ON A TYPE PROBLEM

JAMES A. JENKINS

Considerable interest has attached to the problem of determining the type of a Riemann surface obtained by performing an identification between the edges of a strip or a half-strip **(1, 2, 4, 5, 8)**. A fairly thorough analysis was made in 1946 by Volkovyskii **(6)** who gave various sufficient conditions for parabolic and hyperbolic type. The object of the present paper is to show that his principal sufficient condition for hyperbolic type can be substantially improved.

We regard the half-strip S in the z-plane, $z = x + iy$

$$x > a, 0 < y < b$$

where a and b are finite real numbers, $b > 0$. (The case of a full strip is entirely equivalent.) We denote its edges $x > a$, $y = 0$ and $x > a$, $y = b$ respectively by L_1 and L_2. We consider the identification on L_1 and L_2 determined by the mapping (defined for $x \geqslant a$)

$$T(x) = f(x) + ib$$

where $f(x)$ is an increasing function of x with $f(a) > a$. We will suppose that this identification determines a Riemann surface $\mathscr{R}$ which is then doubly-connected with one boundary component C determined by the segments

$$x = a,\ 0 \leqslant y \leqslant b;\ a \leqslant x \leqslant f(a),\ y = b.$$

For this to hold it is necessary and sufficient that for each $x_0 > a$ there exist a disc $|w| < 1$ divided by a simple open arc λ into two domanins D_1 and D_2 and neighbourhoods E_1 and E_2 of x_0 and $T(x_0)$ relative to S such that there exist conformal mappings $w = \psi_i(z)$, $i = 1, 2$, of E_i onto D_i each admitting a homeomorphic extension to an open boundary arc γ_i of E_i and L_i which it carries onto λ and such that

$$\psi_1(x) = \psi_2(T(x)), \qquad x \in \gamma_1.$$

Non-trivial necessary and sufficient conditions on $f(x)$ for the identification T to determine a Riemann surface are not known. Some sufficient conditions were given by Volkovyskii **(7)**. An easily verified sufficient condition is that $f(x)$ should possess a continuous derivative which does not take the value

Received August 20, 1958. Research supported in part by the National Science Foundation, Mathematics Section, through the University of Notre Dame and by the Office of Ordnance Research under contract No. DA-36-034-ORD-2453 through the Institute for Advanced Study.

zero. For our purposes here we assume that $f(x)$ and its inverse are absolutely continuous, that the identification T does determine a Riemann surface and that the extension of $\psi_i(z)$, $i = 1, 2$, to γ_i is continuously differentiable for each x_0 together with its inverse mapping. Without some restriction it is not *a priori* certain that such a surface is uniquely determined by the identification T. Volkovyskii **(7)** gave some sufficient conditions for this to hold but they are probably far from necessary.

A Riemann surface $\mathscr{R}$ determined by the above identification can be mapped conformally on the circular ring

$$1 < |\zeta| < R$$

where C corresponds to the boundary component $|\zeta| = 1$ and $R \leqslant \infty$. We will distinguish the cases according as $R < \infty$ or $R = \infty$ as being respectively hyperbolic or parabolic.

In order that we have the hyperbolic case it is sufficient that $\mathscr{R}$ have finite module for the family of curves Γ separating its boundary components **(3**, p. 13**)**. That means that if $\rho\,(w)|dw|$ ranges over all conformally invariant metrics on $\mathscr{R}$ (w denotes a local uniformizing parameter) such that for any locally rectifiable simple closed curve γ separating the boundary components of $\mathscr{R}$

$$\int_\gamma \rho|dw| \geqslant 1$$

then ($w = u + iv$)

$$\min \iint_{\mathscr{R}} \rho^2\, du\, dv$$

is finite. In particular it is enough to manifest one admissible metric for which the above integral is finite. First, however, we wish to study the images in $\bar{S}$ of the family of curves Γ on $\mathscr{R}$.

To curves on $\mathscr{R}$ in Γ correspond sets on $\bar{S}$ which may display quite considerable complication. We will denote the family of such sets by Γ^*. Points of such a set γ^* on L_1 are accompanied by their images on L_2 under the mapping T. For any set ω in $\bar{S}$ we will call the (orthogonal) projection of $\omega - L_2$ on L_1 the projection $\pi(\omega)$ of ω. We denote the function $f(x)$ iterated n times by $f_n(x)$, also we take $f_0(x) \equiv x$. Points $(x', 0)$, $(x'', 0)$ on L_1 such that

$$x'' = f_n(x')$$

for some n will be called congruent points. The essential property of sets in Γ^* is given in the following lemma.

LEMMA. *For every $\gamma^* \in \Gamma^*$ there exists a value c, $c \geqslant a$, such that $\pi(\gamma^*)$ contains a point congruent to each point in the interval $[c, f(c))$.*

The set γ^* consists of arcs running from L_1 to L_2, intervals on L_1 and L_2 and arcs running from L_1 back to L_1 or from L_2 back to L_2. We note first that since the corresponding curve $\gamma \in \Gamma$ is compact there can be only a finite number of arcs running from L_1 to L_2. Further we can replace these arcs by rectilinear segments with the same end-points not increasing their projections. If we replace an arc running from $L_1(L_2)$ back to $L_1(L_2)$ by the segment joining its end-points in the projection we at most replace a segment by a congruent segment. Finally a number of segments on $L_1(L_2)$ described consecutively (possibly overlapping) joining two points can be replaced by a single segment joining these points without increasing the projection. Thus it is enough to prove the result of the lemma when γ^* consists of a finite number of segments joining L_1 and L_2 and lying on L_1 and L_2.

There must be at least one segment joining L_1 and L_2. Let then P_1 be the end-point of such a segment farthest to the right on L_1 and P_2 be the end-point of such a segment farthest to the right on L_2. If $P_2 = T(P_1)$, γ^* consists of a single segment and the result of the lemma is evident. If P_2 is to the right of $T(P_1)$ it must be either the end-point of both a segment joining L_1 and L_2 and a segment on L_2 or the end-point of two segments joining L_1 and L_2. In the first instance replacing the segments by a segment forming with them a triangle (and deleting the corresponding segment on L_1) we obtain a new $\gamma' \in \Gamma^*$ with one less side (counting only one side for a pair of corresponding segments on L_1 and L_2) and a not larger projection. In the second instance replacing the segments by the segment on L_1 forming with them a triangle (and inserting the corresponding segment on L_2) we obtain a new $\gamma'' \in \Gamma^*$ with one less side and a not larger projection. Similarly if P_2 is to the left of $T(P_1)$, P_1 must be either the end-point of both a segment joining L_1 and L_2 and a segment on L_1 or the end-point of two segments joining L_1 and L_2. Proceeding as before the same conclusions apply. Since the result of the lemma is true for γ^* consisting of a single segment it follows in full generality by induction.

We now make our final assumption, that

$$\lim_{n\to\infty} f_n(a) = \infty. \tag{1}$$

We denote by I_n, $n = 0, 1, \ldots$, the region

$$f_n(a) \leqslant x < f_{n+1}(a),\ 0 \leqslant y < b.$$

Let $\phi_n(x)$ denote the function inverse to $f_n(x)$. Let $\mu(x)$ be a non-negative integrable function defined for $a \leqslant x < f(a)$ with

$$0 < \int_a^{f(a)} \mu(x)\, dx.$$

Then we consider in S the metric $\rho(z)|dz|$ where

$$\rho(z) = \mu(\phi_n(x))\phi_n'(x), \qquad z \in I_n.$$

Let $\gamma^* \in \Gamma^*$. Under assumption (1) it follows from our lemma that $\pi(\gamma^*)$ contains a point congruent to every point of the interval $[a, f(a))$. Thus

$$\int_{\gamma^*} \rho(z)\,|dz| \geqslant \int_a^{f(a)} \mu(x)\,dx.$$

On the other hand

$$\begin{aligned}\iint_S \rho^2(z)\,dx\,dy &= \sum_{n=0}^{\infty} \iint_{I_n} [\mu(\phi_n(x))\,\phi_n'(x)]^2\,dx\,dy \\ &= b\sum_{n=0}^{\infty} \int_{f_n(a)}^{f_{n+1}(a)} [\mu(\phi_n(x))\,\phi_n'(x)]^2\,dx \\ &= b\sum_{n=0}^{\infty} \int_a^{f(a)} \frac{(\mu(x))^2}{f_n'(x)}\,dx \\ &= b\int_a^{f(a)} (\mu(x))^2 \left(\sum_{n=0}^{\infty} \frac{1}{f_n'(x)}\right) dx\end{aligned}$$

provided that the operations involved are legitimate. This will be the case if

$$\sum_{n=0}^{\infty} \frac{1}{f_n'(x)}$$

converges at those points where $\mu(x)$ is positive and the last integral is convergent. In these circumstances the identification determined by f comes under the hyperbolic case.

We state our result as follows.

THEOREM. *If the function $f(x)$ defined for $a \leqslant x$ is absolutely continuous together with its inverse, if the identification it provides on the strip S determines a Riemann surface $\mathscr{R}$, if the corresponding functions $\psi_i(z)$, $i = 1, 2$, admit extensions to the open boundary arcs of E_i on L_i which are continuously differentiable together with their inverse mappings, if $f(x)$ satisfies the condition*

$$\lim_{n\to\infty} f_n(a) = \infty \tag{1}$$

and if the series

$$\sum_{n=0}^{\infty} \frac{1}{f_n'(x)}$$

converges on a set of positive measure on $[a, f(a))$ then the identification comes under the hyperbolic case.

Indeed this sum is greater than or equal to one on the set of positive measure in question; thus we can take $\mu(x)$ in the preceding argument as the reciprocal of the sum on that set and elsewhere zero.

This result represents a substantial improvement of one of Volkovyskii's basic results which requires that the above series have a bounded sum on an interval as a sufficient condition for the hyperbolic case in addition to other requirements on the function $f(x)$ some of which are not germane to the present problem. It is immediately seen that the present considerations extend similarly to the case of identification of two strips also discussed by Volkovyskii **(6)**.

References

1. C. Blanc, *Les surfaces de Riemann des fonctions méromorphes*, Comm. Math. Helvet., *9* (1936–7), 193–216, 335–68.
2. ———, *Les demi-surfaces de Riemann. Application au problème du type*, Comm. Math. Helvet., *10* (1938–9), 130–50.
3. James A. Jenkins, *Univalent functions and conformal mapping* (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958).
4. R. Nevanlinna, *Ueber die Polygondarstellung einer Riemannschen Fläche*, Ann. Acad. Sci. Fenn. Ser. AI, no. 122 (1952).
5. ———, "Polygonal representation of Riemann surfaces," *Lectures on functions of a complex variable* (Michigan, 1955), 65–70.
6. L. I. Volkovyskii, *On the problem of type of simply-connected Riemann surfaces*, Mat. Sbornik, *18* (N.S.) (1946), 185–211 (Russian, English summary).
7. ———, *Quasiconformal mapping and the problem of conformal pasting*, Ukrain. Mat. Z., *3* (1951), 39–52 (Russian).
8. E. M. Wirth, *Ueber die Bestimmung des Typus einer Riemannschen Fläche*, Comm. Math. Helvet., *31* (1956), 90–107.

University of Notre Dame
Institute for Advanced Study