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A NEW PROOF OF THE d-CONNECTEDNESS OF d-POLYTOPES

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ABSTRACT. Balinski has shown that the graph of a d-polytope is d-connected. In this note we give a new proof of Balinski's theorem.

A fundamental result in the theory of convex polytopes is the following theorem of M. Balinski [1]:

THEOREM. *The graph $\mathcal{G}(P)$ of a d-polytope P is d-connected.*

Here a d-polytope is a convex polytope P in $\mathbf{R}^n$ whose affine hull aff P has dimension d. The graph $\mathcal{G}(P)$ is the graph whose vertices are the vertices of P and whose edges are the edges of P. A graph $\mathcal{G}$ is d-connected provided that is has at least $d+1$ vertices and the removal of as many as $d-1$ vertices (and the edges incident to a removed vertex) does not destroy connectedness. For background information on convex polytopes, including Balinski's theorem, the reader may consult A. Brøndsted [2] and B. Grünbaum [3].

The proofs of Balinski's theorem in [2] and [3] are variants of Balinski's original proof. The aim of this note is to present a new proof based on a different idea.

We need a little terminology. Let P be a d-polytope. Identifying aff P with $\mathbf{R}^d$, we may assume that P lies in $\mathbf{R}^d$. For any facet, i.e. $(d-1)$-dimensional face, F of P we denote by $K(F)$ the closed supporting halfspace of P bounded by the hyperplane aff F. It is a standard fact that

$$P = \bigcap \{K(F) | F \in \mathcal{F}_{d-1}(P)\}, \tag{1}$$

where $\mathcal{F}_{d-1}(P)$ denotes the set of facets of P, see e.g. [2, Corollary 9.6].

For any vertex v of P we denote by $C(P, v)$ the intersection of all closed halfspaces $K(F)$ such that F is a facet of P and v is a vertex of F. It is trivial that

$$P \subseteq C(P, v) \subseteq K(F) \tag{2}$$

for all facets F of P and all vertices v of F. Combining (1) and (2) we obtain

$$P = \bigcap \{C(P, v) | v \in \text{vert } P\}, \tag{3}$$

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where vert P denotes the set of vertices of P.

Our proof of Balinski's theorem is based on the following lemma which shows that we may omit as many as $d-1$ vertices from the right hand side of (3):

LEMMA. Let V be a set of at most $d-1$ vertices of a d-polytope P in $\mathbf{R}^d$. Then

(4) $$P = \bigcap \{C(P,v) | v \in (\text{vert } P) \setminus V\}$$

PROOF. The inclusion $\subseteq$ is a trivial consequence of (3). To prove $\supseteq$, let x be a point not in P. Then by (1) there is a facet F of P such that $x \notin K(F)$. Being a $(d-1)$-polytope, F has at least d vertices, whence at least one vertex v of F is not in V. Using (2) we see that $x \notin C(P,v)$. Hence, x is not in the right hand side of (4). This completes the proof of the lemma. □

We next turn to

PROOF OF THE THEOREM. Let V be any set of at most $d-1$ vertices of P. We shall prove that the subgraph $\mathcal{G}'$ of $\mathcal{G}(P)$ spanned by $(\text{vert } P)\setminus V$ is connected. Let $\mathcal{G}''$ be a connected component of $\mathcal{G}'$; we reach the desired conclusion by showing that $\mathcal{G}' = \mathcal{G}''$.

We denote the vertex set of a graph $\mathcal{G}$ by vert $\mathcal{G}$. Then vert $\mathcal{G}' = (\text{vert } P)\setminus V$. Let Q denote the convex polytope spanned by $(\text{vert } \mathcal{G}'') \cup V$; then vert $Q = (\text{vert } \mathcal{G}'') \cup V$.

Let $v \in \text{vert } \mathcal{G}'' = (\text{vert } Q)\setminus V$. Since $\mathcal{G}''$ is a connected component of $\mathcal{G}'$, any vertex of P adjacent to v is in $(\text{vert } \mathcal{G}'') \cup V = \text{vert } Q$. In other words:

(5) Any $v \in (\text{vert } Q)\setminus V$ has the same adjacent vertices in Q as in P.

The following is a standard fact, cf. [2, Corollary 11.7]:

(6) For any convex polytope R and any vertex u of R, the affine hull of u and its adjacent vertices in R is aff R.

Now, since P is a d-polytope, it follows from (5) and (6) that Q is also a d-polytope. Noting that $V \subseteq \text{vert} Q$, we may then apply the lemma to Q, obtaining

(7) $$Q = \bigcap \{C(Q,v) | v \in (\text{vert } Q)\setminus V\}.$$

We next claim that

(8) $$P \subseteq C(Q,v) \text{ for all } v \in (\text{vert } Q)\setminus V.$$

To prove (8), let $v \in (\text{vert } Q)\setminus V$. Let x be an arbitrary point of P. If $x = v$, then $x \in C(Q,v)$ as desired. If $x \neq v$, we choose a hyperplane H which separates v from the remaining vertices of P and from x. The sets $P' := H \cap P$ and $Q' := H \cap Q$ are then $(d-1)$-polytopes, cf. [2, Theorem 11.2]. The vertices of P' are the points where the edges of P connecting v and its adjacent vertices in P intersect H, cf. [2, Theorem

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11.2]. Similarly for Q'. It then follows from (5) that P' and Q' have the same vertices, whence $P' = Q'$. Let x' be the point where the segment from v to x intersects H. Then obviously $x' \in P'$. Since $P' = Q'$, it follows that $x' \in Q'$, whence $x' \in Q$, and so $x' \in C(Q, v)$. Since $C(Q, v)$ is a cone with vertex v, it follows that the entire halfline emanating from v and passing through x' is in $C(Q, v)$. In particular, $x \in C(Q, v)$ as desired.

Combining (7) and (8) we obtain $P \subseteq Q$, whence

$$P = Q \tag{9}$$

since $Q \subseteq P$ is obvious.

Finally, let $v \in \text{vert } \mathcal{G}' = (\text{vert } P)\backslash V$. Then $v \in (\text{vert } Q)\backslash V = \text{vert } \mathcal{G}''$ by (9). Hence vert $\mathcal{G}' =$ vert $\mathcal{G}''$. By the nature of $\mathcal{G}'$ and $\mathcal{G}''$ this implies $\mathcal{G}' = \mathcal{G}''$ as desired. □

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