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FUNCTORIAL ASPECTS OF THE RECONSTRUCTION OF LIE GROUPOIDS FROM THEIR BISECTIONS

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Abstract

To a Lie groupoid over a compact base M, the associated group of bisection is an (infinite-dimensional) Lie group. Moreover, under certain circumstances one can reconstruct the Lie groupoid from its Lie group of bisections. In the present article we consider functorial aspects of these construction principles. The first observation is that this procedure is functorial (for morphisms fixing M). Moreover, it gives rise to an adjunction between the category of Lie groupoids over M and the category of Lie groups acting on M. In the last section we then show how to promote this adjunction to almost an equivalence of categories.

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1. Introduction

The Lie group structure on bisection Lie groups was constructed in [17, 18], along with a smooth action of the bisections on the arrow manifold of a Lie groupoid. Furthermore, in [19] we have established a tight connection between Lie groupoids and infinite-dimensional Lie groups. Namely, several (re-)construction principles for Lie groupoids from their group of bisections and from infinite-dimensional Lie group actions on a compact manifold were provided. The present paper considers the categorical aspects of these constructions.

It turns out that the construction principles are functorial, that is, they induce functors on suitable categories of (possibly infinite-dimensional) Lie groupoids and Lie groups. Moreover, we show that the (re-)construction functors together with (a suitable version of) the bisection functor discussed in [18] yield adjoint pairs.

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Consequently, we deduce properties of the bisection functor from these results. These properties are interesting in themselves to understand the connection between a Lie groupoid and its associated Lie group of bisections. Note that we neglect all higher structures on the category of Lie groupoids; we will always work with the 1-category of Lie groupoids over a fixed base manifold.

We now go into some more detail and explain the main results. Suppose that $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid. This means that G, M are smooth manifolds, equipped with submersions $\alpha, \beta : G \to M$ and an associative and smooth multiplication $G \times_{\alpha,\beta} G \to G$ that admits a smooth identity map $1 : M \to G$ and a smooth inversion $\iota : G \to G$. Then the bisections $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ are the sections $\sigma : M \to G$ of α such that $\beta \circ \sigma$ is a diffeomorphism of M. This becomes a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \quad \text{for } x \in M.$$

If M is compact, G is modelled on a metrisable space and the groupoid $\mathcal{G}$ admits an adapted local addition (cf. [11, 18]), then this group is a submanifold of the space of smooth maps $C^\infty(M, G)$ and thus a Lie group (cf. [18]). The additional structure provided by a local addition allow us to turn spaces of smooth maps into (infinite-dimensional) manifolds. Using this, one can circumvent to a certain degree that the category Man of (possibly infinite-dimensional) manifolds is not cartesian closed. Moreover, the map $\beta_* : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M), \sigma \mapsto \beta \circ \sigma$ is a Lie group morphism which induces a canonical action of the bisections on M. This Lie group morphism and the associated action are the key ingredients to define the (re-)construction functors and suitable versions of the bisection functor (cf. [18, Section 3]).

Our first aim is to investigate the so-called reconstruction functor. Since $\mathrm{Bis}(\mathcal{G})$ acts on M, we can associate an action groupoid $\mathcal{B}(\mathcal{G}) := (\mathrm{Bis}(\mathcal{G}) \ltimes M \rightrightarrows M)$ to this action. Denote by $\mathsf{LieGroupoids}_M^\Sigma$ the (1-)category with objects locally metrisable Lie groupoids with object manifold M which admit an adapted local addition and arrows the identity-on-objects functors. Then the construction of the action groupoid gives rise to an endofunctor

$$\mathcal{B} : \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroupoids}_M^\Sigma.$$

In [19, Theorem 2.21], we have already observed that certain Lie groupoids $\mathcal{G}$ can be recovered as a quotient in $\mathsf{LieGroupoids}_M^\Sigma$ from the action groupoid $\mathcal{B}(\mathcal{G})$. Furthermore, the joint evaluations

$$\mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to G, \quad (\sigma, m) \mapsto \sigma(m)$$

(for $\mathcal{G} = (G \rightrightarrows M)$) induce a natural transformation from $\mathcal{B}$ to the identity.

To understand the relation of the endofunctor $\mathcal{B}$ to the identity, consider the slice category $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ of morphisms from locally metrisable Lie groups into $\mathrm{Diff}(M)$. Then the construction of the bisection Lie group induces a pair of functors

$$\mathrm{Bis} : \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroups}_{\mathrm{Diff}(M)} \quad \text{and}$$
$$\ltimes : \mathsf{LieGroups}_{\mathrm{Diff}(M)} \to \mathsf{LieGroupoids}_M^\Sigma.$$

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The functor Bis sends a groupoid $\mathcal{G}$ to the morphism $\beta_* : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$, while $\ltimes$ constructs the action groupoid associated to the Lie group morphism $K \to \mathrm{Diff}(M)$ and the natural action of $\mathrm{Diff}(M)$ on M. Note that by construction we have $\mathcal{B} = \ltimes \circ \mathrm{Bis}$. We discuss now the relation of the three functors and obtain the following result.

Theorem A. *The functor $\ltimes$ is left adjoint to the functor* Bis. *The adjunction is given by mapping the morphism $f : K \ltimes M \to \mathcal{G} = (G \rightrightarrows M)$, which is given by a smooth map $f : K \ltimes M \to G$, to the adjoint map $f^\wedge : K \to C^\infty(M, G)$, which happens to be an element of* $\mathrm{Bis}(\mathcal{G})$. *Consequently, the endofunctor $\mathcal{B} = \ltimes \circ \mathrm{Bis}$ is a comonad on* $\mathsf{LieGroupoids}_M^\Sigma$.

The endofunctor $\mathcal{B}$ is called a '*re*construction functor', as one can show that under certain assumptions the groupoid $\mathcal{G}$ is the quotient (in the category $\mathsf{LieGroupoids}_M^\Sigma$) of $\mathcal{B}(\mathcal{G})$ (see Proposition 3.11 for the exact statement).

We afterwards turn to the question of what additional data is needed in addition to a Lie group action $H \to \mathrm{Diff}(M)$ in order to build a Lie groupoid that has the Lie group H as its bisections. This is what we call the 'construction functor'. Observe that the reconstruction functor had a Lie groupoid as its input, which we then *re*constructed. So, the present question is significantly different. In order to simplify matters (for the moment), we will consider this question only for transitive (or more precisely locally trivial) Lie groupoids. Although locally trivial Lie groupoids correspond to gauge groupoids of principal bundles, many morphisms under consideration cannot be described as morphisms of principal bundles (with fixed structure group). However, one can treat these maps as morphisms of (locally trivial) Lie groupoids, whence we prefer the groupoid perspective in contrast to the principal bundle perspective.

To obtain the construction functor, we need to define first the notion of a *transitive pair*. Having already fixed the compact manifold M, we now choose and fix once and for all an element $m \in M$. From now on, we will assume that M is connected. A transitive pair (θ, H) consists of a transitive Lie group action $\theta : K \times M \to M$ and a normal subgroup H of the m-stabiliser K_m of θ, such that H is a regular and co-Banach Lie subgroup of K_m. Regularity (in the sense of Milnor) of Lie groups roughly means that a certain class of differential equations can be solved on the Lie group. This is an essential prerequisite for infinite-dimensional Lie theory (see [6] for more information). Up to this point, all known Lie groups modelled on suitably complete, that is, Mackey complete, spaces are regular. The guiding example here is the transitive pair

$$\overline{\mathrm{Bis}}(\mathcal{G}) := (\mathrm{Bis}(\mathcal{G}) \times M \to M,\ (\sigma, m) \mapsto \beta(\sigma(m)), \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\})$$

induced by the natural action of the bisections of a locally trivial Banach–Lie groupoid $\mathcal{G}$ over a connected manifold M. Transitive pairs (over M with respect to $m \in M$) together with a suitable notion of morphism form a category $\mathsf{TransPairs}_M$. Note that the functor sending a transitive pair (θ, H) to the adjoint morphism $\theta^\wedge : K \to \mathrm{Diff}(M)$ induces a forgetful functor from $\mathsf{TransPairs}_M$ to $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$.

Restricting our attention to the category $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ of locally trivial Banach–Lie groupoids over a connected manifold M, we obtain the *augmented bisection functor*

$$\overline{\mathrm{Bis}} : \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \to \mathsf{TransPairs}_M.$$

The augmented bisection functor descends via the forgetful functor to the category $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ to the functor Bis (restricted to locally trivial Banach–Lie groupoids).

The notion 'transitive pair' is tailored in exactly such a way that one can define a *construction functor*

$$\mathcal{R} : \mathsf{TransPairs}_M \to \mathsf{LieGroupoids}_M^{\Sigma},$$

which associates to a transitive pair a locally trivial Banach–Lie groupoid (see Section 4 for details). Moreover, the augmented bisection functor and the construction functor are closely connected. If we apply the functor $\mathcal{R}$ to $\overline{\mathrm{Bis}}(\mathcal{G})$, we obtain the open subgroupoid of $\mathcal{G}$ of all elements which are contained in the image of a bisection. Hence, if we assume that $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid with bisections through each arrow, that is, for all $g \in G$, there exists a bisection σ with $\sigma(\alpha(g)) = g$, then $\mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G}))$ is isomorphic to $\mathcal{G}$. Denote by $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ the full subcategory of all locally trivial Banach–Lie groupoids with bisections through each arrow. Then our results subsume the following theorem.

THEOREM B. *Let M be a connected and compact manifold. The functor $\mathcal{R}$ is left adjoint to the functor $\overline{\mathrm{Bis}}$. Furthermore, the functors induce an equivalence of categories*

$$\mathsf{TransPairs}_M \supseteq \overline{\mathrm{Bis}}(\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}) \cong \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}.$$

The category equivalence in Theorem B shows that transitive pairs completely describe locally trivial Banach–Lie groupoids which admit sections through each arrow. Hence, the geometric information can be reconstructed from the transitive pair. This result connects infinite-dimensional Lie theory and groupoid theory (or equivalently principal bundle theory) and we explore first applications of this result in [19, Section 5]. However, equally important is the fact that the functors $\mathcal{R}$ and $\overline{\mathrm{Bis}}$ form an adjoint pair. The adjointness relation allows one to generate a wealth of geometrically interesting morphisms from (infinite-dimensional) Lie groups into groups of bisections of locally trivial Banach–Lie groupoids.

Finally, we remark that results similar to Theorem B can also be obtained for a nonconnected manifold M. However, then one has to deal with several technical difficulties in the construction, forcing one to restrict to certain full subcategories. We have avoided this to streamline the exposition but will briefly comment on these results at the end of Section 5.

2. Locally convex Lie groupoids and the bisection functor

In this section, the Lie theoretic notions and conventions used throughout this paper are recalled. We refer to [10] for an introduction to (finite-dimensional) Lie groupoids and the associated group of bisections. The notation for Lie groupoids and their structural maps also follows [10]. However, we do not restrict our attention to finite-dimensional Lie groupoids. Hence, we have to augment the usual definitions with several comments. Note that we will work all the time over a fixed base manifold M.

We use the so-called Bastiani calculus (often also called Keller's C^r_c-theory). As the present paper explores the functorial aspects of certain (re)construction principles, details of the calculus (see [3, 4]) are of minor importance and thus omitted. However, there is a chain rule for smooth maps in this setting, whence there exists an associated concept of a locally convex manifold, that is, a Hausdorff space that is locally homeomorphic to open subsets of locally convex spaces with smooth chart changes. Recall that the associated category Man of locally convex (possibly infinite-dimensional) manifolds with smooth maps is not cartesian closed. See [4, 14, 20] for more details.

Definition 2.1. Let M be a smooth manifold. Then M is called a *Banach* (or *Fréchet*) manifold if all its modelling spaces are Banach (or Fréchet) spaces. The manifold M is called *locally metrisable* if the underlying topological space is locally metrisable (equivalently if all modelling spaces of M are metrisable).

Definition 2.2. Let $\mathcal{G} = (G \rightrightarrows M)$ be a groupoid over M with source projection $\alpha : G \to M$ and target projection $\beta : G \to M$. Then $\mathcal{G}$ is a *(locally convex and locally metrisable) Lie groupoid over M* if:

(a) the objects M and the arrows G are locally convex and locally metrisable manifolds;
(b) the smooth structure turns α and β into surjective submersions, that is, they are locally projections. This implies that the occurring fibre products are submanifolds of the direct products; see [20, Appendix C];
(c) the partial multiplication $m : G \times_{\alpha,\beta} G \to G$, the object inclusion $1 : M \to G$ and the inversion $\iota : G \to G$ are smooth.

Define $\mathsf{LieGroupoids}_M$ to be the category of all (locally convex and locally metrisable) Lie groupoids over M, where the morphisms are given by Lie groupoid morphisms over id_M, that is, the Lie groupoid morphisms $\phi : \mathcal{G} \to \mathcal{H}$ with $\phi \circ 1_{\mathcal{G}} = 1_{\mathcal{H}}$ (cf. [10, Definition 1.2.1]).

Definition 2.3. Let $\mathcal{G}$ be a locally convex and locally metrisable Lie groupoid. The *group of bisections* $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ is given as the set of sections $\sigma : M \to G$ of α such that $\beta \circ \sigma : M \to M$ is a diffeomorphism. This is a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \quad \text{for } x \in M.$$

The object inclusion $1 : M \to G$ is then the neutral element and the inverse element of σ is

$$\sigma^{-1}(x) := \iota(\sigma((\beta \circ \sigma)^{-1}(x))) \quad \text{for } x \in M.$$

If $\mathcal{G}$ is a Lie groupoid over a compact base M, then [18] establishes Lie group structure on the group of bisections if $\mathcal{G}$ admits a certain type of local addition. We recall these results now. A local addition is a tool used to construct a manifold structure on a space of smooth mappings, whence under certain circumstances we can circumvent the fact that Man is not cartesian closed (see for example [8, 11, 20]).

Remark 2.4. A natural question at this point is whether there is any higher categorical structure around that the bisections are a part of. However, in the present setup, bisections form a group, without any additional categorical structure. This comes from the fact that their categorical description is as *atlas automorphisms* of the atlas $M \to \mathfrak{G}$ of the smooth stack $\mathfrak{G} : \mathsf{Man} \to \mathsf{Grpds}$, $X \mapsto \mathrm{Bun}(X, \mathcal{G})$ that is derived from the Lie groupoid $\mathcal{G}$ (cf. [7]). In this setup, a bisection is given by an automorphisms of $M \to \mathfrak{G}$ in the slice category $\mathsf{SmoothStacks}_{\mathfrak{G}}$, that is, by a commuting triangle

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr] & \arrow[d, phantom, "\overset{\Leftarrow}{\sigma}"] & M \arrow[dl] \\ & \mathfrak{G} & \end{tikzcd}$$

for a diffeomorphism φ and a smooth natural transformation σ. This is exactly the same information as in the above definition of a bisection of $\mathcal{G}$.

Since there are no 2-morphisms between 1-morphisms from M to M (because M is 0-truncated), the endomorphisms in $\mathsf{SmoothStacks}_{\mathfrak{G}}$ of $M \to \mathfrak{G}$ form a monoid (rather than a monoidal category). Thus, bisections naturally form a group.

Definition 2.5. Suppose that N is a smooth manifold. Then a *local addition* on N is a smooth map $\Sigma : U \Subset TN \to N$, defined on an open neighbourhood U of the submanifold $N \subseteq TN$, such that:

(a) $\pi \times \Sigma : U \to N \times N$, $v \mapsto (\pi(v), \Sigma(v))$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta N \subseteq N \times N$; and
(b) $\Sigma(0_n) = n$ for all $n \in N$.

We say that N *admits a local addition* if there exists a local addition on N.

To turn the subset $\mathrm{Bis}(\mathcal{G})$ of $C^\infty(M, G)$ into a manifold, we need to require that the local addition is adapted to the groupoid structure maps.

Definition 2.6 (cf. [11, Section 10.6]). Let $s : Q \to N$ be a surjective submersion. Then a *local addition adapted to* s is a local addition $\Sigma : U \Subset TQ \to Q$ such that the fibres of s are additively closed with respect to Σ, that is, $\Sigma(v_q) \in s^{-1}(s(q))$ for all $q \in Q$ and $v_q \in T_q s^{-1}(s(q))$ (note that $s^{-1}(s(q))$ is a submanifold of Q).

Definition 2.7. A Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ admits an *adapted local addition* if G admits a local addition which is adapted to the source projection α (or, equivalently, to the target projection β). Denote by $\mathsf{LieGroupoids}_M^\Sigma$ the full subcategory of $\mathsf{LieGroupoids}_M$ whose objects are Lie groupoids over M that admit an adapted local addition.

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An adapted local addition exists for every Banach–Lie groupoid by [18, Proposition 3.12]. Thus, for the category $\mathsf{BanachLieGpds}_M$ of Banach–Lie groupoids over M, this implies that

$$\mathsf{BanachLieGpds}_M \subseteq \mathsf{LieGroupoids}_M^\Sigma.$$

Furthermore, we recall the following facts on locally convex Lie groupoids and the Lie group structure of their bisection groups (cf. [18, Section 3]).

Proposition 2.8. *Suppose that M is compact and $\mathcal{G} = (G \rightrightarrows M)$ is a locally convex and locally metrisable Lie groupoid over M which admits an adapted local addition. Then* $\mathrm{Bis}(\mathcal{G})$ *is a submanifold of $C^\infty(M, G)$ and this structure turns* $\mathrm{Bis}(\mathcal{G})$ *into a Lie group.*

This construction gives rise to the bisection functor.

Definition 2.9. Suppose that M is a compact manifold. Then the construction of the Lie group of bisections gives rise to a functor

$$\mathrm{Bis} : \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroups},$$

sending a groupoid $\mathcal{G}$ to $\mathrm{Bis}(\mathcal{G})$ and a groupoid morphism $\varphi : \mathcal{G} \to \mathcal{G}'$ to $\mathrm{Bis}(\varphi) : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G}')$, $\sigma \mapsto \varphi \circ \sigma$. Here $\mathsf{LieGroups}$ denotes the category of locally convex Lie groups.

In the following sections, we will study the bisection functor and its relation to functors, which we call (re-)construction functors. The leading idea here is that to a certain extent it is possible to reconstruct Lie groupoids from their group of bisections (cf. [19]).

3. The bisection functor and the reconstruction functor

In this section, we study the reconstruction functor, which arises from the canonical action of the bisections on the base manifold. We will assume throughout this section that M is compact and $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid in the category $\mathsf{LieGroupoids}_M^\Sigma$, that is, $\mathcal{G}$ is a locally metrisable Lie groupoid which admits an adapted local addition.

Definition 3.1 (The bisection action groupoid). The Lie group $\mathrm{Bis}(\mathcal{G})$ has a natural smooth action on M, induced by $(\beta_{\mathcal{G}})_* : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$ and the natural action of $\mathrm{Diff}(M)$ on M. This gives rise to the action Lie groupoid $\mathcal{B}(\mathcal{G}) := \mathrm{Bis}(\mathcal{G}) \ltimes M$, with source and target projections defined by $\alpha_{\mathcal{B}}(\sigma, m) = m$ and $\beta_{\mathcal{B}}(\sigma, m) = \beta_{\mathcal{G}}(\sigma(m))$. The multiplication on $\mathcal{B}(\mathcal{G})$ is defined by

$$(\sigma, \beta_{\mathcal{G}}(\tau(m))) \cdot (\tau, m) := (\sigma \star \tau, m).$$

Remark 3.2. Clearly, any morphism $f : \mathcal{G} \to \mathcal{H}$ of Lie groupoids over M induces a morphism $\mathrm{Bis}(f) \times \mathrm{id}_M : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of Lie groupoids.

Moreover, $\mathcal{B}(\mathcal{G})$ also admits an adapted local addition (for $\alpha_{\mathcal{B}}$ and thus also for $\beta_{\mathcal{B}}$). In fact, this is the case for the Lie group $\mathrm{Bis}(\mathcal{G})$ and the finite-dimensional manifold

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M separately and, on $\mathrm{Bis}(\mathcal{G}) \times M$, one can simply take the product of these local additions. We may thus interpret $\mathcal{B}$ as an endofunctor

$$\mathcal{B} : \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroupoids}_M^\Sigma.$$

In addition, the evaluation map $\mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to G, (\sigma, m) \mapsto \sigma(m)$ is a morphism of Lie groupoids over M:

$$\left(\begin{array}{c} \mathrm{Bis}(\mathcal{G}) \times M \\ \alpha_{\mathcal{B}} \downarrow\ \downarrow \beta_{\mathcal{B}} \\ M \end{array}\right) \xrightarrow{\mathrm{ev}} \left(\begin{array}{c} G \\ \alpha_{\mathcal{G}} \downarrow\ \downarrow \beta_{\mathcal{G}} \\ M \end{array}\right),$$

which we may interpret as a natural transformation $\mathrm{ev} : \mathcal{B} \Rightarrow \mathrm{id}$.

In order to understand the categorical structure of the functor $\mathcal{B}$ and the natural transformation $\mathrm{ev} : \mathcal{B} \Rightarrow \mathrm{id}$, we augment the bisection functor to a functor into a certain slice category.

Definition 3.3. Define the slice category $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$, in which objects are locally convex and locally metrisable Lie groups K that are equipped with a homomorphism $\varphi : K \to \mathrm{Diff}(M)$. A morphism from $\varphi : K \to \mathrm{Diff}(M)$ to $\varphi' : K \to \mathrm{Diff}(M)$ is a morphism of locally convex Lie groups $\psi : K \to K'$ such that $\varphi = \varphi' \circ \psi$.

Clearly, the functor Bis induces a functor

$$\begin{aligned} \mathrm{Bis} : \mathsf{LieGroupoids}_M^\Sigma &\to \mathsf{LieGroups}_{\mathrm{Diff}(M)}, \\ (G \rightrightarrows M) &\mapsto (\beta_* : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)). \end{aligned}$$

By abuse of notation, we will also call this functor the *bisection functor*, as the bisection functor defined in Definition 2.9 can be recovered by an application of the forgetful functor $\mathsf{LieGroups}_{\mathrm{Diff}(M)} \to \mathsf{LieGroups}$.

Definition 3.4. For each object $\varphi : K \to \mathrm{Diff}(M)$ of $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$, we can construct the action groupoid $K \ltimes M$, which admits an adapted local addition by the same argument as for the bisection action groupoid in Remark 3.2. This gives rise to a functor

$$\ltimes : \mathsf{LieGroups}_{\mathrm{Diff}(M)} \to \mathsf{LieGroupoids}_M^\Sigma$$

which maps a morphism ψ in $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ to $\psi \times \mathrm{id}_M$.

The construction of $\ltimes$ and the bisection functor above is tailored to yield $\mathcal{B} = \ltimes \circ \mathrm{Bis}$.

Theorem 3.5. *The functor $\ltimes$ is left adjoint to the functor* Bis. *The adjunction is given by mapping the morphism $f : K \ltimes M \to \mathcal{G} = (G \rightrightarrows M)$, which is given by a smooth map $f : K \ltimes M \to G$, to the adjoint map $f^\wedge : K \to C^\infty(M, G)$, which happens to be an element of* $\mathrm{Bis}(\mathcal{G})$.

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PROOF. It is clear that $f \mapsto f^\wedge$ is natural and injective, since this is also the case on the level of morphisms of sets. Thus, it remains to show that it is surjective and well defined (that is, $f^\wedge(k)$ is in fact a bisection for each $k \in K$ and $k \mapsto f^\wedge(k)$ is a homomorphism).

To show surjectivity, let $\psi : K \to \mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M, G)$ be some Lie group morphism. Then the other adjoint $\psi^\vee : K \times M \to G$ is smooth and satisfies

$$\begin{aligned}\psi^\vee((kk', m)) &= \psi(kk')(m) = (\psi(k) \star \psi(k'))(m)\\ &= \psi(k)(\beta(\psi(k')(m))) \cdot \psi(k')(m)\\ &= \psi(k)(k'.m) \cdot \psi(k')(m) = \psi^\vee((k, k'.m)) \cdot \psi^\vee(k', m).\end{aligned} \tag{3.1}$$

Thus, $\psi^\vee$ is a morphism of Lie groupoids and we have $\psi = (\psi^\vee)^\wedge$.

To verify that $f \mapsto f^\wedge$ is well defined, we first observe that for each $k \in K$, the map $m \mapsto \beta(f^\wedge(k)(m)) = \varphi(k)(m)$ is a diffeomorphism by the assumption $\varphi : K \to \mathrm{Diff}(M)$. It is clear that $f^\wedge$ is smooth and that it is a homomorphism of groups follows from the adjoint equation to (3.1). □

REMARK 3.6. The co-unit of the adjunction $\ltimes \dashv \mathrm{Bis}$ is given by the natural transformation

$$\mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \ltimes M \to G.$$

Indeed, if we take the adjoint $\mathrm{id}^\vee$ of the identity $\mathrm{id} : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$, then we get exactly ev. Likewise, the unit of $\ltimes \dashv \mathrm{Bis}$ is given by the natural transformation

$$\mathrm{const} : (\varphi : K \to \mathrm{Diff}(M)) \longrightarrow ((\beta_{K \ltimes M})_* : \mathrm{Bis}(K \ltimes M) \to \mathrm{Diff}(M))$$

which maps an element $k \in K$ to the 'constant' bisection $m \mapsto (k, m)$ of $K \ltimes M$.

COROLLARY 3.7. *The functor* $\mathcal{B} = \ltimes \circ \mathrm{Bis}$ *gives rise to a comonad* $(\mathcal{B}, \mathrm{ev} : \mathcal{B} \Rightarrow \mathrm{id}, \ltimes(\mathrm{const}\,\mathrm{Bis})) : \mathcal{B} \Rightarrow \mathcal{B} \circ \mathcal{B})$ *(see* [2, Chapter 3, Proposition 1.6]*). Here, for* $\mathcal{G} \in \mathsf{LieGroupoids}_M^\Sigma$, *the natural transformation* $\ltimes(\mathrm{const}_{\mathrm{Bis}(\mathcal{G})})$ *is given by the formula*

$$\begin{aligned}\ltimes(\mathrm{const}_{\mathrm{Bis}(\mathcal{G})}) : \mathrm{Bis}(\mathcal{G}) \ltimes M &\to \mathrm{Bis}(\mathrm{Bis}(\mathcal{G}) \ltimes M) \ltimes M,\\ (\sigma, m) &\mapsto (x \mapsto (\sigma, x), m).\end{aligned}$$

COROLLARY 3.8. *The functor* $\mathrm{Bis} : \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroups}_{\mathrm{Diff}(M)}$ *preserves limits. In particular, kernels and pull-backs are preserved.*

At the end of this section, we would like to recall briefly some results from [19] to make sense of the term 'reconstruction functor' for the endofunctor $\mathcal{B}$. The idea behind this is that in certain circumstances, one can recover the groupoid $\mathcal{G}$ from its bisection action groupoid $\mathcal{B}(\mathcal{G})$ via the natural transformation ev. To make this explicit, recall the notion of a quotient object in a category.

REMARK 3.9. Each category carries a natural notion of quotient object for an internal equivalence relation. If C is a category with finite products and $R \subseteq E \times E$ is an

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internal equivalence relation, then the quotient $E \to E/R$ in C (uniquely determined up to isomorphism) is, if it exists, the co-equaliser of the diagram

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} E\,. \tag{3.2}$$

If, in the case that the quotient exists, (3.2) is also the pull-back of $E \to E/R$ along itself, then the quotient $E \to E/R$ is called *effective* (see [9, Appendix 1] for details).

In general, $\mathcal{G}$ will only be a quotient of $\mathcal{B}(\mathcal{G})$ if there are enough bisections, that is, if for every arrow $g \in G$ (with $\mathcal{G} = (G \rightrightarrows M)$, there is a bisection $\sigma_g \in \mathrm{Bis}(\mathcal{G})$ with $\sigma_g(\alpha(g)) = g$. If $\mathcal{G}$ is a Lie groupoid with this property, we say that $\mathcal{G}$ *admits bisections through each arrow*. A sufficient criterion for this is that the groupoid $\mathcal{G}$ is source connected, that is, that for each $m \in M$, the source fibre $\alpha^{-1}(m)$ is a connected manifold.

LEMMA 3.10 ([19, Theorem 2.14]). *If $\mathcal{G}$ is a source connected Lie groupoid in* $\mathsf{LieGroupoids}^{\Sigma}_{M}$ *over a compact base M, then $\mathcal{G}$ admits bisections through each arrow.*

Not all Lie groupoids admit bisections through each arrow. For example, the pair groupoid of a disjoint union of nondiffeomorphic manifolds does not admit bisections through each arrow (see [19, Remark 2.18(b)] for the details). Notice that the situation is quite different if one replaces bisections with local bisections. A *local bisection* of a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is a smooth map $\sigma : U \Subset M \to G$ such that $\alpha_{\mathcal{G}} \circ \sigma = \mathrm{id}_U$ and $\beta_{\mathcal{G}} \circ \sigma$ is a diffeomorphism onto its open image in M. One can prove that every locally convex Lie groupoid over a finite-dimensional base admits local bisections through each arrow (cf. [19, Appendix A]).

In [19], we were then able to prove the following result on groupoids with enough bisections.

PROPOSITION 3.11 ([19, Theorem 2.21]). *If $\mathcal{G}$ is a Lie groupoid with a bisection through each arrow in G, for example if $\mathcal{G}$ is source connected, then the morphism* $\mathrm{ev} : \mathcal{B}(\mathcal{G}) \to \mathcal{G}$ *is the quotient in* $\mathsf{LieGroupoids}^{\Sigma}_{M}$ *of $\mathcal{B}(\mathcal{G})$ by*

$$R = \{(\sigma, m), (\tau, m) \in \mathrm{Bis}(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G}) \times M \mid \sigma(m) = \tau(m)\}.$$

4. Transitive pairs and the construction functor

In this section, we define categories of Lie groups with transitive actions and functors between these categories and categories of Lie groupoids over M. As always, M will be a fixed compact manifold and we shall consider only Lie groupoids in $\mathsf{LieGroupoids}^{\Sigma}_{M}$.

DEFINITION 4.1 (Transitive pair). Choose and fix once and for all a point $m \in M$. Let $\theta : K \times M \to M$ be a transitive (left-)Lie group action of a Lie group K modelled on a metrisable space and H be a subgroup of K.

Then we call (θ, H) a *transitive pair* (over M with base point m) if the following conditions are satisfied:

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(P1) the map $\theta_m := \theta(\cdot, m)$ is a surjective submersion;
(P2) H is a normal Lie subgroup of the stabiliser K_m of m and this structure turns H into a regular Lie group which is co-Banach as a submanifold in K_m.

The largest subgroup of H which is a normal subgroup of K is called the *kernel* of the transitive pair. Note that by [19, Proposition 4.16], every transitive pair admits a kernel (cf. the comments before Corollary 4.13). By standard arguments for topological groups, the kernel is a closed subgroup. In general, this will not entail that it is a closed Lie subgroup (of the infinite-dimensional Lie group K). If the action of K on M is also n-fold transitive, then we call (θ, H) an *n-fold transitive pair*.

Transitive pairs have been studied in [19] in the context of reconstructions of locally trivial Lie groupoids. Conceptually, they are closely related to Klein geometries and we refer to [19] for more information on this topic. In the context of the present paper, we reconsider the construction of a locally trivial Lie groupoid from a transitive pair. It will turn out that this construction is functorial on a suitable category of transitive pairs, which we introduce now.

Definition 4.2 (Category of transitive pairs). We define the category $\mathsf{TransPairs}_M$ which has as objects transitive pairs (over M with base point m).

A morphism in $\mathsf{TransPairs}_M$ from $(\theta : K \times M \to M, H)$ to $(\theta' : K' \times M \to M, H')$ is a smooth morphism of Lie groups $\varphi : K \to K'$ such that $\varphi(H) \subseteq H'$ and $\theta'^{\wedge} \circ \varphi = \theta^{\wedge}$. Here $\theta^{\wedge} : K \to \mathrm{Diff}(m), \quad k \mapsto \theta(k, \cdot)$ is the Lie group morphism associated to θ via the exponential law (see [20, Theorem 7.6]).

Remark 4.3. By construction, there is a forgetful functor

$$\mathrm{For} : \mathsf{TransPairs}_M \to \mathsf{LieGroups}_{\mathrm{Diff}(M)}$$

which sends a transitive pair $(\theta : K \times M \to M, H)$ to $\theta^{\wedge} : K \to \mathrm{Diff}(M)$ and a morphism $\varphi : (\theta, H) \to (\theta', H')$ to the underlying Lie group morphism.

Lemma 4.4. *A morphism*

$$\varphi : (\theta : K \times M \to M, H) \to (\theta' : K' \times M \to M, H')$$

in $\mathsf{TransPairs}_M$ *is an isomorphism if and only if the underlying morphism of Lie groups* $\varphi : K \to K'$ *is an isomorphism with* $\varphi^{-1}(H') \subseteq H$.

Proof. The condition is clearly necessary. Conversely, assume that $\varphi : K \to K'$ is an isomorphism of Lie groups with inverse $\psi : K' \to K$. Now ψ maps H' into H and it induces a morphism of transitive pairs since

$$\theta^{\wedge} \circ \psi = \theta'^{\wedge} \circ \varphi \circ \psi = \theta'^{\wedge}. \qquad \square$$

Let us now exhibit two examples of transitive pairs. To this end, recall the notion of a locally trivial Lie groupoid.

Definition 4.5. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then we call $\mathcal{G}$ *locally trivial* if the *anchor map* $(\beta, \alpha) : G \to M \times M, g \mapsto (\beta(g), \alpha(g))$ is a surjective submersion.

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EXAMPLE 4.6.

(a) Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid over a compact manifold M. Denote by Vert_m the vertex subgroup of the groupoid $\mathcal{G}$. By [18, Proposition 3.12], $\mathcal{G}$ admits an adapted local addition, whence $\mathrm{Bis}(\mathcal{G})$ becomes a Lie group and [19, Propositions 3.2 and 3.4] shows that

$$\mathrm{Loop}_m(\mathcal{G}) := \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) \in \mathrm{Vert}_m = \alpha^{-1}(m) \cap \beta^{-1}(m)\},$$
$$\mathrm{Bis}_m(\mathcal{G}) := \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}$$

are regular Lie subgroups of $\mathrm{Bis}(\mathcal{G})$ such that $\mathrm{Bis}_m(\mathcal{G})$ is a normal co-Banach Lie subgroup of $\mathrm{Loop}_m(\mathcal{G})$. Note that $\beta \circ \mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to M$ is a Lie group action whose m-stabiliser is $\mathrm{Loop}_m(\mathcal{G})$. Recall from [19, Example 4.2(a)] that the pair $(\beta \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$ is a transitive pair if $\beta \circ \mathrm{ev}$ is a transitive Lie group action. In particular, $(\beta \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$ is a transitive pair if M is a connected manifold (see Lemma 5.1) or, for each $g \in G$, there is a bisection σ_g with $\sigma_g(\alpha(g)) = g$.

The first example motivated the definition of a transitive pair and is closely tied to the quotient process outlined in the previous section. However, one has considerable freedom in choosing the ingredients for such a pair.

(b) Let $\mathrm{Diff}(M)$ be the diffeomorphism group of a compact and connected manifold M. Consider an extension of Lie groups

$$\mathbf{1} \longrightarrow B \xrightarrow{i} K \xrightarrow{q} \mathrm{Diff}(M) \longrightarrow \mathbf{1}$$

of $\mathrm{Diff}(M)$ by a Banach–Lie group B, that is, K is a smooth (locally trivial) principal B-bundle over $\mathrm{Diff}(M)$. We refer to [14, Section V] and [12, 13, 15] for more details of extensions of infinite-dimensional Lie groups.
Now the natural action of $\mathrm{Diff}(M)$ on M induces a transitive K-action via $\theta : K \times M \to M, (k, m) \mapsto q(k)(m)$. Fix $m \in M$ and observe that θ_m is a submersion as $\theta_m = \mathrm{ev}_m \circ q$ is the composition of submersions $\mathrm{ev}_m : \mathrm{Diff}(M) \to M$ and q (cf. [5, Lemma 1.3]). By construction, $K_m = q^{-1}(\mathrm{Diff}_m(M))$. Observe that since q is a submersion, K_m is a closed Lie subgroup of K and indeed a Lie group extension of $\mathrm{Diff}_m(M)$ by B. Since B and $\mathrm{Diff}_m(M)$ are regular Lie groups and regularity is an extension property (see [16, Appendix B]), $H := K_m$ is a regular (and normal) Lie subgroup of K_m. Here we have used that $\mathrm{Diff}(M) = \mathrm{Bis}(\mathcal{P}(M))$ is regular and $\mathrm{Bis}_m(\mathcal{P}(M)) = \mathrm{Diff}_m(M)$. Moreover, B is regular as a Banach–Lie group [14]. We conclude that $(\theta, q^{-1}(\mathrm{Diff}_m(M)))$ is a transitive pair. By construction, the kernel of the transitive pair contains $i(B)$.

(c) Consider a transitive action $K \times M \to M$ of a finite-dimensional Lie group on a compact manifold. Then, by [19, Remark 4.3], a transitive pair (θ, H) is given by any normal subgroup H of K_m. As a special case, consider the canonical action $SO(3) \times S^2 \to S^2$ of the special orthogonal group $SO(3)$ on the 2-sphere (canonically embedded in $\mathbb{R}^3$). This action is transitive with abelian stabiliser $SO(2) \cong S^1$. Hence, we can choose as H any closed subgroup of S^1.

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In particular, choose as H either S^1 or the cyclic subgroups generated by an element with $x^n = 1$ for some $n \in \mathbb{N}$.

Our goal is now to obtain a functor which associates to a transitive pair a locally trivial Lie groupoid. To this end, we have to recall some results from [19, Section 4].

Proposition 4.7. *Let (θ, H) be a transitive pair.*

(a) *Then the quotients K/H and $\Lambda_m := K_m/H$ are Banach manifolds (such that the quotient maps become submersions). Moreover, the map θ_m induces a Λ_m-principal bundle $\pi : K/H \to M, kH \mapsto \theta(k, m)$.*

(b) *We denote the gauge groupoid associated to the Λ_m-principal bundle by*

$$\mathcal{R}(\theta, H) := \left(\begin{array}{c} \frac{K/H \times K/H}{\Lambda_m} \\ \alpha_{\mathcal{R}} \downarrow \downarrow \beta_{\mathcal{R}} \\ M. \end{array} \right).$$

Its structure maps are given by $\alpha_{\mathcal{R}}(\langle gH, kH \rangle) = \pi(kH)$ and $\beta_{\mathcal{R}}(\langle gH, kH \rangle) = \pi(gH)$.

Remark 4.8. Let (θ, H) be a transitive pair with $\theta : K \times M \to M$.

(a) Observe that as K/H is a Banach manifold, the groupoid $\mathcal{R}(\theta, H)$ is a Banach–Lie groupoid and thus $\mathcal{R}(\theta, H) \in \mathsf{LieGroupoids}^{\Sigma}_{M}$ by [18, Proposition 3.12]. Moreover, the gauge groupoid $\mathcal{R}(\theta, H)$ is source connected if and only if K/H is connected.

(b) Choose a section atlas $(\sigma_i, U_i)_{i \in I}$ of $\theta_m : K \xrightarrow{K_m} M$, that is, a family of sections of θ_m such that $M = \bigcup_{i \in I} U_i$ (since θ_m is a submersion, such an atlas exists). Composing the σ_i with the quotient map $p : K \to K/H$, we obtain a section atlas $s_i := p \circ \sigma_i : U_i \to \pi^{-1}(U_i) \subseteq K/H$ of $\pi : K/H \xrightarrow{\Lambda_m} M$ and thus identify the bisections of $\mathcal{R}(\theta, H)$ with bundle automorphisms via

$$\begin{aligned} \operatorname{Aut}(\pi : K/H \to M) &\to \operatorname{Bis}(\mathcal{R}(\theta, H)), \\ f &\mapsto (m \mapsto \langle f(s_i(m)), s_i(m) \rangle, \text{ if } m \in U_i \end{aligned} \tag{4.1}$$

(cf. [18, Example 3.16]).

(c) Recall that the sections described in (b) also induce manifold charts for $(K/H \times K/H)/\Lambda_m$ via

$$\begin{aligned} \frac{\pi^{-1}(U_i) \times \pi^{-1}(U_j)}{\Lambda_m} &\to U_i \times U_j \times \Lambda_m, \\ \langle p_1, p_2 \rangle &\mapsto (\pi(p_1), \pi(p_2), \delta(s_i(\pi(p_1)), p_1)\delta(s_j(\pi(p_2), p_2)^{-1}), \end{aligned} \tag{4.2}$$

where $\delta : K/H \times_{\pi} K/H \to \Lambda_m$ is the smooth map mapping a pair (p, q) to the element $p^{-1} \cdot q \in \Lambda_m$ which maps p to q (via the Λ_m-right action).

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In Section 2, we have seen that the category $\mathsf{BanachLieGpds}_M$ of all Banach–Lie groupoids over M is contained in $\mathsf{LieGroupoids}_M^\Sigma$. Let us define now a suitable subcategory of $\mathsf{BanachLieGpds}_M$ together with a functor.

DEFINITION 4.9. (a) Define the full subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ of $\mathsf{BanachLieGpds}_M$ whose objects are

$$\mathrm{Ob}\,\mathsf{BanachLieGpds}_M^{\mathrm{triv}} := \text{locally trivial Banach–Lie groupoids over } M.$$

(b) Define a functor

$$\mathcal{R} : \mathsf{TransPairs}_M \to \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \subseteq \mathsf{LieGroupoids}_M^\Sigma$$

which maps (θ, H) to the locally trivial Lie groupoid $\mathcal{R}(\theta, H)$ (cf. Proposition 4.7) and a morphism $\varphi : (\theta, H) \to (\theta', H')$ in $\mathsf{TransPairs}_M$ to $\mathcal{R}(\varphi) : \mathcal{R}(\theta, H) \to \mathcal{R}(\theta', H'), \langle kH, gH\rangle \mapsto \langle \varphi(k)H', \varphi(g)H'\rangle$.

The functor $\mathcal{R}$ constructs Lie groupoids from transitive pairs. These Lie groupoids are intimately connected to the transitive action of the transitive pair on M. To see this, we recall some results on a natural Lie group morphism induced by first applying $\mathcal{R}$ and then the bisection functor.

LEMMA 4.10 (see [19, Lemma 4.11]). *Let (θ, H) be a transitive pair. Then the action of K on K/H by left multiplication gives rise to a group homomorphism $K \to \mathrm{Aut}(\pi : K/H \xrightarrow{\Lambda_m} M)$. With respect to the canonical isomorphism $\mathrm{Aut}(\pi : K/H \xrightarrow{\Lambda_m} M) \cong \mathrm{Bis}(\mathcal{R}(\theta, H))$ of Lie groups from (4.1), this gives rise to the group homomorphism*

$$a_{\theta,H} : K \to \mathrm{Bis}(\mathcal{R}(\theta, H)), \quad k \mapsto (x \mapsto \langle k \cdot s_i(x), s_i(x)\rangle, \text{ for } x \in U_i),$$

where $s_i = p_m \circ \sigma_i, i \in I$ are the sections from 4.8(b). Moreover, $a_{\theta,H}$ is smooth and makes the diagram

$$\begin{tikzcd} K \arrow[r, "a_{\theta,H}"] \arrow[dr, "\theta^\wedge"'] & \mathrm{Bis}(\mathcal{R}(\theta, H)) \arrow[d, "(\beta_{\mathcal{R}})_*"] \\ & \mathrm{Diff}(M) \end{tikzcd}$$

commutative.

In general, the Lie group morphism $a_{\theta,H}$ will neither be injective nor surjective (this reflects that the notion of a transitive pair is quite flexible). However, one may understand the construction of $a_{\theta,H}$ as a way to obtain an interesting Lie group morphism from a transitive pair into the bisections of suitable locally trivial Lie groupoids over M. Moreover, under some assumptions on the transitive pair, the Lie group morphism $a_{\theta,H}$ lifts to a morphism of transitive pairs.

LEMMA 4.11. *Let (θ, H) be a transitive pair such that another transitive pair is given by $(\beta_{\mathcal{R}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{R}(\theta, H)))$. Then $a_{\theta,H}$ induces a morphism of transitive pairs $(\theta, H) \to (\beta_{\mathcal{R}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{R}(\theta, H)))$.*

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Proof. In view of Lemma 4.10, we only have to prove that for $h \in H$, we have $a_{\theta,H}(h) \in \mathrm{Bis}_m(\mathcal{R}(\theta, H))$. By definition of $a_{\theta,H}$, we have $a_{\theta,H}(h)(m) = \langle hs_i(m), s_i(m)\rangle$ for the sections $s_i : U_i \to K/H$ of $\pi : K/H \to M$ discussed in Remark 4.8(b). Recall that π is induced by the action θ. Thus, $m = \pi(s_i(m)) = \theta(s_i(m), m)$ holds and $s_i(m) = k_m H \in K_m/H$. Using the K_m/H-principal bundle structure, the image in the gauge groupoid becomes $a_{\theta,H}(h)(m) = \langle hH, 1_K H\rangle = \langle 1_K H, 1_K H\rangle = 1_{\mathcal{R}(\theta,H)}(m)$. Summing up, $a_{\theta,H}(H) \subseteq \mathrm{Bis}_m(\mathcal{R}(\theta, H))$ and thus $a_{\theta,H}$ is a morphism of transitive pairs. □

Proposition 4.12. *Consider a morphism $\varphi : (\theta, H) \to (\tilde{\theta}, \tilde{H})$ in the category* $\mathsf{TransPairs}_M$. *Then, for $a_{\theta,H}$ as in Lemma 4.10, the following diagram in* $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ *commutes:*

$$\begin{tikzcd} K \arrow[r, "a_{\theta,H}"] \arrow[d, "\varphi"] & \mathrm{Bis}(\mathcal{R}(\theta, H)) \arrow[d, "\mathrm{Bis} \circ \mathcal{R}(\varphi)"] \\ \tilde{K} \arrow[r, "a_{\tilde{\theta},\tilde{H}}"] & \mathrm{Bis}(\mathcal{R}(\tilde{\theta}, \tilde{H})) \end{tikzcd}$$

Hence, $(a_{\theta,H})_{(\theta,H)\in \mathsf{TransPairs}_M}$ defines a natural transformation $\mathrm{For} \Rightarrow \mathrm{Bis} \circ \mathcal{R}$ *in* $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ *(where* For *is the forgetful functor from Remark 4.3).*

Proof. Choose a section atlas $(s_i : U_i \to \pi^{-1}(U_i) \subseteq K/H)_{i\in I}$ of the Λ_m-principal bundle $\pi : K/H \to M$ as in Remark 4.8(b). From the definition of a morphism in $\mathsf{TransPairs}_M$, we infer that $\tilde{\theta} \circ (\varphi \times \mathrm{id}_M) = \theta$. Thus, $\tilde{\pi} \circ (\varphi \circ \sigma_i) = \tilde{\theta}_m \circ (\varphi \circ \sigma_i) = \theta_m \circ \sigma_i = \pi \circ \sigma_i = \mathrm{id}_{U_i}$ holds and $(\varphi \circ \sigma_i)_{i\in I}$ is a section atlas of the $\tilde{K}_m$-bundle $\tilde{\pi} : \tilde{K} \to M$. This section atlas descends to a section atlas of $\tilde{K}/\tilde{H} \xrightarrow{\tilde{\Lambda}_m} M$, which we denote by abuse of notation as $(\varphi \circ s_i)_{i\in I}$. In the following, we will assume that $x \in U_i$ and the mappings are represented as in Remark 4.8(b) with respect to the section atlases $(s_i)_{i\in I}$ and $(\varphi \circ s_i)_{i\in I}$. Let us now compute the composition $a_{\tilde{\theta},\tilde{H}} \circ \varphi$ given in the diagram. Then we obtain for $k \in K$ the formula

$$\begin{aligned} a_{\tilde{\theta},\tilde{H}} \circ \varphi(k)(x) &= (x \mapsto \langle \varphi(k)\varphi(s_i(x)), \varphi(s_i(x))\rangle) \\ &= (x \mapsto \langle \varphi(k)\varphi(\sigma_i(x))\tilde{H}, \varphi(\sigma_i(x))\tilde{H}\rangle). \end{aligned}$$

Now we compute the other composition $(\mathrm{Bis}(\mathcal{R}(\varphi))) \circ a_{\theta,H}$ of morphisms in the diagram. We obtain

$$\begin{aligned} \mathrm{Bis}(\mathcal{R}(\varphi)) \circ a_{\theta,H}(k) &= (x \mapsto \mathcal{R}(\varphi)(\langle k \cdot s_i(x), s_i(x)\rangle)) \\ &= (x \mapsto \langle \varphi(k \cdot \sigma_i(x))\tilde{H}, \varphi(\sigma_i(x))\tilde{H}\rangle). \end{aligned}$$

Comparing the right-hand sides, the diagram commutes since $\varphi : K \to \tilde{K}$ is a Lie group morphism.

Lemma 4.10 implies that the morphisms $a_{\theta,H}$ (for (θ, H) in $\mathsf{TransPairs}_M$) are morphisms in $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$. Moreover, we have just seen that the family $(a_{\theta,H})_{(\theta,H)\in \mathsf{TransPairs}_M}$ is natural and thus defines a natural transformation $\mathrm{For} \Rightarrow \mathrm{Bis} \circ \mathcal{R}$ in $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$. □

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In [19, Proposition 4.16], the kernel of a transitive pair (θ, H) has been identified with the kernel of the Lie group morphism $a_{\theta,H}$. Hence, the preceding proposition shows that morphisms of transitive pairs map kernels of transitive pairs into kernels of transitive pairs.

Corollary 4.13. *Let $\varphi : (\theta, H) \to (\tilde{\theta}, \tilde{H})$ be a morphism of transitive pairs. Then φ maps the kernel of the transitive pair (θ, H) into the kernel of the transitive pair $(\tilde{\theta}, \tilde{H})$.*

Proof. Recall from [19, Proposition 4.16] that the kernel of the transitive pair (θ, H) coincides with the kernel of the Lie group morphism $a_{\theta,H}$. Hence, the assertion follows from the commutative diagram in Proposition 4.12. $\square$

In the next section, we will see that the family $(a_{\theta,H})_{(\theta,H)\in \mathsf{TransPairs}_M}$ induces a natural transformation in $\mathsf{TransPairs}_M$. This will establish a close connection between $\mathcal{R}$ and an augmented version of the bisection functor.

5. The augmented bisection functor and the locally trivial construction functor

In this section, we define the augmented bisection functor and establish its connection to the functor $\mathcal{R}$. Unless stated explicitly otherwise, we will assume throughout the whole section that M is a compact and *connected* manifold and consider only groupoids in $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$. Moreover, we choose and fix once and for all $m \in M$. Taking M to be a connected manifold allows us to ignore certain technicalities in the definition of the augmented bisection functor (see Definition 5.3). With suitable care, one can extend the results outlined in this section also for nonconnected M. However, then one has to restrict the occurring functors to suitable full subcategories of $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ and the statements become a lot more technical. We will briefly comment on this in Remark 5.16.

We have already seen in Example 4.6(a) that for a locally trivial Lie groupoid which satisfies an additional assumption, the action of the bisections and a certain subgroup yield a transitive pair. If M is a connected manifold, no additional assumption is needed and we obtain the following result.

Lemma 5.1. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid. Then $(\beta \circ \mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to M, \mathrm{Bis}_m(\mathcal{G}))$ is a transitive pair.*

Proof. The groupoid $\mathcal{G}$ is a locally trivial Lie groupoid, whence the image of $\beta_* : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M), \sigma \mapsto \beta \circ \sigma$ contains the identity component $\mathrm{Diff}(M)_0$ of $\mathrm{Diff}(M)$ (cf. [18, Example 3.16]). Since M is connected, [19, Corollary 2.17] implies that the Lie group action $\gamma : \mathrm{Diff}(M) \times M \to M, (\varphi, m) \mapsto \varphi(m)$ restricts to a transitive action of the group $\mathrm{Diff}(M)_0$. We deduce from $\beta \circ \mathrm{ev} = \gamma \circ \beta_*$ that $\beta \circ \mathrm{ev}$ is transitive, whence $(\beta \circ \mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to M, \mathrm{Bis}_m(\mathcal{G}))$ is a transitive pair by Example 4.6(a). $\square$

Note that the lemma asserts that for a locally trivial Lie groupoid $\mathcal{G}$ over a connected manifold, the canonical action of the bisection together with $\mathrm{Bis}_m(\mathcal{G})$ always yields a transitive pair. For a nonconnected base, this statement is false (see [19, Remark 2.18(b)] for an example).

Lemma 5.2. *Let* $\psi : \mathcal{G} \to \mathcal{G}'$ *be a morphism in* $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$. *Then* $\mathrm{Bis}(\psi) : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G}')$ *in* $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ *induces a morphism of transitive pairs* $(\beta_{\mathcal{G}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G})) \to (\beta_{\mathcal{G}'} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}'))$.

Proof. We only have to prove that $\mathrm{Bis}(\psi)$ maps the subgroup

$$\mathrm{Bis}_m(\mathcal{G})) = \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) = 1_{\mathcal{G}}(m)\}$$

to $\mathrm{Bis}_m(\mathcal{G}')$. However, since ψ is a groupoid morphism over M, $\psi \circ 1_{\mathcal{G}} = 1_{\mathcal{G}'}$. Hence, $\mathrm{Bis}(\psi)(\sigma)(m) = \psi \circ \sigma(m) = 1_{\mathcal{G}'}(m)$ for all $\sigma \in \mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Bis}(\varphi)(\mathrm{Bis}_m(\mathcal{G})) \subseteq \mathrm{Bis}_m(\mathcal{G}')$. $\square$

Using Lemmas 5.1 and 5.2, we can now define the augmented bisection functor.

Definition 5.3. The *augmented bisection functor* on $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ is defined as

$$\begin{aligned} \overline{\mathrm{Bis}} : \mathsf{BanachLieGpds}_M^{\mathrm{triv}} &\to \mathsf{TransPairs}_M, \\ \mathcal{G} &\mapsto (\beta \circ \mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to M, \mathrm{Bis}_m(\mathcal{G})), \\ (\mathcal{G} \xrightarrow{\psi} \mathcal{G}') &\mapsto \mathrm{Bis}(\psi). \end{aligned}$$

Remark 5.4. Composing the functor $\overline{\mathrm{Bis}}$ with the forgetful functor For from Remark 4.3, we obtain precisely the functor

$$\mathrm{Bis}\,|_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}} : \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \to \mathsf{LieGroups}_{\mathrm{Diff}(M)}.$$

The functor $\overline{\mathrm{Bis}}$ is closely related to the functor $\mathcal{R}$, which we discussed in the last section. To make this explicit, we recall results from [19].

Example 5.5. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid. We apply $\mathcal{R}$ to the associated transitive pair $\overline{\mathrm{Bis}}(\mathcal{G}) = (\beta \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$ to obtain the gauge groupoid $\mathcal{R}(\beta \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$. This gauge groupoid is related to the locally trivial Lie groupoid $\mathcal{G}$ via the groupoid homomorphism

$$\begin{aligned} \chi_{\mathcal{G}} : \frac{\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})}{\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})} &\to G, \\ \langle \sigma \,\mathrm{Bis}_m(\mathcal{G}), \tau \,\mathrm{Bis}_m(\mathcal{G}) \rangle &\mapsto \sigma(m) \cdot (\tau(m))^{-1}. \end{aligned} \tag{5.1}$$

Recall from [19, Lemma 4.21] that $\chi_{\mathcal{G}}$ induces an isomorphism of the gauge groupoid $\mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G}))$ onto the open and wide subgroupoid $\mathrm{ev}(\mathrm{Bis}(\mathcal{G}) \times M) = \{g \in G \mid g = \sigma(\alpha(g))$ for some $\sigma \in \mathrm{Bis}(\mathcal{G})\}$ (see [19, Theorem 2.14]).

By Lemma 4.11, the map $a_{\overline{\mathrm{Bis}}(\mathcal{G})} : \overline{\mathrm{Bis}}(\mathcal{G}) \to \overline{\mathrm{Bis}}(\mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G})))$ is a morphism of transitive pairs. Moreover, it is an isomorphism of transitive pairs with inverse $\overline{\mathrm{Bis}}(\chi_{\mathcal{G}})$. This follows from [19, Lemma 4.21 b)], which asserts that $a_{\overline{\mathrm{Bis}}(\mathcal{G})} : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G})))$ is an isomorphism of Lie groups with inverse $\mathrm{Bis}(\chi_{\mathcal{G}})$.

We will now establish that the (re-)construction functor $\mathcal{R}$ defined in the last section and the augmented bisection functor form a pair of adjoint functors. To this end, we will prove first that the morphisms $\chi_{\mathcal{G}}$ induce a natural transformation from $\mathcal{R} \circ \overline{\mathrm{Bis}}$ to the identity functor of $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$.

LEMMA 5.6. *The family* $(\chi_{\mathcal{G}})_{\mathcal{G}\in\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$ *of groupoid morphisms over* M *defined as in (5.1) forms a natural transformation* $\chi : \mathcal{R}\circ\overline{\mathrm{Bis}} \Longrightarrow \mathrm{id}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$.

PROOF. Consider a Lie groupoid morphism $\psi : \mathcal{G} \to \mathcal{G}'$. To see that $\chi = (\chi_{\mathcal{G}})_{\mathcal{G}\in\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$ is a natural transformation, we pick an element in $\mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G}))$ and compute

$$\begin{aligned}
\chi_{\mathcal{G}'} &\circ \mathcal{R}(\mathrm{Bis}(\psi))(\langle \sigma \,\mathrm{Bis}_m(\mathcal{G}), \tau \,\mathrm{Bis}_m(\mathcal{G})\rangle) \\
&= \chi_{\mathcal{G}'}(\langle (\psi\circ\sigma)\,\mathrm{Bis}_m(\mathcal{G}'), (\psi\circ\tau)\,\mathrm{Bis}_m(\mathcal{G}')\rangle) \\
&= \psi(\sigma(m))\cdot(\psi(\tau(m))^{-1} = \psi(\sigma(m)\cdot(\tau(m))^{-1}) \\
&= \psi\circ\chi_{\mathcal{G}}(\langle \sigma\,\mathrm{Bis}_m(\mathcal{G}), \tau\,\mathrm{Bis}_m(\mathcal{G})\rangle).
\end{aligned}$$

□

Combining Lemma 5.6 with the fact that $\mathrm{Bis}(\chi_{\mathcal{G}})$ is an isomorphism, one immediately obtains the following result.

COROLLARY 5.7. *The natural transformation* $\overline{\mathrm{Bis}}(\chi) = (\overline{\mathrm{Bis}}(\chi_{\mathcal{G}}))_{\mathcal{G}\in\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$ *is a natural isomorphism*

$$\overline{\mathrm{Bis}}(\chi) : \overline{\mathrm{Bis}}\circ\mathcal{R}\circ\overline{\mathrm{Bis}} \overset{\cong}{\Longrightarrow} \overline{\mathrm{Bis}}$$

whose inverse is given by $(a_{\overline{\mathrm{Bis}}(\mathcal{G})})_{\mathcal{G}\in\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$.

Having dealt with natural transformations in $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$, we now turn to natural transformations in $\mathsf{TransPairs}_M$. We will lift the natural transformation $(a_{\theta,H})_{\mathsf{TransPairs}_M}$ in $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$ (see Proposition 4.12) to a natural transformation in $\mathsf{TransPairs}_M$.

LEMMA 5.8. *Let* (θ, H) *and* $(\tilde\theta.\tilde H)$ *be transitive pairs and* $\varphi : (\theta, H) \to (\tilde\theta, \tilde H)$ *be a morphism in* $\mathsf{TransPairs}_M$.

(a) *Then we obtain a commutative diagram in* $\mathsf{TransPairs}_M$

$$\begin{tikzcd}
(\theta, H) \arrow[r, "a_{\theta,H}"] \arrow[d, "\varphi"] & \overline{\mathrm{Bis}}(\mathcal{R}(\theta, H)) \arrow[d, "\overline{\mathrm{Bis}}\circ\mathcal{R}(\varphi)"] \\
(\tilde\theta, \tilde H) \arrow[r, "a_{\tilde\theta,\tilde H}"] & \overline{\mathrm{Bis}}(\mathcal{R}(\tilde\theta, \tilde H))
\end{tikzcd} \tag{5.2}$$

Hence, the family $(a_{\theta,H})_{(\theta,H)\in\mathsf{TransPairs}_M}$ *forms a natural transformation* $\mathrm{id}_{\mathsf{TransPairs}_M} \Rightarrow \overline{\mathrm{Bis}}\circ\mathcal{R}$ *in the category* $\mathsf{TransPairs}_M$.

(b) *The map* $\mathcal{R}(a_{\theta,H}) : \mathcal{R}(\theta, H) \to \mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{R}(\theta, H)))$ *is a Lie groupoid isomorphism with inverse* $\chi_{\mathcal{R}(\theta,H)}$.

PROOF. (a) The Lie group map $a_{\theta,H}$ is a morphism of transitive pairs by Lemma 4.11. Hence, (5.2) makes sense as a diagram in $\mathsf{TransPairs}_M$. Applying the forgetful functor For to (5.2), Proposition 4.12 shows that we obtain a commutative diagram in $\mathsf{LieGroups}_{\mathrm{Diff}(M)}$. We conclude that (5.2) must also be commutative as a diagram in $\mathsf{TransPairs}_M$.

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(b) From Example 5.5, we deduce that $\chi_{\mathcal{R}(\theta,H)}$ will be an isomorphism of Lie groupoids if it is surjective. Hence, the assertion will follow if we can prove that $\chi_{\mathcal{R}(\theta,H)} \circ \mathcal{R}(a_{\theta,H}) = \mathrm{id}_{\mathcal{R}(\theta,H)}$. Let us use again the section atlas $(s_i)_{i\in I}$ of $\pi : K/H \to M$ from part (a). We evaluate the composition of both maps in an element and obtain

$$\begin{aligned}
&\chi_{\mathcal{R}(\theta,H)} \circ \mathcal{R}(a_{\theta,H})(\langle kH, gH\rangle) \\
&\qquad = \chi_{\mathcal{R}(\theta,H)}(\langle a_{\theta,H}(k)\,\mathrm{Bis}_m(\mathcal{R}(\theta,H)), a_{\theta,H}(g)\,\mathrm{Bis}_m(\mathcal{R}(\theta,H))\rangle) \\
&\qquad = a_{\theta,H}(k)(m) \cdot (a_{\theta,H}(g)(m))^{-1} = \langle k \cdot s_i(m), g \cdot s_i(m)\rangle = \langle kH, gH\rangle.
\end{aligned}$$

Note that this holds since the gauge groupoid operations are given as $\langle kH, gH\rangle \cdot \langle gH, lH\rangle = \langle kH, lH\rangle$ and $\langle kH, gH\rangle^{-1} = \langle gH, kH$ (cf. [10, Example 1.1.15]). Summing up, $\mathcal{R}(a_{\theta,H})$ is an isomorphism with inverse $\chi_{\mathcal{R}(\theta,H)}$. □

Corollary 5.9. *For a transitive pair (θ, H) and an arrow g in the Lie groupoid $\mathcal{R}(\theta, H)$, there is a bisection $\sigma \in \mathrm{Bis}(\mathcal{R}(\theta, H))$ with $\sigma(\alpha(g)) = g$.*

Proof. By Lemma 5.8, $\chi_{\mathcal{R}(\theta,H)} : \mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{R}(\theta, H)) \to \mathcal{R}(\theta, H)$ is an isomorphism, whence the assertion follows from [19, Lemma 4.21(a)]. □

Proposition 5.10. *The functor $\mathcal{R}$ is left adjoint to the functor $\overline{\mathrm{Bis}}$. The unit of the adjunction is $(a_{\theta,H})_{\mathsf{TransPairs}_M}$ and the co-unit of the adjunction is $\chi = (\chi_{\mathcal{G}})_{\mathcal{G}\in\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}$.*

Proof. Define for $(\theta, H) \in \mathsf{TransPairs}_M$ and $\mathcal{G} \in \mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ the mapping

$$\begin{aligned}
I_{(\theta,H),\mathcal{G}} : \mathrm{Hom}_{\mathsf{TransPairs}_M}((\theta, H), \overline{\mathrm{Bis}}(\mathcal{G})) &\to \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}(\mathcal{R}(\theta, H), \mathcal{G}), \\
\varphi &\mapsto \chi_{\mathcal{G}} \circ \mathcal{R}(\varphi).
\end{aligned}$$

We obtain a family of maps which is natural in $\mathcal{G}$ since the family $\chi = (\chi_{\mathcal{G}})$ is natural by Lemma 5.6. Clearly, the family is also natural in the transitive pair (θ, H). To construct an inverse of $I_{(\theta,H),\mathcal{G}}$, recall from Lemma 4.11 that $a_{\theta,H} : (\theta, H) \to (\beta_{\mathcal{R}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{R}(\theta, H)))$ is a morphism of transitive pairs. Hence, the map

$$\begin{aligned}
J_{\mathcal{G},(\theta,H)} : \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}(\mathcal{R}(\theta, H), \mathcal{G}) &\to \mathrm{Hom}_{\mathsf{TransPairs}_M}((\theta, H), \overline{\mathrm{Bis}}(\mathcal{G})), \\
\psi &\mapsto \overline{\mathrm{Bis}}(\psi) \circ a_{\theta,H}
\end{aligned}$$

makes sense. Clearly, $J_{\mathcal{G},(\theta,H)}$ is natural in $\mathcal{G}$ and it is natural in (θ, H) by Lemma 5.8(a). To see that $J_{\mathcal{G},(\theta,H)}$ and $I_{(\theta,H),\mathcal{G}}$ are mutually inverse, we use the results obtained so far

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and compute

$$\begin{aligned}
J_{\mathcal{G},(\theta,H)} \circ I_{(\theta,H),\mathcal{G}}(\varphi) &= \overline{\mathrm{Bis}}(\chi_{\mathcal{G}} \circ \mathcal{R}(\varphi)) \circ a_{\theta,H} \\
&= \overline{\mathrm{Bis}}(\chi_{\mathcal{G}}) \circ \overline{\mathrm{Bis}}(\mathcal{R}(\varphi)) \circ a_{\theta,H} \\
&\overset{\text{Lemma 5.8(a)}}{=} \underbrace{\overline{\mathrm{Bis}}(\chi_{\mathcal{G}}) \circ a_{\overline{\mathrm{Bis}}(\mathcal{G})}}_{=\mathrm{id}_{\mathrm{Bis}(\mathcal{G})} \text{ by Example 5.5}} \circ \varphi = \varphi, \\
I_{(\theta,H),\mathcal{G}} \circ J_{\mathcal{G},(\theta,H)}(\psi) &= \chi_{\mathcal{G}} \circ \mathcal{R}(\overline{\mathrm{Bis}}(\psi) \circ a_{\theta,H}) \\
&= \chi_{\mathcal{G}} \circ \mathcal{R}(\overline{\mathrm{Bis}}(\psi)) \circ \mathcal{R}(a_{\theta,H}) \\
&\overset{\text{Lemma 5.6}}{=} \psi \circ \underbrace{\chi_{\mathcal{R}(\theta,H)} \circ \mathcal{R}(a_{\theta,H})}_{=\mathrm{id}_{\mathcal{R}(\theta,H)} \text{ by Lemma 5.8(b)}} = \psi.
\end{aligned}$$

□

COROLLARY 5.11. *The functor* $\overline{\mathrm{Bis}} : \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \to \mathsf{TransPairs}_M$ *preserves limits and the functor* $\mathcal{R}$ *preserves colimits. In particular,* $\overline{\mathrm{Bis}}$ *preserves kernels and pull-backs and* $\mathcal{R}$ *preserves cokernels and push-outs.*

The results obtained so far can be used to establish a category equivalence between certain subcategories of transitive pairs and locally trivial Banach–Lie groupoids. Let us first define these subcategories.

DEFINITION 5.12. Define the full subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ of the category $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ whose objects are Lie groupoids $\mathcal{G} = (G \rightrightarrows M)$ which admit a bisection through each arrow, that is, for all $g \in G$ there is $\sigma \in \mathrm{Bis}(\mathcal{G})$ such that $\sigma(\alpha(g)) = g$.

REMARK 5.13.

(a) Corollary 5.9 asserts that the functor $\mathcal{R}$ takes its image in the category $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$. By abuse of notation, we will in the following identify $\mathcal{R}$ with a functor $\mathcal{R} : \mathsf{TransPairs}_M \to \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$.
(b) If a Banach–Lie groupoid $\mathcal{G}$ (with compact base) is source connected, that is, the fibres of the source projection are connected, then $\mathcal{G}$ is contained in $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ by [19, Theorem 2.14].

Objects in the subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ admit 'enough' bisections, that is, by Example 5.5, these groupoids can be completely recovered from the transitive pair constructed from their bisections. Passing to the subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$, we restrict to all Lie groupoids which can be reconstructed by the functor $\mathcal{R}$. Note that Example 5.5 also shows that the functor $\mathcal{R} \circ \overline{\mathrm{Bis}}$ associates to every locally trivial Banach–Lie groupoid an open subgroupoid from $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$. We can view this subgroupoid as a selection of a convenient subobject. Following [1, page 56], this should correspond on the level of categories to $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ being a coreflected subcategory of $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$.

Proposition 5.14. *The functor*

$$\mathcal{E} := \mathcal{R} \circ \overline{\mathrm{Bis}} : \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \to \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$$

is right adjoint to the inclusion $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}} \to \mathsf{BanachLieGpds}_M^{\mathrm{triv}}$. *Thus,* $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ *is a coreflected subcategory of* $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$.

Proof. Consider $\mathcal{K} \in \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ and $\mathcal{G} \in \mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ and define

$$I_{\mathcal{K},\mathcal{G}} : \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}}(\mathcal{K}, \mathcal{E}(\mathcal{G})) \to \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}(\mathcal{K}, \mathcal{G}),$$
$$\varphi \mapsto \chi_{\mathcal{G}} \circ \varphi.$$

The family $(I_{\mathcal{K},\mathcal{G}})_{\mathcal{K},\mathcal{G}}$ is natural in $\mathcal{K}$ and $\mathcal{G}$ by Lemma 5.6. Let us construct an inverse for $I_{\mathcal{K},\mathcal{G}}$. For each Lie groupoid $\mathcal{G}$, the map $\chi_{\mathcal{G}} : \mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G})) \to \mathcal{G}$ restricts by [19, Lemma 4.21] to an isomorphism $\chi_{\mathcal{G}}^{res} : \mathcal{R}(\overline{\mathrm{Bis}}(\mathcal{G})) \to I_{\mathcal{G}} \subseteq \mathcal{G}$ of Lie groupoids onto its image $I_{\mathcal{G}}$. Recall from [19] that $\chi_{\mathcal{K}}$ is an isomorphism for all $\mathcal{K}$ in $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$, that is, $\chi_{\mathcal{K}} = \chi_{\mathcal{K}}^{res}$ and thus $I_{\mathcal{K}} = \mathcal{K}$. Now, for a Lie groupoid morphism $\psi : \mathcal{K} \to \mathcal{G}$, we use that every arrow k of $\mathcal{K}$ is in the image of a bisection to obtain

$$\psi(k) = \psi(\sigma_k(\alpha(k)) = \underbrace{\mathrm{Bis}(\psi)(\sigma_k)}_{\in \mathrm{Bis}(\mathcal{G})}(\alpha(k)) \in I_{\mathcal{G}}.$$

Hence, the mapping

$$J_{\mathcal{G},\mathcal{K}} : \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv}}}(\mathcal{K}, \mathcal{G}) \to \mathrm{Hom}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}}(\mathcal{K}, \mathcal{E}(\mathcal{G})),$$
$$\psi \mapsto (\chi_{\mathcal{G}}^{res})^{-1} \circ \psi$$

makes sense and the maps $J_{\mathcal{G},\mathcal{K}}$ and $I_{\mathcal{K},\mathcal{G}}$ are mutual inverses. Moreover, the family $(J_{\mathcal{G},\mathcal{K}})_{\mathcal{G},\mathcal{K}}$ is natural in $\mathcal{K}$ and $\mathcal{G}$, again by Lemma 5.6. Thus, $\mathcal{E}$ is right adjoint to the inclusion $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}} \to \mathsf{BanachLieGpds}_M^{\mathrm{triv}}$. □

Finally, we will state the equivalence of categories already alluded to.

Theorem 5.15. *Let C be the essential image of $\overline{\mathrm{Bis}}$. Then we obtain an induced equivalence of categories*

$$\begin{tikzcd} \mathsf{TransPairs}_M \arrow[r, shift left, "\mathcal{R}"] & \mathsf{BanachLieGpds}_M^{\mathrm{triv}} \arrow[l, shift left, "\overline{\mathrm{Bis}}"] \\ C \arrow[u, hook] \arrow[r, "\cong"] & \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}} \arrow[u, hook] \end{tikzcd}$$

Proof. By Remark 5.13, the functor $\mathcal{R}$ can be restricted to a functor $\mathcal{R}^{res} : C \to \mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ and, by construction, $\overline{\mathrm{Bis}}$ restricts to a functor $\overline{\mathrm{Bis}}^{res} := \overline{\mathrm{Bis}}\,|^{C}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}}$. From Corollary 5.9, we deduce that $\overline{\mathrm{Bis}}^{res} \circ \mathcal{R}^{res} \cong \mathrm{id}_C$. Conversely, Lemma 5.6 implies that $\mathcal{R}^{res} \circ \overline{\mathrm{Bis}}^{res} \cong \mathrm{id}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}}$. □

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Remark 5.16.

(a) In this section, we have so far always assumed that M is connected. The reason for this was that in the proof of Lemma 5.1, connectivity was needed to ensure the transitivity of the natural action of $\mathrm{Diff}(M)$ on M. Consequently, the natural action of $\mathrm{Bis}(\mathcal{G})$ on M was also transitive and $(\beta \circ \mathrm{ev} : \mathrm{Bis}(\mathcal{G}) \times M \to M, \mathrm{Bis}_m(\mathcal{G}))$ was concluded to be a transitive pair.
Without the assumption on M being connected, the natural action of $\mathrm{Bis}(\mathcal{G})$ on M is not transitive in general (see [19, Remark 2.18(b)]). However, restricting to the category $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$, we still obtain a transitive pair $(\beta_{\mathcal{G}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$. Thus, $\overline{\mathrm{Bis}}$ makes sense on this subcategory also for nonconnected M.

(b) If we restrict $\mathcal{R}$ and $\overline{\mathrm{Bis}}$ to the full subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$, all proofs carry over verbatim to show the results of this section for $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ instead of $\mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ also in the case of a nonconnected manifold M. On $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$, the natural transformation χ (see Lemma 5.6) then even induces a natural *isomorphism* $\mathcal{R} \circ \overline{\mathrm{Bis}}\,|_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}} \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}}$. Using this isomorphism, one can then show that the functor $\overline{\mathrm{Bis}}$ restricted to the subcategory reflects isomorphisms.

(c) Unfortunately, the subcategory $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$ is not easy to describe, since the existence of bisections through each arrow can be ensured or destroyed by many different independent properties (see the following example). Likewise, the essential image C of $\overline{\mathrm{Bis}}$ (which is equivalent to $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}}$) is not easily described by an auxiliary property that is easy to handle.

Example 5.17. In general, the question of the existence of bisections through each arrow of a Lie groupoid G is very hard to characterise. In fact, if $\mathcal{G}$ is locally trivial (that is, the gauge groupoid of some principal bundle $P \to M$), then this property is equivalent to the transitivity of the action of bundle automorphisms $\mathrm{Aut}(P)$ on P. We give some examples of phenomena that can occur here.

(a) If $P \to M$ is a principal A-bundle with A an *abelian* Lie group, then $\mathrm{Aut}(P) \cong C^\infty(M, A) \rtimes \mathrm{Diff}(M)$ and $\mathrm{Aut}(P)$ acts transitively on P if and only if $\mathrm{Diff}(M)_{[P]}$ does so (the latter denotes the open subgroup of $\mathrm{Diff}(M)$ that fixes the isomorphism class $[P]$ in $\mathrm{Bun}(M, A) \cong \check{H}^1(M, A)$).

(b) Let M be a compact connected manifold with nonabelian fundamental group $\pi_1(M) := \Delta$. Then the universal covering $\widetilde{M} \to M$ is a principal Δ-bundle and the gauge transformation group of this principal bundle is given by

$$\mathrm{Gau}(\widetilde{M}) \cong C^\infty(\widetilde{M}, \Delta)^\Delta,$$

where Δ acts on functions by point-wise conjugation. Since $\widetilde{M}$ is connected and Δ is discrete, we have $C^\infty(\widetilde{M}, \Delta)^\Delta = \Delta^\Delta = Z(\Delta)$ and, since each diffeomorphism of M lifts to $\widetilde{M}$,

$$\mathrm{Aut}(\widetilde{M}) \cong Z(\Delta) \times \mathrm{Diff}(M).$$

The elements of $Z(\Delta)$ act on $\widetilde{M}$ by the usual action of the fundamental group on $\widetilde{M}$, which is not transitive on the fibres of $\widetilde{M} \to M$ since $Z(\Delta) \neq \Delta$. Consequently, $\mathrm{Aut}(\widetilde{\Delta})$ does not act transitively on $\widetilde{M}$. This shows in particular that the inclusion $\mathsf{BanachLieGpds}_M^{\mathrm{triv,ev}} \hookrightarrow \mathsf{BanachLieGpds}_M^{\mathrm{triv}}$ is not essentially surjective.

(c) If $P \to M$ is a trivial principal K-bundle, then $\mathrm{Aut}(P) \cong C^\infty(M, K) \rtimes \mathrm{Diff}(M)$ acts transitively on P if and only if $\mathrm{Diff}(M)$ acts transitively on M.

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