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GLOBAL ANALYSIS OF ONE-DIMENSIONAL VARIATIONAL PROBLEMS

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From the global analytical point of view a one-dimensional variational problem consists in extremising a differentiable action/cost function $f : X \to \mathbb{R}$, where X is an infinite-dimensional manifold of paths in a manifold M, over a subset $\Omega \subset X$ of admissible paths, for example those satisfying some regularity conditions, boundary conditions or other constraints. Thus, a solution to the variational problem is a critical point of the restriction $f \,|\, \Omega$.

A standard criterion for existence of critical points is the Palais–Smale condition. If this condition is satisfied then the gradient flow associated with f is well behaved, and we are guaranteed not only existence of critical points but also existence of a minimum. Moreover it is then possible to relate the total number of critical points to topological properties of Ω.

This thesis is about methods for proving that a one-dimensional variational problem satisfies the Palais–Smale condition. The methods are demonstrated with examples motivated by interpolation, approximation, geometric and optimal control problems in Riemannian manifolds. To begin with we consider *conditional extremals*: the critical points of $\frac{1}{2}\int_I \|\dot{x} - A\|^2\, dt$, where I is the unit interval, $x : I \to M$ is a path on M, $\dot{x}$ is the tangent vector along x, A is an arbitrary vector field on M and the admissible paths satisfy fixed boundary conditions. Our results on this topic have appeared in [3]. Next we treat problems with higher order covariant derivatives in the action, such as Riemannian cubics in tension: the critical points of $\frac{1}{2}\int_I \|\nabla_t \dot{x}\|^2 - \tau^2\|\dot{x}\|^2\, dt$ with $\tau \in \mathbb{R}$ constant. These results have appeared in [1]. This is followed by an investigation of curves with minimum total squared curvature $\int_I k^2\, ds$ subject to a fixed length constraint (see [2]). Such curves are known as *elastica* and this is the first example we encounter with a constraint that is not a boundary condition. Finally, we consider a

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class of problems known as sub-Riemannian, where the admissible paths are required to be tangent to a nonintegrable distribution on M.

The thesis itself is available at UWA's research repository via the link: http://research-repository.uwa.edu.au/en/publications/global-analysis-of-onedimensional-variational-problems(00d99236-fb92-430b-abef-500fda3fbdd2).html.

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