A CHARACTERIZATION OF 2-BETWEENNESS IN 2-METRIC SPACES

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1. Introduction. The topology of abstract 2-metric (area-metric) spaces has been the object of study in recent papers of Gähler **(1)** and Froda **(2)**. The geometric properties of such spaces, however, have remained largely untouched since the initial work of Menger **(3)**. As in ordinary metric spaces, a notion of 2-betweenness, or interiorness, can be easily defined in 2-metric spaces. In abstract metric spaces the betweenness relation is characterized among all relations defined on each triple of points of every metric space by six natural properties (**4**, pp. 33–40; **5**). The purpose of this paper is to prove a similar theorem characterizing the relation of 2-betweenness in 2-metric spaces.

2. Preliminary notions. By a *2-metric space* is meant an abstract set S together with a function pqr, called the 2-metric or area, on triples of points of S into the non-negative real numbers, satisfying the relations

(1) if $p, q \in S$, there is a point r of S with $pqr \neq 0$, and

(2) each set of four points $p, q, r, s \in S$ is congruently embeddable in the three-dimensional Euclidean space, E_3, i.e., the points p, q, r, s can be placed in a 1–1 area-preserving correspondence with a quadruple p', q', r', s' of points of E_3.

We denote this 2-congruence by $p, q, r, s \approx_2 p', q', r', s'$, and the 2-congruent embedding by $p, q, r, s \ll_2 E_3$. In particular, a 2-congruence of a 2-metric space onto itself is called a *2-motion*.

A 2-metric space is said to satisfy the *simplex inequality* provided that for each quadruple p, q, r, s of its points we have $pqr \leqslant pqs + prs + qrs$.

The relation of *2-betweenness* or *interiorness* is now defined as follows. Letting p, q, r, s be points of a 2-metric space S, we say p is a *2-betweenpoint* of q, r, s, denoted $pIqrs$, provided $(pqr)(pqs)(prs)(qrs) \neq 0$, and

$$qrs = pqr + prs + pqs.$$

If q, r, s is a triple of points of S with non-vanishing 2-metric, we denote by $I(q, r, s)$ the set of 2-betweenpoints of q, r, s. Similarly, suppressing the requirement that no triple have vanishing 2-metric, we say that p is a *weak 2-betweenpoint* of q, r, s (or is *weakly interior* to q, r, s) provided $qrs = pqr + prs + pqs$. The corresponding set of weak 2-betweenpoints is denoted by $\bar{I}(q, r, s)$.

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A useful property in 2-metric spaces is the following property of transitivity. We say a 2-metric space satisfies *property* T provided that for each quintuple p, q, r, s, t of its points, $pIqrs$ and $tIpqr$ imply $tIqrs$.

Finally, a notion of convergence can be introduced in 2-metric spaces in the following way. A sequence $\{p_n\}$ in a 2-metric space S is *weakly 2-convergent* to p in S provided $\lim p_n pt = 0$ for each t in S. As the weak 2-convergence topology does not provide continuity of the 2-metric, a stronger notion of convergence is introduced. We say $\{p_n\}$ is strongly 2-convergent to p in S provided

(1) $\{p_n\}$ is weakly 2-convergent to p, and

(2) for each point q of S and each sequence $\{q_n\}$ weakly 2-convergent to q, we have $\lim pp_m q_n = 0$ $(m, n \to \infty)$.

In the strong 2-convergence topology, the 2-metric is a continuous function. It can be shown further that in any 2-metric space in which the 2-metric is a continuous function, strong 2-convergence and weak 2-convergence are equivalent. Clearly in Euclidean spaces the notions of weak 2-convergence, strong 2-convergence, and metric convergence are equivalent.

3. Properties of 2-betweenness. The main theorem will provide a characterization of the relation $pIqrs$ in 2-metric spaces with property T, by the following properties:

(1) $(pqr)(prs)(pqs)(qrs) \neq 0$ (non-vanishing).

(2) If q', r', s' denote any permutation of the points q, r, s, then $pIqrs$ implies $pIq'r's'$ (symmetry).

(3) $pIqrs$ implies not $qIprs$, not $rIpqs$, and not $sIpqr$ (special interior point).

(4) $pIqrs$ and $tIpqr$ implies $tIqrs$ (transitivity).

(5) The set $\bar{I}(q, r, s)$ is closed (closure).

(6) Given a 2-metric space consisting of q, r, s, and given $\epsilon > 0$, there exists a 2-metric space with property T containing q, r, s and a point p such that $pqr < \epsilon$, $pqs < \epsilon$, and $pIqrs$ (extension).

(7) If $p, q, r, s \approx_2 p', q', r', s'$, then $pIqrs$ implies $p'Iq'r's'$ (2-congruence invariant).

It can easily be seen that in any 2-metric space with property T, the relation I has these properties. Indeed, properties (1), (2), (4), and (7) are obvious by definition of I, and by the assumption of property T, while property (5) follows by continuity of the 2-metric.

Property (3) follows directly, for if both $pIqrs$ and $qIprs$, then

$$qrs = pqr + pqs + prs = 2(pqr + pqs) + qrs$$

and hence $pqr = 0$, contrary to (1).

To prove property (6), let S be a 2-metric space with property T consisting of points q, r, s. Given $\epsilon > 0$, consider the space consisting of q, r, s, and a point p such that $p, q, r, s \approx_2 p', q', r', s'$ where p', q', r', s' are points of E_2 with $q'r's' = qrs$ and $p'Iq'r's'$ such that $p'q'r' < \epsilon$ and $p'q's' < \epsilon$. Since the points p, q, r, s form a 2-metric space that satisfies property T vacuously, the desired 2-metric space has been constructed.

4. Some lemmas. In order to prove that properties (1)–(7) characterize the relation of 2-betweenness in 2-metric spaces with property T, we let R denote any relation on quadruples of every 2-metric space, satisfying (1)–(7), and show the equivalence of R with 2-betweenness. To accomplish this, some additional properties of 2-metric spaces are needed, as well as properties of such relations R.

LEMMA 1. *Let p, q, r, s be points and pqr, pqs, prs, qrs any positive real numbers satisfying the simplex inequality. Considering these numbers as values of a 2-metric, we have $p, q, r, s \ll_2 E_3$.*

Proof. Suppose pqr and qrs are the maximum and the minimum of the four numbers, respectively. If $pqr \geqslant pqs + prs$, then letting p', q', r' be vertices of an equilateral triangle of E_2 such that $p'q'r' = pqr$, consideration of the loci of points having a given 2-metric with pairs of p', q', r' shows that a point s' of E_3 can be found with $p, q, r, s \approx_2 p', q', r', s'$.

If $pqr < pqs + prs$, let $[q^*r^*s^*]$ be a Euclidean triangle with $q^*r^*s^* = qrs$, such that the ratio of sides is as follows:

$$\frac{q^*r^*}{pqr} = \frac{r^*s^*}{prs} = \frac{q^*s^*}{pqs},$$

where q^*r^*, r^*s^*, and q^*s^* denote the respective Euclidean distances of the points. Then on the line perpendicular to the plane of $[q^*r^*s^*]$ at the incentre of the triangle is found a point p^* such that $p^*q^*r^* = pqr$, $p^*q^*s^* = pqs$, and $p^*r^*s^* = prs$, and hence $p, q, r, s \approx_2 p^*, q^*, r^*, s^*$, completing the proof.

It is noted that if every quadruple of a 2-metric space satisfies the simplex inequality in the strict sense, then the space has property T vacuously.

Letting R now represent any relation satisfying (1)–(7), defined on all quadruples of any 2-metric space S with property T, we note immediately that if $qRprs$, all points of the quadruple p, q, r, s must be distinct. In addition we have the following results.

LEMMA 2. *If $p, q, r, s \in S$, then $qRprs$ implies $pqs \neq prs$, $pqr \neq prs$, and $qrs \neq prs$.*

Proof. If $pqs = prs$, then $p, q, r, s \approx_2 p, r, q, s$, and hence from $qRprs$ follows $rRpqs$, contrary to property (3). Similarly a contradiction is reached in the other two cases.

In the following lemma, it is useful to have a measure of the deviation of a quadruple from satisfying one of the four possible 2-betweenness relations.

Such a measure is the *defect* of a quadruple p, q, r, s, defined as the minimum of the quantities

$$pqr + qrs + rsp - spq, \qquad qrs + rsp + spq - pqr,$$
$$rsp + spq + pqr - qrs, \qquad spq + pqr + qrs - rsp.$$

Using this concept, the final lemma can now be proved.

LEMMA 3. *If $p, q, r, s \in S$ and $qRprs$, then $pIqrs$ or $qIprs$ or $rIpqs$ or $sIpqr$.*

Proof. Suppose the contrary and let p', q', r', s' be points of E_3 (or E_2) such that $p', q', r', s' \approx_2 p, q, r, s$. Since no triple of the primed points can have vanishing 2-metric, there exists a positive number δ such that for all y' for which $y'p'q' < \delta$, $y'p's' < \delta$, it follows that the defect of q', r', s', y' is greater than $|y'r's' - p'r's'|$, that $p'r'y' < p'r's'$ and $p's'y' < p'r's'$, and neither $r'Iq's'y'$ nor $s'Iq'r'y'$.

Given p', q', s', there exists by the extension property a 2-metric space with property T with points p'', q'', s'', x'' with $x''Rp''q''s''$, $p''q''s'' = p'q's'$, and $p''q''x'' < \delta$ and $p''s''x'' < \delta$. Since $p'', q'', s'', x'' \ll_2 E_3$, let p^*, q^*, s^*, x^* be their images and let x' denote a point into which x^* is sent under a 2-motion sending p^*, q^* s^* into p', q', s', i.e., $p, q, s, x \approx_2 p', q', s', x'$. Consider a quintuple $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{x}$ such that the 2-metric of each triple is defined by

$$\bar{p}, \bar{q}, \bar{r}, \bar{s} \approx_2 p', q', r', s'; \qquad \bar{p}, \bar{q}, \bar{r}, \bar{x} \approx_2 p', q', r', x';$$
$$\bar{p}, \bar{q}, \bar{s}, \bar{x} \approx_2 p', q', s', x';$$

and $\bar{r}\bar{x}\bar{s} = r'p's'$. Hence each quadruple of the barred points is 2-congruently embeddable in E_3, with the possible exceptions of $\bar{p}$, $\bar{r}$, $\bar{s}$, $\bar{x}$ and $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{x}$.

The quadruple $\bar{p}$, $\bar{r}$, $\bar{s}$, $\bar{x}$ has no triple with vanishing 2-metric, since $x'Rp'r's'$. It also satisfies the simplex inequality since $\bar{p}\bar{r}\bar{s} = \bar{r}\bar{s}\bar{x}$ and $\bar{p}\bar{r}\bar{x}$, $\bar{p}\bar{s}\bar{x}$ are each less than $\bar{p}\bar{r}\bar{s}$, and by Lemma 1 is 2-congruently embeddable in E_3.

Considering the quadruple $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{x}$, it is observed that no triple has vanishing 2-metric and

$$\begin{aligned}\bar{r}\bar{x}\bar{s} = p'r's' &< r's'x' + \mathrm{Defect}(q', r', s', x')\\ &\leqslant q'r's' + q'r'x' + q's'x' = \bar{q}\bar{r}\bar{s} + \bar{q}\bar{r}\bar{x} + \bar{q}\bar{s}\bar{x},\end{aligned}$$

while

$$\begin{aligned}\bar{q}\bar{r}\bar{s} = q'r's' &\leqslant q'r'x' + q's'x' + r's'x' - \mathrm{Defect}\,(q', r', s', x')\\ &< q'r'x' + q's'x' + p'r's' = \bar{q}\bar{r}\bar{x} + \bar{q}\bar{s}\bar{x} + \bar{r}\bar{s}\bar{x}.\end{aligned}$$

Similarly, if $\bar{q}\bar{r}\bar{x}$ or $\bar{q}\bar{s}\bar{x}$ is isolated, the simplex inequality is satisfied.

The above shows that all forms of the simplex inequality for three of the five quadruples of the barred points are satisfied in a strict sense. Hence the barred points form a 2-metric space with property T provided that the quadruples $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{x}$ and $\bar{p}$, $\bar{q}$, $\bar{s}$, $\bar{x}$ do not satisfy the antecedent of property T.

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Considering the corresponding primed points of E_3 it follows, upon consideration of the cases involved, and making use of the fact that $p's'x'$ and $p'r'x'$ are both less than $p'r's'$ while neither $r'Iq's'x'$ nor $s'Iq'r'x'$ can hold, that property T is satisfied vacuously by the barred points.

Hence the space of the barred points is a 2-metric space with property T, and therefore $\bar{q}R\bar{p}\bar{r}\bar{s}$ and $\bar{x}R\bar{p}\bar{q}\bar{s}$ imply $\bar{x}R\bar{p}\bar{r}\bar{s}$. However $\bar{x}\bar{r}\bar{s} = p'r's' = \bar{p}\bar{r}\bar{s}$, contrary to Lemma 2.

5. The characterization theorems. The final result can now be proved in the form of Theorems 1 and 2. As before, we let S be a 2-metric space having property T, and R any relation satisfying (1)–(7) defined on quadruples of every 2-metric space with property T.

THEOREM 1. *If $p, q, r, s \in S$ and $qRprs$, then $qIprs$.*

Proof. By Lemma 3, $qRprs$ implies $pIqrs$ or $qIprs$ or $rIpqs$ or $sIpqr$. Suppose $pIqrs$. There exists a 2-metric space with property T containing p, q, r, and a point x such that $xRpqr$. If $xIpqr$, then, as in Lemma 3, there exist points p', q', r', s', x' of E_3 such that

$$p, q, r, s \approx_2 p', q', r', s', \quad \text{and} \quad p, q, r, x \approx_2 p', q', r', x',$$

which, since $x'Ip'q'r'$ and $p'Iq'r's'$, implies that p', q', r', s', x' are contained in an E_2. Let $\bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{x}$ be points whose 2-metric is defined such that $\bar{p}, \bar{q}, \bar{r}, \bar{s} \approx_2 p', q', r', s'$ and $\bar{p}, \bar{q}, \bar{r}, \bar{x} \approx_2 p', q', r', x'$ while $\bar{p}\bar{x}\bar{s} = 0$,

$$\bar{s}\bar{x}\bar{r} = p'x'r' + p'r's',$$

$\bar{q}\bar{s}\bar{x} = q'p'x' + q'p's'$. Since $\bar{x}\bar{q}\bar{r} + \bar{x}\bar{r}\bar{s} + \bar{x}\bar{s}\bar{q} = \bar{q}\bar{r}\bar{s}$, we have $\bar{q}, \bar{r}, \bar{s}, \bar{x} \ll_2 E_2$. Also since $\bar{r}\bar{x}\bar{s} = \bar{p}\bar{r}\bar{x} + \bar{p}\bar{r}\bar{s}$ while $\bar{p}\bar{x}\bar{s} = 0$, the quadruple $\bar{p}, \bar{r}, \bar{x}, \bar{s}$ is also embeddable. Hence, since in addition $\bar{x}I\bar{q}\bar{r}\bar{s}$, the quintuple is a 2-metric space with property T, and therefore $\bar{q}R\bar{p}\bar{r}\bar{s}$ and $\bar{x}R\bar{p}\bar{q}\bar{r}$ imply $\bar{x}R\bar{p}\bar{r}\bar{s}$, thus contradicting $\bar{p}\bar{x}\bar{s} = 0$.

If not $xIpqr$, let $\bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{x}$ be points such that $\bar{p}, \bar{q}, \bar{r}, \bar{s} \approx_2 p, q, r, s$ and $\bar{p}, \bar{q}, \bar{r}, \bar{x} \approx_2 p, q, r, x$ while

$$\bar{q}\bar{s}\bar{x} = \bar{p}\bar{s}\bar{x} = \bar{r}\bar{s}\bar{x} = \max(pqr, pqs, pqx, prx, qrs, prs, qrx).$$

Hence none of the 10 triples of barred points has non-vanishing 2-metric and therefore, since all quadruples of the barred points satisfy the simplex inequality in a strict sense, the quintuple is a 2-metric space with property T, and hence $\bar{q}R\bar{p}\bar{r}\bar{s}$ and $\bar{x}R\bar{p}\bar{q}\bar{r}$ imply $\bar{x}R\bar{p}\bar{r}\bar{s}$. However, the quadruple $\bar{p}, \bar{r}, \bar{s}, \bar{x}$ satisfies the simplex inequality in a strict sense, contradicting $\bar{x}R\bar{p}\bar{r}\bar{s}$.

Suppose $rIpqs$. Interchanging p and r in the above argument yields a contradiction. Similarly if $sIpqr$, interchanging s and p again yields a contradiction and the result $qIprs$ follows.

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THEOREM 2. *If $p, q, r, s \in S$ and $qIprs$, then $qRprs$.*

Proof. Let p, q, r, s be points of S such that $qIprs$. Then there exist points p', q', r', s' of E_2 such that p', q', r', s' are 2-congruent with p, q, r, s. Given p', r', s', there exists a 2-metric space containing p', r', s' and a point t'' such that $t''Rp'r's'$. The preceding theorem implies that $t''Ip'r's'$; hence there exists a point t' of the plane of p', r', s' such that $t'Ip'r's'$ and

$$p', t', r', s' \approx_2 p', t'', r', s'.$$

Assuming the falsity of $q'Rp'r's'$, then since $\bar{I}(p', r', s')$ is closed, there exists a circle C with centre q' with the property that for at least one u' on C, $u'Rp'r's'$, whereas this fails for all points interior to both the circle and to $[p'r's']$. For at least one of the points p', r', s', the segment joining it to u' is a secant of C, say s'. By a now familiar process, given p', u', s', and $\epsilon > 0$, there exists a point x' of the plane such that $x'Rp'u's'$ (i.e., $x'Ip'u's'$) and $u's'x' < \epsilon$, $p's'x' < \epsilon$, where ϵ is chosen so as to guarantee that the ray $u'x'$ is a secant of C. Applying this process to x', u', s', we obtain a point y' such that $y'Rx'u's'$ and y' is interior to C. Now $y'Rx'u's'$ and $x'Rp'u's'$ imply $y'Rp'u's'$ which, with $u'Rp'r's'$, implies $y'Rp'r's'$, contrary to y' being interior to C, and the proof is complete.

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