \text{and} \quad \mathfrak{h}' = \oplus \mathfrak{h}'_i.$$

Define

$$\Sigma'_i = \{\alpha' \in \Sigma' | f_Z^*(\alpha')/k_{\alpha'} \in \Sigma_i\}$$

and

$$\mathfrak{a}'_i = \{a' \in \mathfrak{a}' | f^*(a') \in \mathfrak{h}_i\} \subset \mathfrak{h}'_i.$$

Notice that $\Sigma' = \Sigma'_1 \vee \ldots \vee \Sigma'_s$ because of Proposition 6.1. We claim that Σ'_i is a root system for the pair $(\mathfrak{h}'_i, \mathfrak{a}'_i)$

For each $\beta' \in \Sigma' \backslash \Sigma'_i$, $\beta' \perp \mathfrak{h}'_i$. Indeed for $x \in \mathfrak{h}'_i$,

$$f^*(r_{\beta'} x) = \psi(r_{\beta'})f^*(x) = f^*(x)$$

since $f^*(x) \in \mathfrak{h}_i$ and $f^*_Z(\beta')/k_{\beta'} \notin \Sigma_i$. Now $\{\Sigma'_i\}_k$ is a non-singular space (relative to $(\,.\,,\,.\,)'$) and its orthogonal complement $\mathfrak{d}'_i$ in $\mathfrak{h}'_i$ is thus orthogonal to all the roots of Σ'. Thus $\mathfrak{d}'_i$ is in $\mathfrak{h}'^{\perp} = \mathfrak{a}'$ and it follows that $\mathfrak{d}'_i = \mathfrak{a}'_i$ and

$$\mathfrak{h}'_i = \{\Sigma'_i\}_k \perp \mathfrak{a}'_i.$$

In view of this remark we can restrict our attention to the case when Σ is indecomposable. Henceforth we make this assumption. Notice that it can still happen that Σ' is decomposable even when Σ is indecomposable. Let

$$\Sigma' = \Sigma'_1 \vee \ldots \vee \Sigma'_s$$

be a decomposition of Σ' into indecomposable root systems and let

$$\mathfrak{h}' = \mathfrak{h}'_l \perp \ldots \perp \mathfrak{h}'_s$$

be the corresponding orthogonal decomposition of $\mathfrak{h}'$ as above.

Let $(\,.\,,\,.\,)$ be a non-degenerate W-invariant bilinear form on $\mathfrak{h}$. Then

$$(\,.\,,\,.\,)' : \mathfrak{h}' \times \mathfrak{h}' \to k$$

defined by

$$(x, y)' \mapsto (f^*(x), f^*(y))$$

is W'-invariant and hence on each $\mathfrak{h}'_i$, when restricted to $\{\Sigma_i\}_k$, is a multiple of the canonical W'-invariant form coming from the Killing form on $\mathfrak{g}'$. Thus in using $(\,.\,,\,.\,)'$ on $\mathfrak{h}'$ we do not alter the geometry of the system; accordingly, from now on we assume that f^* is an isometry.

Proposition 6.2. *For each component Σ'_i there is a positive integer k_1 such that $\alpha' \mapsto k_i\alpha$ for all long roots α' of Σ'_i (in the case where all roots are of equal length they are considered long). In addition if there are two root lengths in Σ'_i and the ratio of lengths, long to short, is $\sqrt{r}$ ($r = 2$ or 3) then one of the two following holds*:

(i) *long* (*resp. short*) *roots of* Σ_i' *go to long* (*resp. short*) *roots of* Σ *and*

$\beta' \to k_i\beta$ *for all* $\beta' \in \Sigma_i'$

(ii) *long* (*resp. short*) *roots of* Σ_i' *go to short* (*resp. long*) *roots of* Σ. *Furthermore*

$\alpha' \mapsto k_i\alpha$ *for* α' *long*

$\alpha' \mapsto (k_i/r)\alpha$ *for* α' *short.*

Proof. Let $\alpha' \in \Sigma_i'$ with $\alpha' \mapsto k_\alpha\alpha$. Let $w' \in W'$. Then from

$$w'\alpha' \mapsto \psi(w')k_{\alpha'}\alpha = k_{\alpha'}\psi(w')\alpha$$

we conclude that

$$k_{\alpha'} = k_{w'\alpha'}.$$

Thus $k_{\alpha'}$ is constant on W'-orbits. Thus with α' long in Σ_i',

$$k_{\beta'} = k_{\alpha'} \quad \text{for all long } \beta' \in \Sigma_i'.$$

Suppose that Σ_i' also has short roots and $k_{\beta'} = l_i$ for these short roots.

With α' long, β' short, $\alpha' \mapsto k_i\alpha$ and $\beta' \mapsto l_i\beta$ we have

$$r = \frac{(\alpha', \alpha')'}{(\beta', \beta')'} = \left(\frac{k_i}{l_i}\right)^2\frac{(\alpha, \alpha)}{(\beta, \beta)}.$$

Since r is not a square $(\alpha, \alpha) \neq (\beta, \beta)$ and there are two possibilities

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = r \text{ or } \frac{1}{r}.$$

These give $k_i = l_i$ and $k_i = rl_i$ respectively.

Proposition 6.3. *If* $f_{\mathbf{Z}}^*:\mathfrak{h}_{\mathbf{Z}}' \to \mathfrak{h}_{\mathbf{Z}}$ *and* $\psi: W' \to W$ *are group isomorphisms then* $f_{\mathbf{Z}}^*(\Sigma') = \Sigma$ *and* $f_{\mathbf{Z}}^*$ *is an isomorphism of root systems.*

Proof. Since roots $\alpha' \in \Sigma'$ are indivisible in $\mathfrak{h}_{\mathbf{Z}}'$ so then are $f_{\mathbf{Z}}^*(\alpha') = k_\alpha\alpha$. Thus each $k_\alpha = 1$ and $f_{\mathbf{Z}}^*(\Sigma') \subset \Sigma$. Symmetrically $f_{\mathbf{Z}}^{*-1}(\Sigma) \subset \Sigma'$.

Proposition 6.4. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be two reductive Lie algebras and suppose that* $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ *is a* k*-linear* λ-ring *isomorphism. Then* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are isomorphic. In particular, two semisimple Lie algebras are isomorphic if and only if their representation rings are* λ*-isomorphic.*

Proof. The first part is clear by our last proposition. The second follows from the fact that if $\mathfrak{g}'$ is semisimple then any λ-ring morphism $f:R(\mathfrak{g}) \to R(\mathfrak{g}')$ is k-linear.

Remark. If $\mathfrak{g}$ and $\mathfrak{g}'$ are reductive and $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ is a λ-ring isomorphism (not necessarily k-linear) it can still be shown that their respective semisimple parts are isomorphic. More generally if $\mathfrak{g}$ and $\mathfrak{g}'$ are any two Lie algebras and $\mathfrak{s}$ and $\mathfrak{s}'$ denote maximal semisimple Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively then

$$R(\mathfrak{g}) \overset{\lambda}{\simeq} R(\mathfrak{g}') \Rightarrow \mathfrak{s} \simeq \mathfrak{s}' \quad \text{(Appendix, 3, Theorem 1).}$$

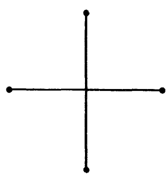
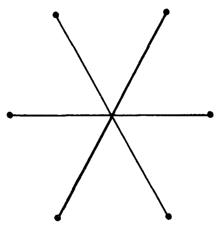
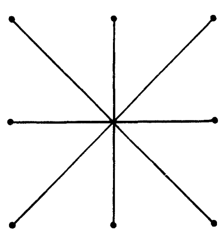
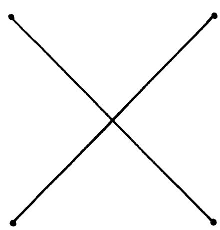
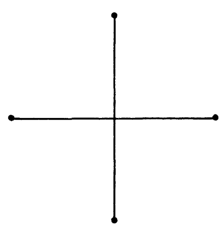
7. Subroot systems. Let Σ be a root system and let Σ' be a non empty subset of Σ. We say that Σ' is a *subroot system* of Σ if for all $\alpha \in \Sigma'$,

$$r_\alpha \Sigma' = \Sigma'.$$

Then Σ' is a root system in its own right. We say that Σ' is *closed in* Σ if

$$(\Sigma' + \Sigma') \cap \Sigma \subset \Sigma'.$$

We have the following list of proper rank two subroot systems of the rank two root systems.

Σ		Closed	Non closed
$A_1 \times A_1$		None	None
A_2		None	None
B_2			

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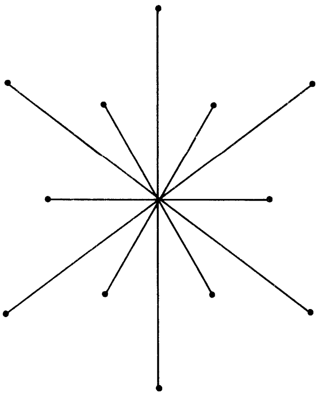
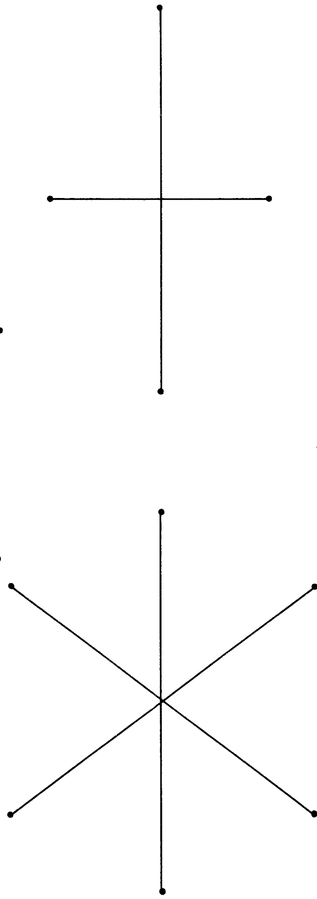
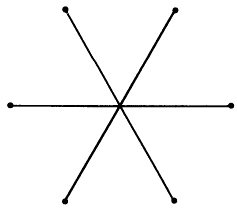
Σ	Closed	Non closed
G_2 		

PROPOSITION 7.1. *Let Σ' be a subroot system of an indecomposable root system Σ. Then*

$$\Delta = ((\Sigma' + \Sigma') \cap \Sigma) \cup \Sigma'$$

is a subroot system of Σ.

Proof. If Σ' is of rank one the result is clear. Suppose rank $\Sigma' > 1$. Then Δ necessarily satisfies the integral property

$$2(\alpha, \beta)/(\alpha, \alpha) \in \mathbf{Z}$$

for all α, $\beta \in \Delta$. We prove that Δ is stable by reflections r_α, $\alpha \in \Delta$. Suppose Σ' not closed. We can see by inspection that the result is true for the non-closed A_2 in G_2 so we assume Σ is not of type G_2.

If $\alpha \in \Sigma'$ then r_α stabilizes Σ' and Σ, hence Δ.

Let $\alpha = \alpha_1 + \alpha_2 \in (\Sigma' + \Sigma') \cap \Sigma \setminus \Sigma'$, where $\alpha_1, \alpha_2 \in \Sigma'$. Then

$$(\mathbf{Q}\alpha_1 + \mathbf{Q}\alpha_2) \cap \Sigma$$

is a rank two root system and

$$(\mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2) \cap \Sigma'$$

is a subroot system. According to the discussion of rank two root systems, α_1 and α_2 are short, $\alpha_1 \perp \alpha_2$ and $\alpha_1 + \alpha_2$ is long in Σ. We are assuming that $(\varphi, \varphi) = 2$ for long roots, so $(\alpha_i, \alpha_i) = 1$, $i = 1, 2$; $(\alpha, \alpha) = 2$. Let $\beta \in \Sigma'$. Then

$$(*) \quad r_\alpha\beta = \beta - (\beta, \alpha)\alpha = \beta - (\beta, \alpha_1 + \alpha_2)\alpha_1 - (\beta, \alpha_1 + \alpha_2)\alpha_2.$$

We have $r_\alpha\alpha_1 = -\alpha_2$, $r_\alpha\alpha_2 = -\alpha_1$, both of which belong to Σ'. We assume $\beta \neq \pm\alpha_1; \pm\alpha_2$. Next

$$(1) \quad (\beta, \alpha_1 + \alpha_2) = 0 \Rightarrow r_\alpha\beta = \beta \in \Sigma'$$

$$(2) \quad (\beta, \alpha_1) = (\beta, \alpha_2) \Rightarrow r_\alpha\beta = \beta - 2(\beta, \alpha_1)\alpha_1 - 2(\beta, \alpha_2)\alpha_2$$
$$= r_{\alpha_2}r_{\alpha_1}\beta \in \Sigma'.$$

Suppose that β is long. Then

$$(\beta, \alpha_i) = 0, \pm 1, \quad i = 1, 2.$$

Because of (1) and (2) we are reduced to the case

$$(\beta, \alpha_1) = \pm 1, \quad (\beta, \alpha_2) = 0$$

or the same with α_1 and α_2 interchanged. Then by (*)

$$r_\alpha\beta = \beta \mp \alpha_1 \mp \alpha_2.$$

However

$$r_{\alpha_1}\beta = \beta \pm 2\alpha_1$$

and since $\alpha_1, \beta \in \Sigma'$, we know by root strings that $\beta \mp \alpha_1 \in \Sigma'$ and it is short. Thus $r_\alpha\beta \in \Sigma' + \Sigma'$.

Suppose that β is short. Then

$$(\beta, \alpha_i) = 0, \pm\frac{1}{2}.$$

However

$$2(\beta, \alpha)/(\alpha, \alpha) = (\beta, \alpha_1 + \alpha_2)$$

is an integer and hence one of (1) or (2) holds and $r_\alpha\beta \in \Sigma'$.

Finally suppose that

$$\beta \in (\Sigma' + \Sigma') \cap \Sigma \setminus \Sigma'.$$

Then $\beta = \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in \Sigma'$ are short and $r_\alpha\beta_i \in \Sigma'$ whence $r_\alpha\beta \in \Sigma' + \Sigma'$.

The root system Δ of Proposition 7.1 is called the closure of Σ' and is denoted by $\langle \Sigma' \rangle$.

COROLLARY 7.2. *If Σ' is a maximal proper subroot system of Σ and Σ' is not closed then*

$$\Sigma = ((\Sigma' + \Sigma') \cup \Sigma) \cup \Sigma'.$$

8. Maximal equal rank subjoining. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be reductive and suppose that $f:R(\mathfrak{g}) \rightarrow R(\mathfrak{g}')$ is an equal rank subjoining. In Section 6 we have seen that for the purpose of studying equal rank subjoinings we can assume that $\mathfrak{g}$ is simple. If this is the case then $\mathfrak{g}'$ will still be, in general, of the form $\mathfrak{g}' = \mathfrak{s}' \times \mathfrak{a}'$ with $\mathfrak{s}'$ semi-simple but not necessarily simple and $\mathfrak{a}'$ possibly trivial.

The natural way to start investigating maximal equal rank subjoinings is to look at the maximal equal rank subalgebras. This situation was completely described by Borel and de Siebenthal in [**1**].

THEOREM. *Let $\mathfrak{g}$ be a simple Lie algebra of rank l with Coxeter-Dynkin diagram X. Let $\widetilde{X}$ be the extended Coxeter-Dynkin diagram obtained by adjoining a node corresponding to the negative of the highest root*

$$\phi = \sum_{i=1}^{l} n_i\alpha_i$$

with respect to some Cartan subalgebra and some base of the corresponding root system. Let the nodes of $\widetilde{X}$ be indexed by the n_i and the new node by 1. *Then the maximal subalgebras $\mathfrak{g}'$ of rank l of $\mathfrak{g}$ are enumerated by the diagrams obtained by*:

(1) *Deleting any one node whose index is a prime number. In this case $\mathfrak{g}' = \mathfrak{s}'$ is semisimple and rank $\mathfrak{s}' = l$.*

(2) *Deleting the extension node and any other one node whose index is* 1. *In this case the diagram obtained corresponds to a semisimple Lie algebra $\mathfrak{s}'$ of rank $l - 1$ and we have $\mathfrak{g}' = \mathfrak{s}' \times k$.*

Actual specimens of subalgebras (1) as above may be obtained by adjoining to a standard generating system $e_1, \ldots, e_l; f_1, \ldots, f_l$ two new generators

$$e_0 \in \mathfrak{g}^{-\phi} \quad \text{and} \quad f_0 \in \mathfrak{g}^{\phi}$$

so that $[e_0, f_0]$ is the coroot corresponding to $-\phi$. Then if i is the deleted index, the subalgebra generated by

$$e_0, \ldots, \hat{e}_i, \ldots, e_l, f_0, \ldots, \hat{f}_i, \ldots, f_l$$

is a subalgebra of the corresponding type.

Subalgebras of type (2) on the other hand are generated by $\mathfrak{h}$ and

$$e_1, \ldots, \hat{e}_i, \ldots, e_l, f_1, \ldots, \hat{f}_i, \ldots, f_l$$

where i is any node indexed by 1.

The statement that $\mathfrak{g}'$ is an equal rank maximal subalgebra of $\mathfrak{g}$ has a simple interpretation at the level of root systems. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$. Then $\mathfrak{h}'$ can be taken as a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and relative to this the root system Δ' of $\mathfrak{g}'$ is a subset of the root system Δ of $\mathfrak{g}$. Since $\mathfrak{g}'$ is a maximal subalgebra of $\mathfrak{g}$, Δ' is necessarily a maximal closed subroot system of Δ. Conversely every maximal closed subroot system of Δ determines a maximal subalgebra of $\mathfrak{g}$.

We return now to the situation of Section 6 where

$$f:R(\mathfrak{g}) \to R(\mathfrak{g}')$$

is an equal rank subjoining and in addition we assume that $\mathfrak{g}$ is simple and f is a maximal proper. Keeping our previous notation and conventions we recall that Σ' and Σ are the coroot systems. After identifying Σ' inside the coroot lattice generated by Σ we have

$$\Sigma' \subset \mathfrak{h}' = \mathfrak{h}, \quad \Sigma \subset \mathfrak{h}$$

and for all $\alpha' \in \Sigma'$, $\alpha' = k_\alpha \alpha$ for some $\alpha \in \Sigma$. Although it is open to abuse we will sometimes write $\Sigma' < \Sigma$ to describe this situation whenever there is no danger of confusion.

Our present discussion breaks into a number of cases:

Case A. Σ' decomposable. In this case

$$\Sigma' = \Sigma'_1 \vee \ldots \vee \Sigma'_s$$

and in general

$$\mathfrak{g}' = \mathfrak{s}' \times \mathfrak{a}' \quad \text{with} \quad \mathfrak{h}' = \{\Sigma'_1\}_k \perp \ldots \perp \{\Sigma'_s\}_k \perp \mathfrak{a}' = \mathfrak{h}.$$

Let $\Sigma_i = \Sigma \cap \{\Sigma'_i\}_k$. Then $\Sigma_1 \vee \ldots \vee \Sigma_s$ is a subroot system of Σ and is of rank $\leqq l$. We have

$$\Sigma'_1 \vee \ldots \vee \Sigma'_s \leqq \Sigma_1 \vee \ldots \vee \Sigma_s \nleq \Sigma.$$

Thus, under the assumption of maximality,

$$\Sigma'_1 \vee \ldots \vee \Sigma'_s = \Sigma_1 \vee \ldots \vee \Sigma_s$$

and $\Sigma_1 \vee \ldots \vee \Sigma_s$ is maximal in Σ.

From the definition of Σ_i, it is closed in Σ.

(i) If $\Sigma_1 \vee \ldots \vee \Sigma_s$ is closed in Σ then we know exactly what possibilities exist by the Theorem of Borel-de Siebenthal.

(ii) If $\Sigma' = \Sigma_1 \vee \ldots \vee \Sigma_s$ is not closed in Σ then by the corollary to Proposition 7.1.

$$\Sigma = ((\Sigma' + \Sigma') \cap \Sigma) \cup \Sigma'.$$

Let $\alpha_1 + \alpha_2 \in \Sigma \setminus \Sigma'$, $\alpha_1, \alpha_2 \in \Sigma'$. Then α_1, α_2 lie in different Σ_i's; say Σ_1 and Σ_2. Thus, $\Sigma_1 \vee \Sigma_2$ is not closed in Σ and its closure $\langle \Sigma_1 \vee \Sigma_2 \rangle$ gives a proper intermediate subjoining

$$\Sigma_1 \vee \ldots \vee \Sigma_s \lneqq \langle \Sigma_1 \vee \Sigma_2 \rangle \vee \ldots \vee \Sigma_s \leqq \Sigma$$

unless $s = 2$ and $\Sigma = \langle \Sigma_1 \vee \Sigma_2 \rangle$. Thus in this case we have

$$\Sigma = ((\Sigma_1 + \Sigma_2) \cap \Sigma) \cup (\Sigma_1 1 \vee \Sigma_2).$$

We claim that $\Sigma_1^\vee \vee \Sigma_2^\vee$ is maximal closed in $\Sigma^\vee$. In fact $\Sigma^\vee$ consists of the co-coroots $\alpha^\vee := 2\alpha/(\alpha, \alpha)$, $\alpha \in \Sigma$ which are comprised of

$$\alpha^\vee, \quad \alpha \in \Sigma_i$$

$$(\alpha_1 + \alpha_2)^\vee, \quad \alpha_i \in \Sigma_i, \alpha_i \text{ short } i = 1, 2.$$

Since $\alpha_1 + \alpha_2$ is long in Σ, $(\alpha_1 + \alpha_2)^\vee$ is short in $\Sigma^\vee$. Such an element cannot be a sum $\beta_1^\vee + \beta_2^\vee$, $\beta_i^\vee \in \Sigma_i^\vee$. Thus $\Sigma_1^\vee \vee \Sigma_2^\vee$ is closed in $\Sigma^\vee$. The maximality is clear.

Summarizing case A:

If Σ' is decomposable then either

(i) Σ' is maximal closed in Σ

(ii) $\Sigma' = \Sigma_1 \vee \Sigma_2 \subset \Sigma$ where $\Sigma_1^\vee \vee \Sigma_2^\vee$ is maximal closed in $\Sigma^\vee$.

Case B. Σ' indecomposable.

Case B1. Σ has one root length. By Proposition 6.2 Σ' has only one root length and

$$\alpha' \mapsto k\alpha$$

where k is independent of α'. In view of maximality, either k is a prime or $k = 1$ and Σ' is a maximal closed subroot system of Σ.

Case B2. Σ has two root lengths, $\Sigma \neq G_2$.

(a) Σ' has only one root length. In this case

$$\alpha' \to k\alpha \quad k \text{ constant}$$

and

$$\Sigma' < k\,\Sigma \leqq \Sigma$$

so that $k = 1$. If Σ' is not closed then the roots of Σ correspond to short roots in Σ the sum of some pairs of which are long in Σ. Thus $\Sigma'^{\vee}$ consists entirely of long roots in $\Sigma^{\vee}$ and so is closed in $\Sigma^{\vee}$. We conclude that either

Σ' is maximal closed in Σ or

$\Sigma'^{\vee}$ is maximal closed in $\Sigma^{\vee}$.

(b) Σ' has two root lengths. Then either relative root lengths are preserved or reversed by the subjoining.

(b: preserved) We must have k constant. There are again two cases:

$k = 1$. Then $\Sigma' < \Sigma$ (maximal). If Σ' is not closed then from

$$\Sigma = \Sigma' \cup ((\Sigma' + \Sigma') \cap \Sigma)$$

we see that Σ' contains all short roots of Σ. Thus $\Sigma'^{\vee}$ has all the long roots of $\Sigma^{\vee}$ and is maximal closed in $\Sigma^{\vee}$.

$k > 1$. Then $\Sigma' = 2\,k\Sigma$ and k is a prime. If $k = 2$ then

$$\Sigma' = 2\Sigma < \Sigma^{\vee} < \Sigma$$

and Σ' is not maximal. If $k > 2$, however, $k\Sigma$ is maximal in Σ. In fact, if

$$k\Sigma < \Sigma^* < \Sigma$$

for some root system Σ^*, then Σ^* has roots on every ray of Σ.

Let $c\alpha \in \Sigma^* \setminus k\Sigma$ where $\alpha \in \Sigma$, $1 \leqq c < k$. Then $c|k$ so $\alpha \in \Sigma^*$ and all roots of the same length as α lie in Σ^*. Suppose $\gamma \in \Sigma$, $\gamma \notin W\alpha$. Then $(\gamma, \gamma)/(\alpha, \alpha) = 2$ or $1/2$. Now $d\gamma \in \Sigma^*$ for some $d = 1$ or k. However,

$$(d\gamma, d\gamma)/(\alpha, \alpha) = d^2(\gamma, \gamma)/(\alpha, \alpha) = 1/2,\ 1 \text{ or } 2.$$

Since $k \neq 2$ the only possibility is $d = 1$ so we have $\Sigma^* = \Sigma$.

The conclusion reached from B(2) (b: preserved) is:

$$\Sigma' = p\Sigma \quad p \text{ prime} \quad p \neq 2$$

or

$$\Sigma' \text{ maximal closed in } \Sigma$$

or

$$\Sigma'^{\vee} \text{ maximal closed in } \Sigma^{\vee}.$$

(b: reversed) By Proposition 6.2 we have

$$\alpha' \mapsto k\alpha \quad \text{if } \alpha' \text{ is short}$$

$$\alpha' \mapsto 2k\alpha \quad \text{if } \alpha' \text{ is long.}$$

Consider $\Sigma^{\vee} < \Sigma$. For $\alpha \in \Sigma$, if α is long then $\alpha^{\vee} = \alpha$ and if α is short $\alpha^{\vee} = 2\alpha$. Thus

$$\alpha' \mapsto k\alpha^{\vee}$$

in all cases. We have

$$\Sigma' \leqq k\Sigma^{\vee} \leqq \Sigma^{\vee} < \Sigma.$$

By maximality $\Sigma' = \Sigma^{\vee}$.

This concludes our discussion except for the case $\Sigma = G_2$ which is quite similar. Summarizing we have

THEOREM 8.1. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be a simple and a reductive Lie algebra respectively, both of the same rank. Let Σ and Σ' be their respective coroot systems. Then if $\mathfrak{g}'$ is maximally subjoined to $\mathfrak{g}$ we have one of the following (after identification of Σ' in the coroot lattice generated by Σ)*

(1) *Σ' is a maximal closed subroot system of Σ*

(2) *Σ' is a maximal closed subroot system of $\Sigma^{\vee}$*

(3) $\Sigma' = \Sigma^{\vee}$

(4) *$\Sigma' = p\Sigma$ where p is a prime and*

$$p \neq 2 \text{ if } \Sigma \text{ is of type } B_l,\ C_l, \text{ or } F_4$$

$$p \neq 3 \text{ if } \Sigma \text{ is of type } G_2.$$

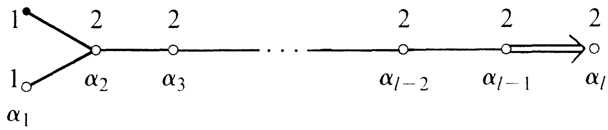
9. Classification of maximal equal rank subjoining. Using Theorem 8.1 we can complete the classification of the maximal equal rank subjoining by looking at each of the various root systems in turn. This is rather straightforward and we have omitted details except for some discussion of the $B_l - C_l$ and the F_4 cases. Since we are working with coroot systems everything has to be dualized at the end to get back to the algebra level. Theorem 9.1 summarizes all our discussion and it is stated in a most general way. In the schemata depicting the relation of the maximal subalgebras and maximal subjoinings to each other we have indicated the subalgebras by enclosing their labels in rectangular boxes.

The root systems and accompanying notations are taken from [**3**]. In particular for each $n \in \mathbf{N}$ the set $\{\epsilon_1, \dots, \epsilon_n\}$ is an orthonormal basis for $\mathbf{R}^n$.

Types X_l:$X = A, D, E$. These are covered by Case A(i) and Case B1. Apart from the maximal subalgebras there are only the subjoinings pX_l, p a prime.

Types B_l and C_l.

B_l:



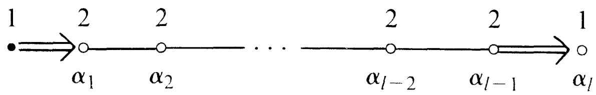
The roots are:

$$\pm \epsilon_i, \pm \epsilon_i \pm \epsilon_j, \quad 1 \leqq i < j \leqq l \quad (\text{extra node } -\epsilon_1 - \epsilon_2).$$

Standard base:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = \epsilon_l$$

C_l:



The roots are:

$$\pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j, \quad 1 \leqq i < j \leqq l \quad (\text{extra node } -2\epsilon_1).$$

Standard base:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = 2\epsilon_l.$$

Given in this way $C_l = B_l^\vee$ but C_l itself does not conform to our hypothesis on root lengths (namely that long roots should have length 2).

The proper maximal closed subroot systems for both cases are given by [**1**]

$$B_l \begin{cases} D_i \vee B_{l-i} & i = 2, 3, \dots, l-1 \\ D_l \\ B_{l-1} \end{cases}$$

$$C_l \begin{cases} C_i \vee C_{l-i} & i = 1, 2, \dots, l-1 \\ A_{l-1} \end{cases}$$

and can be picked out in the obvious way from the above diagrams.

The unusual point is that $D_i \vee B_{l-i}$, $B_{l-1} \times k$, and $A_{l-1} \times k$ are not maximal subjoinings. Indeed $D_i \vee B_{l-i}$ $(2 \leqq i \leqq l - 1)$ has roots

$$\begin{array}{lll} \pm \epsilon_r \pm \epsilon_s & 1 \leqq r < s \leqq i: & D_i \\ \left.\begin{array}{ll} \pm \epsilon_r \pm \epsilon_s & i + 1 \leqq r < s \leqq l \\ \pm \epsilon_r & i + 1 \leqq r \leqq l \end{array}\right\} & & B_{l-i} \end{array}$$

B_{l-1} has roots

$$\left\{\begin{array}{ll} \pm \epsilon_r \pm \epsilon_s & 2 \leqq r < s \leqq l \\ \pm \epsilon_r & 2 \leqq r \leqq l. \end{array}\right.$$

However corresponding to the $C_i \vee C_{l-i}$ in C_l we have

$$\left.\begin{array}{ll} \pm \epsilon_r \pm \epsilon_s & 1 \leqq r < s \leqq i \\ \pm \epsilon_r & 1 \leqq r \leqq i \end{array}\right\} \quad C_i^\vee$$

$$\left.\begin{array}{ll} \pm \epsilon_r \pm \epsilon_s & i + 1 \leqq r < s \leqq l \\ \pm \epsilon_r & i + 1 \leqq r \leqq l \end{array}\right\} \quad C_{l-i}^\vee$$

Thus $D_i \vee B_{l-i} < C_i^\vee \vee C_{l-i}^\vee < B_l$ for all $2 \leqq i \leqq l - 1$ and

$$B_{l-1} \times k < C_1^\vee \vee C_{l-i}^\vee < B_l.$$

As for $A_{l-1} \times k$ in C_l we see that D_l is maximally subjoined to C_l by dualizing D_l as a maximal subalgebra of B_l.

D_l has as roots $\pm \epsilon_r \pm \epsilon_s \quad 1 \leqq r < s \leqq l$ while $A_{l-1} \times k$ is a maximal subalgebra of D_l with root system

$$\pm(\epsilon_r - \epsilon_s) \quad 1 \leqq r < s \leqq l.$$

Thus $A_{l-1} \times k < D_l < C_l$.

Type F_4. The roots are:

$$\pm \epsilon_i \pm \epsilon_j, \pm \epsilon_i, \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \quad \text{with } i \neq j \text{ and } i, j \in \{1, 2, 3, 4\}.$$

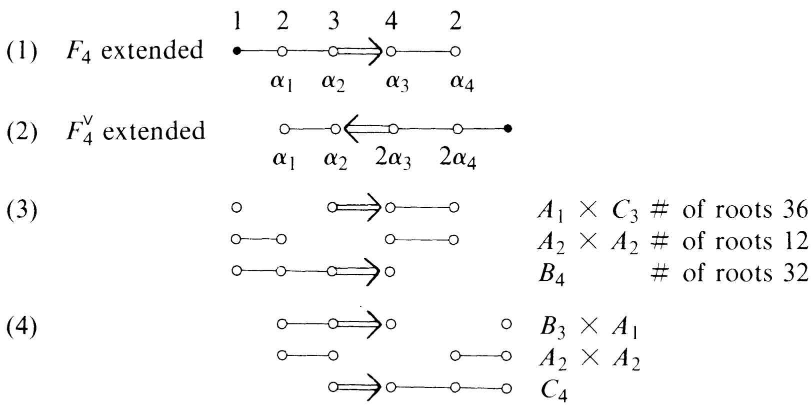
Standard basis: $\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = 1/2(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$

Highest long root: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \epsilon_1 + \epsilon_2$

Highest short root: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = \epsilon_1$.

We proceed as follows. (1) Extend the diagram of F_4 by the negative of the highest long root. (2) Dualize F_4 to obtain $F_4^\vee$ and extend its diagram by the negative of its highest long root, i.e., by the negative of the highest short root of F_4. (3) Compute the maximal closed subroots systems of F_4. (4) The dual of these lie in $F_4^\vee$; identify $F_4^\vee$ with F_4 by turning the diagram over.

The following diagram illustrates the above four steps. The alignment of the nodes is meaningful.



We now have to decide which of these are maximal subjoinings. $F_4^\vee$ is maximally subjoined to F_4. The $B_3 \times A_1$ has the base

$$\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4, -\epsilon_1.$$

It is not subjoined to $F_4^\vee$ (whose lattice does not contain $-\epsilon_1$). Also, observing that its roots are indivisible in the root lattice of F_4 and taking into account the numerology we conclude that it is maximal.

The C_4 is maximal by similar arguments.

The second $A_2 \times A_2$ is closed and hence up to conjugation by the Weyl group is the same as the first.

Type G_2. The analysis is similar to the one of F_4 though rather easier since the dualization of its maximal closed subroot systems (A_2 and $A_1 \times A_1$) does not introduce anything new.

Summarizing:

THEOREM 9.1. (Classification of maximal equal rank subjoinings). *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be two Lie algebras and suppose that*

$$f:R(\mathfrak{g}) \to R(\mathfrak{g}')$$

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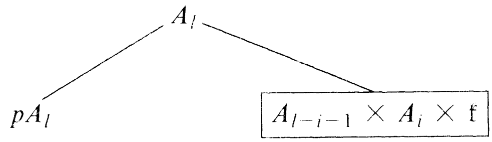
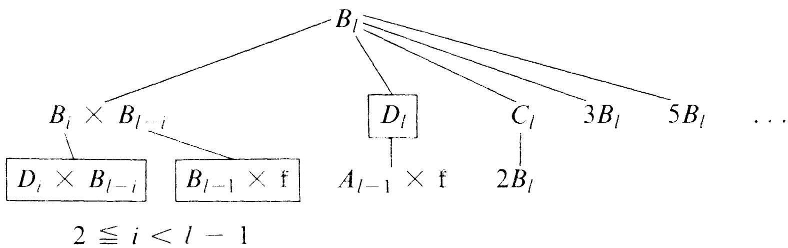
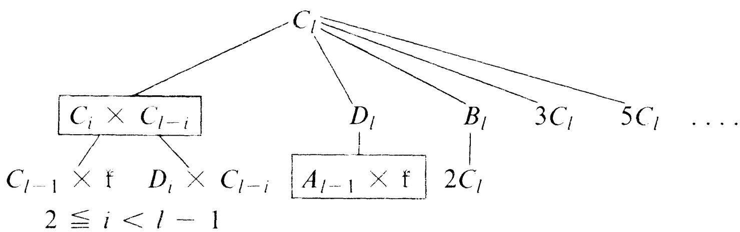
is a maximal equal rank subjoining. Let $\mathfrak{s}$ *and* $\mathfrak{s}'$ *be maximal semisimple Lie subalgebras of* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *respectively and write* $\mathfrak{s} = \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_k$ *as a product of simple Lie algebras. Then (possibly after relabeling) we have*

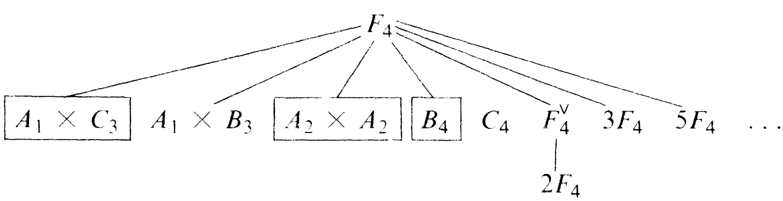
$$\mathfrak{s}' \cong \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_{k-1} \times \mathfrak{m}$$

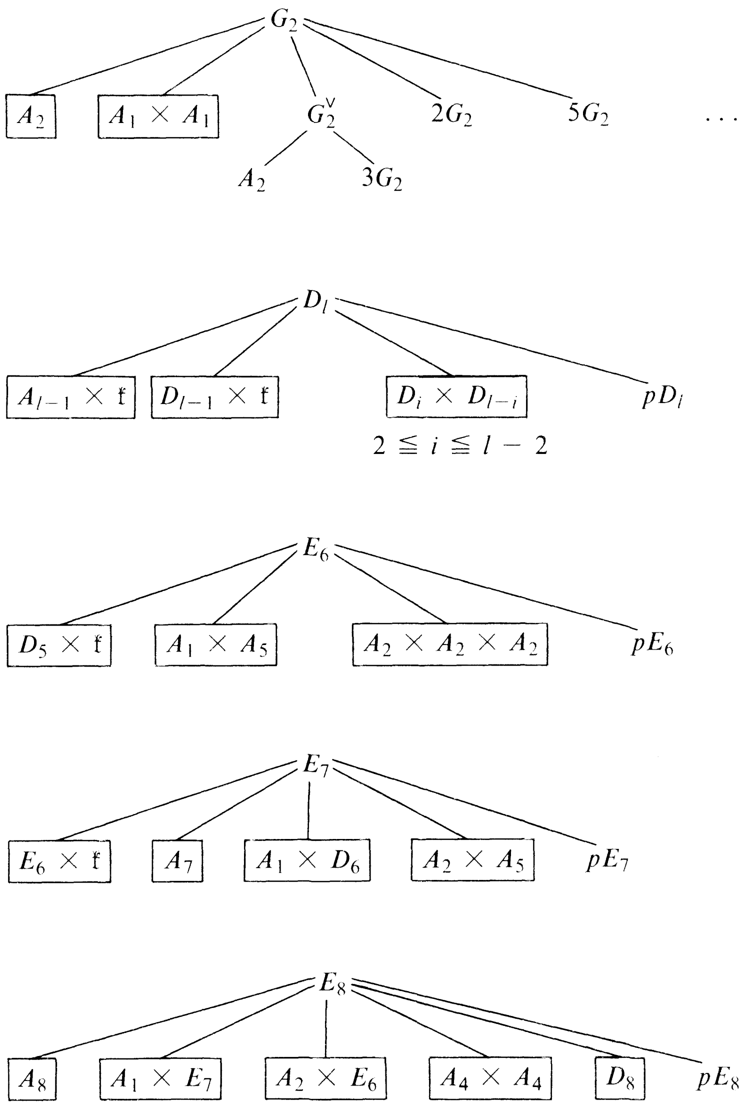
where $\mathfrak{m}$ *is in general semisimple. Moreover,* rank $\mathfrak{m}$ = rank $(\mathfrak{s}_k)$ *or* rank $\mathfrak{m}$ = rank $(\mathfrak{s}_k) - 1$ *and we have that in the first case* $\mathfrak{m}$ *itself and in the second* $\mathfrak{m} \times \mathfrak{k}$ *is maximally subjoined to the simple Lie algebra* $\mathfrak{s}_k$ *according to the following table*:











10. Examples. We present four examples which illustrate the various guises in which subjoinings appear.

Example 1. $G_2^\vee < G_2$

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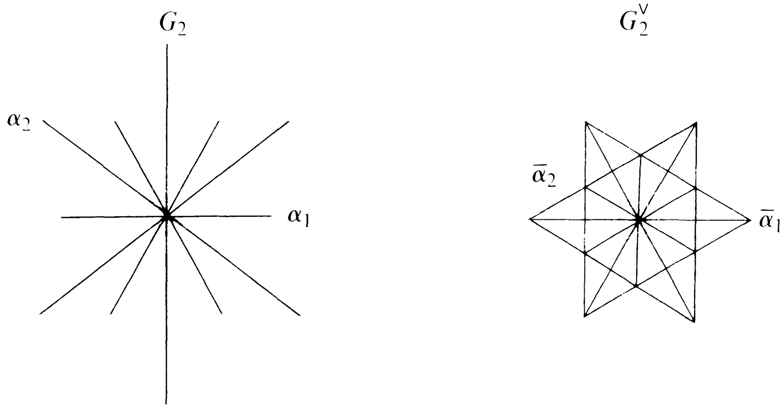

Let α_1, α_2 and $\bar{\alpha}_1$, $\bar{\alpha}_2$ be the fundamental roots of G_2 and $G_2^\vee$ respectively. Similarly ω_1, ω_2 and $\bar{\omega}_1$, $\bar{\omega}_2$ denote their fundamental weights.

To analyze the subjoining of $G_2^\vee$ to G_2 we operate with the coroots of $G_2^\vee$ and G_2 or in other words with the roots of $(G_2^\vee)^\vee$ $(= G_2)$ and $(G_2)^\vee$.

Let

$$f^*: \begin{cases} \bar{\alpha}_1^\vee \mapsto \alpha_1^\vee \\ \bar{\alpha}_2^\vee \mapsto 3\alpha_2^\vee \end{cases}$$

We conclude that the dual map $f_\circ$ is given by

$$f_\circ \begin{cases} \omega_1 \mapsto \bar{\omega}_1 \\ \omega_2 \mapsto 3\bar{\omega}_2 \end{cases}$$

The pair $(f_\circ, \psi)$ with $\psi: W \to W$ the identity map determines a subjoining of $G_2^\vee$ to G_2.

Notice that $L(G_2^\vee) = L_f(e(\bar{\omega}_2))$. The minimal polynomial of $e(\bar{\omega}_2)$ over L_f is

$$X^3 - e(f_\circ(\omega_2)) = 0.$$

We have the following diagram of finite extensions.

$$\begin{tikzcd} L_f \arrow[r, dash, "3"] \arrow[d, dash, "12"'] & L' \arrow[d, dash, "12"] \\ K_f \arrow[r, dash, "3"'] & K' \end{tikzcd}$$

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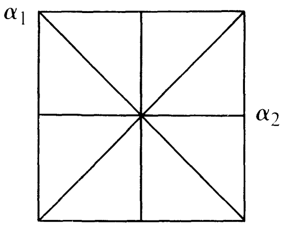
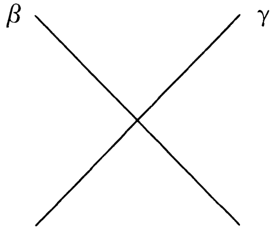
Example 2. $A_1 \times A_1 < B_2$ (Subalgebra)





$$f^*: \begin{cases} (\beta^\vee, 0) & \mapsto \alpha_1^\vee \\ (0, \gamma^\vee) & \mapsto (\alpha_1 + 2\alpha_2)^\vee = \alpha_1^\vee + \alpha_2^\vee \end{cases}$$

$$f_\circ: \begin{cases} \omega_1 & \mapsto (\omega_\beta, \omega_\gamma) \\ \omega_2 & \mapsto (0, \omega_\gamma) \end{cases}$$

$$\psi: \begin{cases} (r_\beta, 1) & \mapsto r_1 \\ (1, r_\gamma) & \mapsto r_2 r_1 r_2 \end{cases}$$

$L_f = L'$ and we have the following diagram

$$\begin{tikzcd} & L_f = L' \arrow[dl, dash, "8"'] \arrow[d, dash, "4"] \\ K_f \arrow[r, dash, "2"'] & K' \end{tikzcd}$$

Example 3. $A_1 < A_1 \times A_1 < B_2$.
We present a scheme of the form

$$\begin{tikzcd} R(B_2) \arrow[r, "f''"] \arrow[dr, "f"'] & R(A_1 \times A_1) \arrow[d, "f'"] \\ & R(A_1) \end{tikzcd}$$

where A_1 is subjoined to $A_1 \times A_1$ and B_2 as a subalgebra while $A_1 \times A_1$ into B_2 is not.

With the notation for B_2 as in the last example, we have

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$$\omega_1 = \alpha_1 + \alpha_2 \quad \text{and} \quad \omega_2 = \frac{1}{2}(\alpha_1 + 2\alpha_2).$$

Let

$$A^\lambda = \mathfrak{g}^{-\lambda} \oplus \{h_\lambda\}_k \oplus \mathfrak{g}^\lambda, \quad \lambda = \beta, \gamma, \delta$$

be three Lie algebras of type A_1.

Step 1. $f:R(B_2) \to R(A_1)$

$$f^*:\delta^\vee = \delta \mapsto (\alpha_1 + \alpha_2)^\vee = 2\alpha_1^\vee + \alpha_2^\vee$$

$$2 = \langle \omega_1, 2\alpha_1^\vee + \alpha_2^\vee \rangle = \langle f_\circ(\omega_1), \delta \rangle,$$

$$1 = \langle \omega_2, 2\alpha_1^\vee + \alpha_2^\vee \rangle = \langle f_\circ(\omega_2), \delta \rangle.$$

Thus

$$f_\circ: \begin{cases} \omega_1 \mapsto 2\omega_\delta \\ \omega_2 \mapsto \omega_\delta \end{cases} \quad \text{and } \mathfrak{j} = \ker f_\circ = \langle \omega_1 - 2\omega_2 \rangle_{\mathbf{Z}} = \langle \alpha_2 \rangle_{\mathbf{Z}}$$

$$W_D = \{1, r_2, r_1r_2r_1, r_2r_1r_2r_1\}$$

$$W_I = \{1, r_2\}$$

$$W_D/W_I = \{\overline{1}, \overline{r_1r_2r_1}\}.$$

$\psi: W' \to W_D/W_I$ is given by $r_\delta \mapsto \overline{r_1r_2r_1}$. $(f_\circ, \psi)$ subjoins A_1^δ to B_2 and under this interpretation A_1^δ is viewed as a subalgebra of B_2.

Step 2.

$$f':R(A_1^\beta \times A_1^\gamma) \to R(A_1^\delta)$$

$$f'^*:\delta^\vee = \delta \mapsto (\beta^\vee, 0) = (\beta, 0).$$

Clearly:

$$f'_\circ \begin{cases} (\omega_\beta, 0) \mapsto \omega_\delta \\ (0, \omega_\gamma) \mapsto 0 \end{cases} ; \ \mathfrak{j}' := \ker f'_\circ = \langle (0, \omega_\gamma) \rangle_{\mathbf{Z}}$$

$$W'' = \{ (1, 1), (r_\beta, 1), (1, r_\gamma), (r_\beta, r_\gamma) \}$$

$$W''_D = W''$$

$$W''_I = \{ (1, 1), (1, r_\gamma) \}$$

$$W''_D/W''_I = \{ \overline{(1, 1)}, \overline{(r_\beta, 1)} \}$$

$$\psi': W' \to W''_D/W''_I \quad \text{with } \psi':r_\delta \mapsto \overline{(r_\beta, 1)}.$$

Step 3.

$$f'':R(B_2) \rightarrow R(A_1^{\beta} \times A_1^{\gamma})$$

$$f''^*:\begin{cases}(\beta^\vee, 0) \mapsto (\alpha_1 + \alpha_2)^\vee = 2\alpha_1^\vee + \alpha_2^\vee \\ (0, \gamma^\vee) \mapsto \alpha_2^\vee\end{cases}$$

$$f''_\circ:\begin{cases}\omega_1 \mapsto (2\omega_\beta, 0) \\ \omega_2 \mapsto (\omega_\beta, \omega_\gamma)\end{cases}$$

$$\psi'':\begin{cases}(r_\beta, 1) \mapsto r_1r_2r_1 \\ (1, r_\gamma) \mapsto r_2\end{cases}$$

We leave to the reader to check that $f_\circ = f'_\circ \circ f''_\circ$ and also that if $w' \in W'$, $w'' \in \psi'(w')$ and $w \in \psi''(w')$ then $w \in \psi(w')$.

At the field level we now have for $A_1 \times A_1 \subset B_2$ that $L' = L_f(e(\omega_\gamma))$. The minimal polynomial of $e(\omega_\gamma)$ over L_f is

$$X^2 - e[(f_\circ(\omega_2))^2/f_\circ(\omega_1)] = 0$$

and the degree of the extensions is

$$\begin{tikzcd} & L' \arrow[d, dash, "4"] \\ L_f \arrow[ur, dash, "2"] \arrow[d, dash, "8"'] & K' \\ K_f \arrow[ur, dash, "4"'] & \end{tikzcd}$$

The reader should compare this with Example 2.

Example 4. $A_{l-1} \times k < A_l$ (subalgebra). Consider a Lie algebra of type A_l and choose a Cartan subalgebra $\mathfrak{h}$. Let $\{h_1, \dots, h_l\}$ be a basis of $\mathfrak{h}$ consisting of coroots $\alpha_1^\vee, \dots, \alpha_l^\vee \in \Sigma'$. Consider next a Lie algebra of type $A_{l-1} \times k$ with Cartan subalgebra $\mathfrak{h}' \times k$ and a basis $\{h'_1, \dots, h'_{l-1}, 1\}$ consisting of coroots $\alpha_1'^\vee, \dots, \alpha_{l-1}'^\vee$ and $1 \in k$. Notice that under the canonical identification of k and k^*, $x \rightarrow x^*$ where $\langle x^*, 1\rangle = x$ for all $x \in k$.

We now view $A_{l-1} \times k$ as a maximal subalgebra A_l as prescribed by Borel-de Siebenthal. Thus

$$f^*:\begin{cases}h'_i \mapsto h_i \\ 1 \mapsto h_\circ\end{cases}$$

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where $h_0 \in \mathfrak{h}$ satisfies $h_0 \perp \{h_1, \ldots, h_{l-1}\}_k$.

Let

$$\lambda := \alpha_1 + 2\alpha_2 + \ldots + (l-1)\alpha_{l-1} + l\alpha_l \in P.$$

Then $\langle \lambda, h_i \rangle = 0$ for all $1 \leqq i \leqq l-1$ so we can further assume that h_0 is chosen so that

$$\langle \mu, h_0 \rangle = (\mu, \lambda) \quad \text{for all } \mu \in \mathfrak{h}^*.$$

For all $1 \leqq j \leqq l-1$, $1 \leqq i \leqq l$ we have

$$\delta_{ij} = \langle \omega_i, h_j \rangle = \langle f_0(\omega_i), h_j' \rangle \quad \text{and}$$

$$i = \langle \omega_i, h_0 \rangle = \langle f_0(\omega_i), 1 \rangle.$$

We conclude that

$$f_0 \begin{cases} \omega_i \mapsto (\omega_i', i^*), & 1 \leqq i \leqq l-1 \\ \omega_l \mapsto (0, l^*). \end{cases}$$

Also $\psi : r_{\alpha_i'} \mapsto r_{\alpha_i}$, $1 \leqq i \leqq l-1$. Notice that $l\omega_i - i\omega_l \mapsto (l\omega_i', 0)$ so that

$$[P' \times \{0\} : f_0(P) \cap (P' \times \{0\})] \leqq l^l < \infty.$$

Also $\mathfrak{a}_{im}'^* = \{f_0(\omega_l)\}_k = k^*$, whence

$$e(0, k^*) \subset K_f \subset L_f.$$

Since $e(\omega_i', i^*) \in f(Z[P])$ and $e(0, -i^*) \in L_f$ it follows that $e(\omega_i', 0) \in L_f$. Thus $L_f = L'$.

Obviously the subjoining $g : R(A_l) \to R(A_{l-1})$ arising from the inclusion $g^* : h_i' \mapsto h_i$ $1 \leqq i \leqq l-1$, is closely related to this. The above arguments show that

$$g_0 : P \to P'$$

is such that

$$g_0 : \omega_i \mapsto \omega_i', \quad 1 \leqq i \leqq l-1$$

and

$$\langle \omega_l \rangle_{\mathbf{Z}} = \ker g_0.$$

Clearly $W_D \simeq W'$ and $W_I = \{1\}$. Thus

$$\psi_g : W' \to W_D / W_I \simeq W'.$$

Returning to our problem we claim that $K_f = K'$ so that

$$\begin{tikzcd} L_f \arrow[r, dash, "="] \arrow[d, dash, "l!"'] & L' \arrow[d, dash, "l!"] \\ K_f \arrow[r, dash, "="] & K' \end{tikzcd}$$

To see this consider the subjoining $k:R(A_{l-1} \times k) \to R(A_{l-1})$ arising from $A_{l-1} \hookrightarrow A_{l-1} \times k$. Clearly

$$k_0:P' \times k^* \to P'$$

is the canonical projection and it is straightforward to see that

$$g = k \circ f \quad \text{(composition of subjoinings).}$$

It will suffice to show then that $g(Z[P]^W)$ covers $Z[P']^{W'}$. This is indeed the case. Let $[\omega_i]$ (resp. $[\omega_i']$) denote the character and/or the module of the i^{th} fundamental representation of A_l (resp. A_{l-1}).
By an argument on dimensions

$$[\omega_1] \mapsto [\omega_1'] + e(0) \quad [\omega_1] - e(0) \mapsto [\omega_1']$$

and therefore $[\omega_1']$ has a preimage in $Z[P]^W$. The result now follows using the fact that

$$\Lambda^i([\omega_1] - e(0)) \to \Lambda^i[\omega_1'] = [\omega_i'].$$

Appendix. On λ-structures and representation rings.

Introduction. The concept of λ-ring was first introduced by Grothendieck in [**5**]. In Appendix 1 we present a brief summary of definitions and results on the theory of λ-rings. In one way or another this material can be found in [**7**], [**12**], P. Berthelot, Generalities sur les λ-anneaux, Seminaire de geometrie algebraique, Lecture Notes in Mathematics 225 (Springer-Verlag), and in M. Atiyah and D. D. Tall, Group representations, λ-rings, and the J-homomorphism, Topology 8 (1969), 253-297.

The representation ring $R(\mathfrak{g})$ of a Lie algebra can easily be given a pre λ-ring structure using exterior powers of $\mathfrak{g}$-module. Usually in the literature no further proof is given that it is in fact a λ-ring. In Appendix 2 we establish this fact.

Concerning Lie algebras we have kept all the conventions of Section 1. All tensor products are taken over the ring **Z** of integers unless otherwise specified.

A1. λ-structures. (In this section ring means commutative ring with identity. Whenever a ring is viewed as a subring of a larger ring both multiplicative units are supposed to be the same.)

Recall from Section 1 that a pre λ-ring is a pair (A, λ_t) where A is a ring and

$$\lambda_t : A \to 1 + A[[t]]^+$$

with action

$$\lambda_t : a \mapsto 1 + \lambda^1(a)t + \lambda^2(a)t^2 + \dots, \quad \forall\, a \in A$$

is such that λ_t is a group morphism and $\lambda^1(a) = a$ for all $a \in A$. We also know that an equivalent definition can be given in terms of a family of mappings $\{\lambda^n\}_{n \in \mathbf{Z}_{\geqq 0}} : A \to A$.

If (A, λ_t) is a λ-structure then λ_t is called the λ-mapping and $\{\lambda^n\}_{n \in \mathbf{Z}_{\geqq 0}}$ the set of λ-powers of the λ-structure.

Also from Section 1 we recall the definitions of λ-ideal, λ-morphism, and λ-degree.

We start by constructing two families of polynomials that are essential for the definition of λ-ring.

Let $\chi_0 \in \mathbf{N}$ be an arbitrary large natural number (see Remark below). Let $\{X_i\}_{1 \leqq i \leqq \chi_0}$ be a family of algebraically independent variables over Z and denote by e_i the i-th elementary symmetric function on $\{X_i\}$.

Consider the polynomial

$$P = \prod_{i,j=1}^{\chi_0} (1 + X_i \otimes X_j t)$$

$$P \in (\mathbf{Z}[X_1, \dots, X_n, \dots] \otimes \mathbf{Z}[X_1, \dots, X_n, \dots])[t].$$

If we write

$$P = \sum_{N=0}^{\chi_0} P_N t^N,$$

then P_N is a polynomial of degree N in $\{X_i \otimes X_j\}$ and it can be shown that

$$P_N \in \mathbf{Z}[e_1, \dots, e_N] \otimes \mathbf{Z}[e_1, \dots, e_N]$$

is a polynomial in $2N$ variables that we write

$$P_N = P_N(e_1, \ldots, e_N; e_1, \ldots, e_N).$$

Moreover, the P_N's are universal in the sense that if $\chi_\circ \geqq n \geqq N$ and we compute

$$P^{(n)} = \prod_{i,j=1}^{n} (1 + (x_i \otimes x_j)t) = \sum_{i=0}^{n} P_i^{(n)} t^i$$

then

$$P_N^{(n)} = P_N(e_1^{(n)}, \ldots, e_N^{(n)}; e_1^{(n)}, \ldots, e_N^{(n)})$$

where $e_i^{(n)}$ denotes the i-th elementary symmetric function on $X_1, \ldots, X_n$.

In a similar way if $D \in \mathbf{Z}_{\geqq 0}$ and we consider

$$P_D = \prod_{1 \leqq i_1 < \ldots < i_D}^{\chi_\circ} (1 + X_{i_1} X_{i_2} \ldots X_{i_D} t) \in \mathbf{Z}[X_1, \ldots, X_n, \ldots][t]$$

and write

$$P_D = \sum_{N=0}^{\chi_\circ} P_{N,D} t^N.$$

Then

$$P_{N,D} \in \mathbf{Z}[e_1, \ldots, e_{ND}]$$

and the $P_{N,D}$'s are universal, as above, provided that $\chi_\circ \geqq n \geqq ND$.

Remark. The more formal mathematical treatment of this material would be the following.

Let A be a ring. We define

$$A_n = A[X_1, \ldots, X_n] \otimes A[X_1, \ldots, X_n]$$

and ring homomorphisms

$$\rho_{m,n}: A_m \rightarrow A_n \quad m \geqq n$$

by

$$\rho_{m,n}: X_i \otimes X_j \mapsto \epsilon_i X_i \otimes \epsilon_j X_j$$

where

$$\epsilon_h = 1 \quad \text{if } 1 \leqq h \leqq n \text{ and } 0 \text{ otherwise.}$$

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Let Λ_n be the subring of A_n consisting of elements invariant under the action of $\mathfrak{S}_n \times \mathfrak{S}_n$ ($\mathfrak{S}_n$ = symmetric group on n symbols).

Let $\Lambda = \varprojlim_n \Lambda_n$ with respect to $\{\rho_{m,n}\}$. Then

$$(P^{(i)})_{i \in \mathbf{Z}_{\geqq 0}} \in \Lambda.$$

Similarly for $\{P_{n,D}\}_{n \in \mathbf{Z}_{\geqq 0}}$ where now $A_n = A[X_1, \ldots, X_{ND}]$.

Definition. A λ-ring is a pre λ-ring (A, λ_t) such that

$$\text{LRI}: \lambda_t(1) = 1 + t$$

$$\text{LRII}: \lambda^n(ab) = P_n(a, \lambda^2(a), \ldots, \lambda^n(a); b, \lambda^2(b), \ldots, \lambda^n(b))$$

$$\text{LRIII}: \lambda^m(\lambda^n(a)) = P_{m,n}(a, \lambda^2(a), \ldots, \lambda^{mn}(a))$$

for all $a, b \in A$ and $m, n \in \mathbf{Z}_{\geqq 0}$.

Definition. A λ-structure (A, λ_t) is said to be *constructable* over a certain set $B \subset A$ if

(i) $\deg_\lambda b = 1 \ \forall \ b \in B$
(ii) $b_1, b_2 \in B \Rightarrow \deg_\lambda(b_1 b_2) = 1$
(iii) $a \in A \Rightarrow \exists b_1, b_2, \ldots, b_n \in B$ such that

$$a = \sum_{i=1}^{n} \epsilon_i b_i \quad \text{where } \epsilon_i = \pm 1.$$

PROPOSITION 1. *Let (A, λ_t) be a pre λ-ring. Let $B \subset A$ and suppose that (A, λ_t) is constructable over B. Then (A, λ_t) is a λ-ring.*

PROPOSITION 2. *Let (R, λ_t) and (S, Λ_t) be λ-structures. Suppose that (R, λ_t) is constructable over a certain set $B \subset R$. Let $f:R \to S$ be a ring morphism such that*

$$\deg_\Lambda f(r) = 1 \quad \text{for all } r \in B \backslash \ker f.$$

Then f is a λ-morphism.

PROPOSITION 3. *Let (R, λ_t) and (S, Λ_t) be two λ-rings. There exists a unique mapping*

$$(\lambda \otimes \Lambda)_t: R \otimes S \to 1 + (R \otimes S)[[t]]^+$$

$$(\lambda \otimes \Lambda)_t: x \mapsto (\lambda \otimes \Lambda)^0(x) + (\lambda \otimes \Lambda)^1(x)t + (\lambda \otimes \Lambda)^2(x)t^2 + \ldots$$

for all $x \in R \otimes S$ having the following properties:

(i) $(\lambda \otimes \Lambda)^n(r \otimes 1) = \lambda^n(r) \otimes 1$ *and* $(\lambda \otimes \Lambda)^n(1 \otimes s) = 1 \otimes \Lambda^n(s)$

(ii) $(\lambda \otimes \Lambda)^n(r \otimes s) = P_n(r \otimes 1, \ldots, (\lambda \otimes \Lambda)^n(r \otimes 1); 1 \otimes s, \ldots, (\lambda \otimes \Lambda)^n(1 \otimes s))$ *for all* $r \in R$, $s \in S$ *and* $n \in \mathbf{Z}_{\geqq 0}$,

(iii) $(R \otimes S, (\lambda \otimes \Lambda)_t)$ *is a* λ*-ring.*

Example. The following result about λ-structures for $\mathbf{Z}$ was used in Proposition 4.6.

Define $\lambda_t:\mathbf{Z} \to 1 + \mathbf{Z}[[t]]^+$ by

$$\lambda_t:n \mapsto (1 + t)^n.$$

Clearly $(\mathbf{Z}, \lambda_t)$ is a pre λ-ring and moreover a λ-ring since $(\mathbf{Z}, \lambda_t)$ is constructable over $B = \{1\}$; on the other hand if $(\mathbf{Z}, \Lambda_t)$ is a λ-ring, then by LRI we have $\Lambda_t(1) = 1 + t$ and therefore $\Lambda_t = \lambda_t$.

We conclude that $\mathbf{Z}$ has a unique λ-ring structure. However $\mathbf{Z}$ has an infinite number of different pre λ-ring structures for if $a_2, a_3, \ldots, a_k \in \mathbf{Z}$ then

$$\lambda_t:n \mapsto (1 + t + a_2t^2 + \ldots + a_kt^k)^n$$

gives $\mathbf{Z}$ a pre λ-ring structure. Notice that Theorem 2.1 uses the λ-ring structure on $R(\mathfrak{g}')$ (see Appendix 2). When $\mathfrak{g}' = \{0\}$ this means that

$$R(\mathfrak{g}') \simeq Z[e(0)] \to 1 + Z[e(0)][[t]]^+$$

is given by

$$ne(0) \mapsto (1 + e(0)t)^n$$

which of course coincides with the unique λ-ring structure of $\mathbf{Z}$ after having identified $\mathbf{Z}$ with the group algebra $\mathbf{Z}[e(0)]$ of the trivial group over $\mathbf{Z}$.

A2. Rings of representations. (If $\mathfrak{g}$ is a Lie algebra by a $\mathfrak{g}$-module we mean a finite dimensional $\mathfrak{g}$-module.)

Let $\mathfrak{g}$ be a Lie algebra. Denote by $s(\mathfrak{g})$ the set of isomorphism classes of simple $\mathfrak{g}$-modules. If V is such a module then we write $[V]$ for its class. Let $R(\mathfrak{g})$ be the free abelian group on $s(\mathfrak{g})$.

Suppose that F is an arbitrary $\mathfrak{g}$-module. Let $(F_n, F_{n-1}, \ldots, F_0)$ be a Jordan-Hölder decomposition series for F. We define

$$[F] = \sum_{i=1}^{n} [F_i/F_{i-1}] \in R(\mathfrak{g}).$$

In $R(\mathfrak{g})$ there exists a unique multiplication that is distributive with respect to the sum and such that for all $\mathfrak{g}$-modules A and B one has

$$[A][B] = [A \otimes_k B].$$

Together with this multiplication $R(\mathfrak{g})$ becomes a commutative ring with identity called the *representation ring* of $\mathfrak{g}$.

If $n \in \mathbf{Z}_{\geqq 0}$ and V is a $\mathfrak{g}$-module let $\Lambda^n(V)$ denote the n-th exterior power of V. The family of mappings

$$\{\Lambda^n\}_{n \in \mathbf{Z}_{\geqq 0}}: R(\mathfrak{g}) \to R(\mathfrak{g})$$

such that

$$\Lambda^n: [V] \mapsto [\Lambda^n(V)]$$

is well defined. Let

$$\Lambda_t: R(\mathfrak{g}) \to 1 + R(\mathfrak{g})[[t]]^+$$

be defined by

$$\Lambda_t \left(\sum_{i=1}^m n_i [V_i] \right) \mapsto \prod_{i=1}^m (\Lambda^0[V_i] + \Lambda^1[V_i]t + \Lambda^2[V_i]t^2 + \dots)^{n_i}.$$

One can easily verify then that $(R(g), \Lambda_t)$ is a pre λ-ring.

Remark. Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$. Any $\mathfrak{g}$-module can be viewed as a $\mathfrak{g}'$-module simply by restriction. We have a ring morphism $f: R(\mathfrak{g}) \to R(\mathfrak{g}')$. Since the underlying vector space structure of V is the same regardless of whether we view V as a $\mathfrak{g}$ or a $\mathfrak{g}'$-module, f as above is a λ-morphism.

In the remaining part of this section, we will study the structure of the representation rings, eventually showing that $(R(\mathfrak{g}), \Lambda_t)$ is actually a λ-ring. The analysis breaks into a number of cases.

Semisimple case. Let $\mathfrak{g} = \mathfrak{s}$ be a semisimple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a base Π for the corresponding root system. Let P be the weight lattice and $Z[P]$ the group algebra on P over $\mathbf{Z}$ written multiplicatively. The Weyl group W acts naturally on $Z[P]$ by

$$w \sum_{\mu \in P} n_\mu e(\mu) = \sum_{\mu \in P} n_\mu e(w\mu)$$

for all $w \in W$ and

$$\sum_{\mu \in P} n_\mu e(\mu) \in Z[P].$$

Let $Z[P]^W$ denote the subring of $Z[P]$ consisting of W-invariant elements. It is known then [4] that the character map

$$\mathrm{ch}:R(\mathfrak{s}) \to Z[P]^W$$

is a ring isomorphism. Define

$$\lambda_t:Z[P] \to 1 + Z[P][[t]]^+$$

by

$$\lambda_t:\Sigma\, n_\mu e(\mu) \mapsto \prod(1 + e(\mu)t)^{n_\mu}.$$

Clearly $(Z[P], \lambda_t)$ is a pre λ-ring. Moreover $(Z[P], \lambda_t)$ is constructable over $B = \{e(\mu):\mu \in P\}$ and therefore is a λ-ring (Proposition 1).

To show that $(Z[P]^W, \lambda_t)$ is a λ-ring it will suffice to show that

$$\lambda_t(Z[P]^W) \subset 1 + Z[P]^W[[t]]^+.$$

If $x = \Sigma\, n_\mu\, e(\mu) \in Z[P]^W$ then $n_\mu = n_{w\mu}$ for all $w \in W$ and $\mu \in P$, therefore the formal power series

$$\prod(1 + e(\mu)t)^{n_\mu}$$

is W-invariant together with all of its coefficients. Thus

$$\lambda_t(x) = \prod(1 + e(\mu)t)^{n_\mu} \in 1 + Z[P]^W[[t]]^+.$$

Finally, let V be an $\mathfrak{s}$-module. Choose a basis $(v_1, \ldots, v_n)$ for V consisting of weight vectors relative to $\mathfrak{h}$. If $k \in \mathbf{Z}_{\geqq 0}$ the set

$$\{v_{i_1} \wedge \ldots \wedge v_{i_k}:1 \leqq i_1 < \ldots i_k \leqq n\}$$

forms a k-basis for $\Lambda^k(V)$ and

$$\mathrm{ch}\ \Lambda^k(V) = \sum_{1 \leqq i_1 < \ldots < i_k \leqq n} e(v_{i1}) \ldots e(v_{ik}).$$

If ch_t denotes the extension of ch to the ring of formal power series, straightforward computation shows that

$$\lambda_t \mathrm{ch} = \mathrm{ch}_t\, \Lambda_t.$$

Therefore

$$(Z[P]^W, \lambda_t) \underset{\mathrm{ch}}{\overset{\lambda}{\simeq}} (R(\mathfrak{s}), \Lambda_t).$$

In particular the latter is a λ-ring.

Commutative case. Let $\mathfrak{a}$ be a commutative Lie algebra. Since k is algebraically closed, every irreducible $\mathfrak{a}$-module is one dimensional and one can easily see that

$$\mathrm{ch}:R(\mathfrak{a}) \rightarrow Z[\mathfrak{a}^*]$$

is a ring isomorphism where $Z[\mathfrak{a}^*]$ denotes the group algebra on $\{e(\varphi):\varphi' \in \mathfrak{a}^*\}$ over $\mathbf{Z}$.

As before; if we define

$$\lambda_t:Z[\mathfrak{a}^*] \rightarrow 1 + Z[\mathfrak{a}^*][[t]]^+$$

by

$$\lambda_t: \sum_{\varphi \in \mathfrak{a}^*} n_\varphi e(\varphi) \mapsto \prod_{\varphi \in \mathfrak{a}^*} (1 + e(\varphi)t)^{n_\varphi}$$

then $(Z[\mathfrak{a}^*], \lambda_t)$ is a pre λ-ring that, again by constructability, is also a λ-ring. Moreover

$$(R(\mathfrak{a}), \Lambda_t) \underset{\mathrm{ch}}{\overset{\lambda}{\simeq}} (Z[\mathfrak{a}^*], \lambda_t).$$

Remark. It is unfortunate that the existence of a λ-ring isomorphism between the representation rings of two commutative Lie algebras does not imply the isomorphism of the algebras. For example let $n, m \in \mathbf{Z}_{>0}$ and $n \neq m$. Then any group isomorphism

$$f:(k^n, +) \rightarrow (k^m, +)$$

(such mappings always exist) extends to a ring isomorphism

$$f:Z[k^{n*}] \rightarrow Z[k^{m*}].$$

Furthermore f is a λ-morphism by Proposition 2. We overcome this problem in the main test by imposing k-linearity on our mappings.

Reductive case. Let $\mathfrak{g}_i:i = 1, 2$ be two Lie algebras. Then for the natural ring structure of $R(\mathfrak{g}_1) \otimes R(\mathfrak{g}_2)$ there exists a unique ring monomorphism

$$i:R(\mathfrak{g}_1) \otimes R(\mathfrak{g}_2) \rightarrow R(\mathfrak{g}_1 \times \mathfrak{g}_2)$$

extending the natural embeddings

$$e_i:R(\mathfrak{g}_1) \hookrightarrow R(\mathfrak{g}_1 \times \mathfrak{g}_2): \quad i = 1, 2.$$

If M_i is a $\mathfrak{g}_i$-module, $i = 1, 2$ then

$$i:[M_1] \otimes [M_2] \mapsto [M_1 \otimes_k M_2].$$

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Suppose now that $\mathfrak{g}_1 = \mathfrak{s}$ is semisimple and $\mathfrak{g}_2 = \mathfrak{a}$ is commutative. If V is a simple $\mathfrak{s} \times \mathfrak{a}$-module then $\mathfrak{a}$ acts on V as scalar multiplication, say

$$x \cdot v = \varphi(x)v \quad \forall x \in \mathfrak{a}, v \in V$$

where $\varphi \in \mathfrak{a}^*$.

Let w be a symbol and kw be the one dimension k vector space with $\{w\}$ as a basis. Make kw into an $\mathfrak{a}$ module by defining

$$x \cdot w = \varphi(x)w \quad \text{for all } x \in \mathfrak{a}.$$

Denote by $\bar{V}$ the $\mathfrak{s}$-module obtained by restricting the action of $\mathfrak{s} \times \mathfrak{a}$ on V to $\mathfrak{s}$. We have

$$[\bar{V}] \otimes [kw] \xrightarrow{i} [\bar{V} \otimes_k kw] = [V].$$

Therefore $i:R(\mathfrak{s}) \otimes R(\mathfrak{a}) \to R(\mathfrak{s} \times \mathfrak{a})$ is a ring isomorphism.

PROPOSITION 4. $(R(\mathfrak{s} \times \mathfrak{a}), \Lambda_t)$ *is a* λ*-ring. Moreover*

$$R(\mathfrak{s}) \otimes R(\mathfrak{a}) \overset{\lambda}{\simeq} R(\mathfrak{s} \times \mathfrak{a}).$$

Proof. We denote by Λ^n, $\Lambda^n_{\mathfrak{s}}$, $\Lambda^n_{\mathfrak{a}}$ the n-th exterior powers of $\mathfrak{s} \times \mathfrak{a}$, $\mathfrak{s}$ and $\mathfrak{a}$-modules respectively.

$(R(\mathfrak{s} \times \mathfrak{a}), \Lambda_t)$ has a pre λ-ring structure that is completely determined by knowing the λ-powers of simple $\mathfrak{s} \times \mathfrak{a}$-modules. We keep the above notation but write N instead of kw. Computation shows that for all $n \in \mathbf{Z}_{\geqq 0}$

$$\Lambda^n(V) \simeq \Lambda^n_{\mathfrak{s}}(\bar{V} \otimes_{\mathfrak{k}} N) \simeq \Lambda^n_{\mathfrak{s}}(\bar{V}) \otimes_{\mathfrak{k}} N^n$$

where

$$N^n = \overset{n}{\otimes}_{\mathfrak{k}} N.$$

On the other hand in the unique λ-ring structure of $R(\mathfrak{s}) \otimes R(\mathfrak{a})$, (according to Proposition 3) we have

$$\begin{aligned}
&(\Lambda_{\mathfrak{s}} \otimes \Lambda_{\mathfrak{a}})^n(\bar{V} \otimes_{\mathbf{Z}} N) \\
&= P_n([\bar{V}] \otimes 1, \ldots, \Lambda^n_{\mathfrak{s}}([\bar{V}]) \otimes 1; 1 \otimes N, 0, \ldots, 0) \\
&= \Lambda^n_{\mathfrak{s}}([\bar{V}]) \otimes [N^n].
\end{aligned}$$

We conclude that Λ_t is the mapping induced from $(\Lambda_{\mathfrak{s}} \otimes \Lambda_{\mathfrak{a}})_t$ via the ring isomorphism i, whence the result.

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General case.

PROPOSITION 5. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{n}$ *its nil-radical. Then*

$$R(\mathfrak{g}/\mathfrak{n}) \simeq R(\mathfrak{g}).$$

Proof. If V is a $\mathfrak{g}/\mathfrak{n}$-module then clearly V can be viewed as a $\mathfrak{g}$-module. If we call this "assignment" φ and write

$$\varphi: V \rightarrow V_{\mathfrak{g}}$$

we then have a ring monomorphism

$$\psi: R(\mathfrak{g}/\mathfrak{n}) \rightarrow R(\mathfrak{g})$$

given by

$$\psi:[V] \rightarrow [V_{\mathfrak{g}}] \quad \text{where} \quad \psi = [\ ] \circ \varphi; \quad \text{i.e.,}$$

$$V \overset{\varphi}{\mapsto} V_{\mathfrak{g}} \overset{[\ ]}{\mapsto} [V_{\mathfrak{g}}].$$

On the other hand if ρ is an irreducible representation of $\mathfrak{g}$ then $\mathfrak{n} \subset \ker \rho$ and ρ can be viewed as a representation of $\mathfrak{g}/\mathfrak{n}$. Thus ψ is also surjective.

PROPOSITION 6. *Let* $\mathfrak{g}$ *be a Lie algebra. Then* $(R(\mathfrak{g}), \Lambda_t)$ *is a* λ*-ring. Moreover*

$$(R(\mathfrak{g}); \Lambda_t) \underset{\psi}{\overset{\lambda}{\simeq}} R(\mathfrak{g}/\mathfrak{n}, \Lambda_t).$$

Proof. Since $\mathfrak{g}/\mathfrak{n}$ is reductive $R(\mathfrak{g}/\mathfrak{n}, \Lambda_t)$ is a λ-ring. By our last proposition it will suffice to show that the exterior powers $\Lambda_{\mathfrak{g}}^n$ of $\mathfrak{g}$-modules are the ones induced from the exterior powers Λ^n of $\mathfrak{g}/\mathfrak{n}$-modules via ψ. Since

$$(\Lambda^n(V))_{\mathfrak{g}} = \Lambda_{\mathfrak{g}}^n(V_{\mathfrak{g}})$$

we have

$$\begin{aligned} \psi(\Lambda^n[V]) &= \psi[\Lambda^n(V)] = [\ ] \circ \varphi(\Lambda^n(V)) = [\ ](\Lambda^n(V))_{\mathfrak{g}} \\ &= [\ ](\Lambda_{\mathfrak{g}}^n(V_{\mathfrak{g}})) = [\Lambda_{\mathfrak{g}}^n(V_{\mathfrak{g}})] = \Lambda_{\mathfrak{g}}^n[V_{\mathfrak{g}}] = \Lambda_{\mathfrak{g}}^n\psi[V]. \end{aligned}$$

A3. Further results.

PROPOSITION 7. *Let* $\mathfrak{g} = \mathfrak{s} \times \mathfrak{a}$ *be reductive. For an element* $X \in R(\mathfrak{s} \times \mathfrak{a})$ *to be of* λ*-degree one it is necessary and sufficient that* $X = [1 \otimes N]$ *for some simple* $\mathfrak{a}$*-module* N.

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Necessity. Let

$$X = \sum_{i \in I} n_i[V_i] \in R(\mathfrak{s} \times \mathfrak{a})$$

where I is finite and for all $i \in I$, $n_i \neq 0$, V_i is a simple $\mathfrak{s} \times \mathfrak{a}$-module and $i \neq j \Rightarrow [V_i] \neq [V_j]$. Let

$$I^+ = \{i \in I:n_i > 0\} \quad \text{and} \quad I^- = \{i \in I:n_i < 0\}.$$

By definition

$$\Lambda_t X = \prod_{i \in I} (1 + [V_i]t + \ldots + \Lambda^n[V_i]t^n + \ldots)^{n_i}.$$

If $\deg_\Lambda X = 1$ then

$$\Lambda_t X = 1 + \left(\sum_{i \in I} n_i[V_i]\right)t.$$

Therefore

$$(1 + \sum_{i \in I} n_i[V_i]t) \prod_{i \in I^-} (1 + [V_i]t + \ldots)^{-n_i}$$

$$= \prod_{i \in I^+} (1 + [V_i]t + \ldots)^{n_i}.$$

Equating the highest powers of t on both sides of the above we obtain

$$\sum_{i \in I} n_i[V_i] \prod_{i \in I^-} (\Lambda^{d(i)}[V_i])^{-n_i} = \prod_{i \in I^+} (\Lambda^{d(i)}[V_i])^{n_i}$$

where $d(i) = \dim_k V_i$ for all $i \in I$. Now

$$\dim_k \Lambda^{d(i)}(V_i) = 1$$

so that

$$[\Lambda^{d(i)}(V_i)] = [1 \otimes N_i]$$

for some simple $\mathfrak{a}$-module N_i. The last equation now reads

$$\left(\sum_{i \in I} n_i[V_i]\right)[1 \otimes N] = [1 \otimes M]$$

for some simple $\mathfrak{a}$-modules N and M. Multiplying both sides of this equation by $[1 \otimes N']$ for a suitable N' gives

$$(\sum n_i[V_i]) = [1 \otimes T]$$

where T is a simple $\mathfrak{a}$-module.

Sufficiency. This is clear.

Definition. A Lie algebra is said to be *perfect* if it coincides with its derived algebra.

PROPOSITION 8. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ and $\mathfrak{n}$ its radical and nil-radical respectively. Then the following conditions are equivalent*:

(i) $\mathfrak{g}$ *is perfect*

(ii) $\mathfrak{r} = \mathfrak{n}$

(iii) $\mathfrak{g}/\mathfrak{n}$ *is semisimple.*

THEOREM 1. (a) *If the representation rings of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are λ-isomorphic then any two maximal semisimple Lie subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

(b) *Suppose $\mathfrak{s}_1 \simeq \mathfrak{s}_2$. Then for $R(\mathfrak{g}_1)$ to be λ-isomorphic to $R(\mathfrak{g}_2)$ it is necessary and sufficient that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be both perfect or both non perfect.*

Proof. Let $\mathfrak{n}_i$ be the nil-radical of $\mathfrak{g}_i$. If $\mathfrak{s}_i$ is maximal semisimple then $\mathfrak{g}_i/\mathfrak{n}_i \simeq \mathfrak{s}_i \times \mathfrak{a}_i$ for $i = 1, 2$.

Suppose that $R(\mathfrak{g}_1) \underset{f}{\overset{\lambda}{\simeq}} R(\mathfrak{g}_2)$. Then we have a λ-ring isomorphism

$$h:R(\mathfrak{s}_1) \otimes R(\mathfrak{a}_1) \rightarrow R(\mathfrak{s}_2) \otimes R(\mathfrak{a}_2)$$

where according to our previous notation

$$h:= i_2'^{-1} \circ \psi_2'^{-1} \circ f \circ \psi_1 \circ i_1.$$

Clearly $1 \otimes R(\mathfrak{a}_i)$ is a sub λ-ring of $R(\mathfrak{g}_i/\mathfrak{n}_i)$. A free basis for $1 \otimes R(\mathfrak{a}_i)$ consists of elements of the form $[1 \otimes N_i]$ where N_i is a simple $\mathfrak{a}_i$-module. By Proposition 7 these are the elements of λ-degree one in $R(\mathfrak{g}_i/\mathfrak{n}_i)$.

Since a λ-isomorphism preserves λ-degrees, the free bases are mapped one onto the other by h so that

$$h_{\mathfrak{a}}:= h|_{1\otimes R(\mathfrak{a}_1)}:R(\mathfrak{a}_1) \rightarrow R(\mathfrak{a}_2)$$

is a λ-ring isomorphism.

The ring morphisms

$$d_i:R(\mathfrak{a}_i) \rightarrow Z$$

defined by

$$d_i:\sum a_m[N_m] \mapsto \Sigma\, a_m$$

are λ-ring morphisms by Proposition 2. Consider next for $i = 1, 2$ the ring morphisms

$$\sigma_i : R(\mathfrak{s}_i) \otimes R(\mathfrak{a}_i) \to R(\mathfrak{s}_i)$$

defined by

$$\left(\sum_n s_n[V_n]\right) \otimes \sum_m a_m[N_m] \mapsto \left(\sum_m a_m\right) \sum_n s_n[V_n].$$

Let $\mathcal{J}_i = \ker d_i$. It is clear that $\ker \sigma_i = R(\mathfrak{s}_i) \otimes \mathcal{J}_i$.

By the way the λ-powers are defined in the tensor product of two λ-rings, in order to show that $\ker \sigma_i$ is a λ-ideal of $R(\mathfrak{g}_i/\mathfrak{n}_i)$ it will suffice to notice that $\mathcal{J}_i$ is a λ-ideal of $R(\mathfrak{a}_i)$. We have

$$R(\mathfrak{g}_i/\mathfrak{n}_i)/R(\mathfrak{s}_i) \otimes \mathcal{J}_i \overset{\lambda}{\simeq} R(\mathfrak{s}_i).$$

Now $h_{\mathfrak{a}} : R(\mathfrak{a}_1) \to R(\mathfrak{a}_2)$ gives a λ-ring isomorphism $\mathcal{J}_1 \overset{\lambda}{\simeq} \mathcal{J}_2$ which leads to a λ-ring isomorphism between the ideals generated by $\mathcal{J}_i$ in $R(\mathfrak{g}_i/\mathfrak{n}_i)$. This last in turn gives us a λ-ring isomorphism between the quotient rings from which we conclude that

$$R(\mathfrak{s}_1) \overset{\lambda}{\cong} R(\mathfrak{s}_2).$$

The result now follows from Proposition 6.4.

The second part of the theorem is almost clear. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are both perfect then $\mathfrak{g}_1/\mathfrak{n}_1 \simeq \mathfrak{s}_1$ and $\mathfrak{g}_2/\mathfrak{n}_2 \simeq \mathfrak{s}_2$ and the result follows from Proposition 6 of this appendix.

If neither $\mathfrak{g}_1$ nor $\mathfrak{g}_2$ is perfect, then

$$\mathfrak{g}_1/\mathfrak{n}_1 \simeq \mathfrak{s}_1 \times \mathfrak{a}_1, \quad \mathfrak{g}_2/\mathfrak{n}_2 \simeq \mathfrak{s}_2 \times \mathfrak{a}_2$$

with $\mathfrak{a}_1 \neq \{0\} \neq \mathfrak{a}_2$.

Let P_i be the weight lattice of $\mathfrak{s}_i$ with respect to a certain Cartan subalgebra $\mathfrak{h}_i$ that we suppose chosen. By hypothesis $\mathfrak{s}_1 \simeq \mathfrak{s}_2$. Let

$$f_{0,\mathfrak{s}} : P_1 \to P_2$$

and

$$f_{0,\mathfrak{a}} : \mathfrak{a}_1 \to \mathfrak{a}_2$$

be group isomorphisms. ($f_{0,\mathfrak{a}}$ is constructed as in Appendix, 2, commutative case.) Define a group isomorphism

$$f_{\mathrm{o}} := f_{\mathrm{o},\mathfrak{s}} + f_{\mathrm{o},\mathfrak{a}} : P_1 \times \mathfrak{a}_1^* \to P_2 \times \mathfrak{a}_2^*.$$

Since the Weyl groups W_1 and W_2 of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are isomorphic the result now follows by applying Theorem 1.2(B) to (f_{o}, ψ), $W_1 \overset{\psi}{\cong} W_2$, and Proposition 6.

Conversely, if $R(\mathfrak{g}_1)$ and $R(\mathfrak{g}_2)$ are λ-isomorphic then there exists an induced group isomorphism between the group of weights of $\mathfrak{g}_1/\mathfrak{n}_1$ and $\mathfrak{g}_2/\mathfrak{n}_2$. These groups are free abelian, if one algebra is perfect and the other is not then the rank of one of them is finite while the other is infinite and therefore the two groups are not isomorphic.

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University of Saskatchewan,
Saskatoon, Saskatchewan