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COMMUTATIVE ABSOLUTE SUBRETRACTS

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Abstract

Directly indecomposable absolute subretracts that are commutative Noetherian rings are described. This is an application of our main result characterizing unital directly indecomposable absolute subretracts which contain a maximal ideal with nonzero annihilator.

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Throughout this paper, all rings are associative and commutative. The variety generated by a ring R is denoted $\mathrm{Var}(R)$ (cf. [7]).

Recently, several authors [2, 4, 5] have studied the notion of absolute subretract. Recall that a ring R is said to be an absolute subretract if for every ring S in $\mathrm{Var}(R)$ and every ring monomorphism $f : R \to S$, there exists a ring morphism $g : S \to R$ such that gf is the identity mapping on R. In [4], Gardner and Stewart characterized directly indecomposable absolute subretracts R with $R^2 = \{0\}$, and gave an example of a special principal ideal ring which is an absolute subretract. Then Jespers [5] obtained necessary and sufficient conditions for a finite special principal ideal ring (of characteristic different from 2^n) to be an absolute subretract, also obtaining results for the infinite and characteristic 2^n cases.

We first show that in a unital directly indecomposable absolute subretract R, the set of zero divisors is a maximal ideal M and R/M is finite. If, moreover, M has nonzero annihilator (denoted $\mathrm{Ann}(M)$), we obtain necessary and sufficient conditions for R to be an absolute subretract. As an immediate consequence, a characterization of Noetherian directly indecomposable

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absolute subretracts is obtained. The latter is then applied to Noetherian contracted monoid algebras.

LEMMA 1 [2, 4]. *Let* R *be a directly indecomposable absolute subretract. If* I *and* J *are nonzero ideals of* R, *then* $I \bigcap J \neq \{0\}$.

PROPOSITION 2. *Let* R *be a unital ring. If* R *is a directly indecomposable absolute subretract, then the set of zero-divisors of* R *is a maximal ideal* M *and* R/M *is finite.*

PROOF. Let M be the set of zero divisors. Because of Lemma 1, one easily verifies that M is an ideal of R. Clearly R/M is a domain. If R/M is finite, then R/M is a field and thus M is a maximal ideal. In the remainder, we show that R/M has to be finite.

Suppose the contrary. Then R/M, being an infinite domain, satisfies only identities $f(X) = 0$ where $f(X) \in p\mathbf{Z}[X]$, $p = \text{char}\,(R/M)$. If $p = 0$, we have shown that R/M, and hence R, satisfies no nontrivial polynomial identities. If $p \neq 0$, any nontrivial polynomial identity satisfied by R must be of the form $0 = p^n f(X) \in \mathbf{Z}[X]$, $n \geq 1$, where $f(X) \notin p\mathbf{Z}[X]$. We claim that in this case $\text{char}(R) \mid p^n$. If not, then $p^n \neq 0$ in R so $f(X)$ is in M for all choices of X and $f(X) = 0$ in R/M. But we saw before that this implies $f(X) \in p\mathbf{Z}[X]$, a contradiction.

In either case (cf. [6]), $\text{Var}(R)$ contains all central extensions of R. As in [8] and [4], we focus our attention on the localization of the polynomial ring in one variable $R[x]$ obtained by inverting all monic polynomials, and denote this by T. Since R is an absolute subretract, there is a homomorphism $g : T \to R$ extending the identity map on R. Say $g(x) = r$. Then, $g(x - r) = 0$, a contradiction since $x - r$ is invertible. This finishes the proof. □

COROLLARY 3. *Let* R *be a unital directly indecomposable absolute subretract and let* M *be the ideal of zero divisors. If* $\text{Ann}(M) \neq 0$, *then* $\text{Ann}(M)$ *is the minimum nonzero ideal of* R.

PROOF. Say $\text{Ann}(M) \nsubseteq I$ for some nonzero ideal I of R. Choose $0 \neq m \in \text{Ann}(M)$ with $m \notin I$. By Lemma 1, there exists $r \in R$ such that $0 \neq rm \in I$. Proposition 2 then says that $r^{n-1} = 1 + \alpha$ for some $\alpha \in M$ where $n = |R/M|$. But this implies that $m = r^{n-1} m \in I$, a contradiction. □

LEMMA 4. *Let* R *be a unital directly indecomposable absolute subretract with* M *the ideal of zero divisors. Assume* $M \neq \{0\}$ *and* $\text{Ann}(M) \neq \{0\}$. *If* $|R/M| > 2$, *then* $M = \text{Ann}(M)$, *that is* $M^2 = \{0\}$.

PROOF. Since R/M has more than two elements, there exist $u, u' \notin M$ with $u - u' \notin M$. Also, by Corollary 3, $\mathrm{Ann}(M) = Rx$ for some $0 \neq x \in R$.

Let $S = \{(a, a+j) \in R \times R \mid j \in M\}$ and let I be the principal ideal generated by $(ux, u'x)$ in S. Note that, since $Mx = 0$, $I = \{(aux, au'x) \mid a \notin M\} \cup \{0\}$, and that $T = S/I$ is in $\mathrm{Var}(R)$. Define $f : R \to T$ by $f(r) = (r, r) + I$. Clearly f is a homomorphism. Suppose $f(r) = 0$ for some $r \neq 0$, that is $(r, r) \in I$. Then $0 \neq r = aux = au'x$ for some $a \notin M$, so $a(u - u')x = 0$. Since $a(u - u') \notin M$, that is, $a(u - u')$ is not a zero divisor, it follows that $x = 0$, a contradiction. Thus f is a monomorphism.

We now show that every principal ideal Tt of T intersects $f(R)$ non-trivially, where $0 \neq t = (a, b) + I$.

First consider the case where $a, b \in Rx$, that is, $a = vx$, $b = wx$ and either v or w is not in M. Let $s = (u' - u)^{n-2}(v - w)$ where $n = |R/M|$. Note that $s(u' - u)x = (u' - u)^{n-1}(v - w)x = (1 + \alpha)(v - w)x$ for some $\alpha \in M$. Hence

$$s(u' - u)x = (v - w)x.$$

Therefore $(v + su)x = (w + su')x$, and so $0 \neq (a, b) + I = (vx, wx) + s(ux, u'x) + I = ((v + su)x, (w + su')x) + I \in f(R)$.

Next assume that either $a \notin Rx$ or $b \notin Rx$, for example, say, $a \notin Rx$. Because $Ma \neq 0$ and Rx is minimum, there exists $r \in M$ with $ra = x$. Hence $(r, 0) \in S$ and $(r, 0)(a, b) + I = (x, 0) + I$ belongs to Tt. Note that $(x, 0) \notin I$ since $(x, 0) = (aux, au'x)$ implies $au'x = 0$. As u' is not a zero divisor, this yields $ax = 0$ and thus $x = aux = 0$, a contradiction. By the previous case, we know that $(x, 0) + I$ belongs to $f(R)$.

Since R is an absolute subretract, it follows that $f(R) = T$. Hence, for every $m \in M$, $(um + x, u'm + x) + I \in f(R)$. Thus $(um + x, u'm + x) = (r, r) + (aux, au'x)$ for some $r \in R$. It follows that $um - u'm \in \mathrm{Ann}(M)$. Since $u - u'$ is not a zero divisor, we conclude that $m \in \mathrm{Ann}(M)$, so $M \subseteq \mathrm{Ann}(M)$. Since $M \neq 0$, $\mathrm{Ann}(M) \subseteq M$, and the result follows. □

We next show that the characteristic 2 case can be settled in the same way.

LEMMA 5. *Let R be a unital directly indecomposable absolute subretract with M the ideal of zero divisors. Assume $M \neq \{0\}$ and $\mathrm{Ann}(M) \neq \{0\}$. If $|R/M| = 2$, then $M = \mathrm{Ann}(M)$.*

PROOF. Let $S = \{(a, b, c) | a, b, c \in R,\ a - b \text{ and } b - c \in M\}$. Note that $S \in \mathrm{Var}(R)$. Let $\mathrm{Ann}(M) = Rx$, $x \neq 0$, and define

$$I = \{(0, 0, 0), (0, x, x), (x, 0, x), (x, x, 0)\}.$$

Observe that I is an ideal of S and let $T = S/I$.

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Define $f : R \to T$ by $f(r) = (r, r, r) + I$. Clearly f is a monomorphism. We claim that every nonzero principal ideal of T intersects $f(R)$ nontrivially. For this, let $0 \neq t = (a, b, c) + I$. If a, b, c are all in $Rx = \{0, x\}$, then since $t \neq 0$, we have $t = (x, x, x) + I = f(x)$. So assume, for example, that $a \notin Rx$. Then $Ma \neq 0$ and thus $Ma \supseteq Rx$ since Rx is minimum. So $ra = x$ for some $r \in M$. Hence $(r, 0, 0) \in M$ and $(r, 0, 0)(a, b, c) + I = (x, 0, 0) + I = (x, x, x) + I = f(x)$ is in Tt, and the claim is proved. Since R is an absolute subretract, we conclude that $f(R) = T$. Hence for every $m \in M$, $(m, m, 0) + I \in f(R)$. Consequently, $m \in Rx$ so $M \subseteq \mathrm{Ann}(M)$. Since $M \neq 0$, $\mathrm{Ann}(M) \subseteq M$ and the result follows. □

We are now ready to prove our main result.

THEOREM 6. *Let R be a unital ring with maximal ideal M such that* $\mathrm{Ann}(M) \neq \{0\}$. *Then R is a directly indecomposable absolute subretract if and only if R is finite and $M = Rx$, $x^2 = 0$, for some x in R.*

PROOF. First we show that the conditions are sufficient. If $M \neq \{0\}$, this follows from Proposition 2 in [5]. The case $M = \{0\}$, that is R is a finite field, is proved in the same way, but we will sketch it here for completeness. So let R be a finite field and say $f : R \to T$ is a monomorphism where $T \in \mathrm{Var}(R)$. Although T itself may not have a multiplicative identity, it has a direct summand T_1 such that $f(R) \subseteq T_1$ and T_1 shares the same multiplicative identity as $f(R)$. Choose N an ideal of T_1 maximal such that $f(R) \cap N = \{0\}$. Then T_1/N is a field satisfying the same polynomial identities as R and R is embedded in T_1/N. Hence $R \simeq T_1/N$ and the result follows.

To prove the necessity of the conditions, note that because of Proposition 2 and the assumptions, M is the set of zero divisors of R and R/M is a finite field. The result then follows from Lemmas 4 and 5. □

COROLLARY 7. *Let R be a unital directly indecomposable ring. Then R is a Noetherian absolute subretract if and only if R is a finite field or R is a finite local ring with maximum ideal $M = Rx$ and $x^2 = 0$ for some $0 \neq x \in R$.*

PROOF. Assume R is a Noetherian absolute subretract. Then by the assumptions and Proposition 2, the set of zero divisors M is a finitely generated maximal ideal. Hence by Lemma 1, $\mathrm{Ann}(M) \neq \{0\}$. The result now follows from Theorem 6. □

We conclude with an application to contracted monoid algebras. For terminology and notation we refer to [1].

COROLLARY 8. *Let k be a field and S a commutative monoid with identity element e and zero element $\theta \neq e$. Then the contracted monoid algebra is a Noetherian directly indecomposable absolute subretract if and only if k is a finite field and one of the following conditions is satisfied.*

(i) $S = \{e, s, \theta\}, s^2 = \theta, s \neq \theta, s \neq e$. *In particular,* $k_0[S] = k[x]/(x^2)$.

(ii) $char(k) = 2, S = \{e, s, \theta\}, s^2 = e$. *In particular,* $k_0[S] \cong k[\mathbf{Z}_2]$, *a group algebra of the cyclic group of order 2.*

(iii) $S = \{e, \theta\}$. *In particular,* $k_0[S] \cong k$.

PROOF. First note that $k_0[S]$ is a field if and only if $S = \{e, \theta]$, which is precisely case (iii). Because of Corollary 7 and the results in [3], the proof now goes exactly as the proof of Corollary 4 in [5]. □

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