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Nagoya Math. J.
Vol. 96 (1984), 61–70

KÄHLERIAN SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE WITH SECOND FUNDAMENTAL FORM OF POLYNOMIAL TYPE

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Let P_N be an N-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature, and M be a Kählerian submanifold in P_N. Let H be the second fundamental tensor of M, and $\overset{+}{\nabla}$ be the covariant derivative of type (1, 0) on M. We proved in [5] that, if M is locally symmetric, then

$$(1)\qquad \overset{+}{\nabla}{}^m H = 0 \qquad \text{for some positive integer } m\,.$$

So it will be a natural question to ask what Kählerian submanifolds satisfy the above condition (1). In this paper we give some partial solutions to it. First we show that the condition (1) is equivalent to

$$(2)\qquad \overset{+}{\nabla}{}^d R = 0 \qquad \text{for some positive integer } d\,,$$

where R denotes the curvature tensor of M. On the other hand, the curvature tensor R of every Kählerian C-space satisfies the condition (2) ([4]). Thus every Kählerian C-space holomorphically embedded in P_N satisfies the condition (1) too. Next we prove that, if M is a Kählerian hypersurface with condition (1) in P_N, then M is totally geodesic or a complex quadric. Finally we give some examples of Kählerian submanifold in P_N satisfying $\overset{+}{\nabla}{}^2 H = 0$ but $\overset{+}{\nabla} H \neq 0$.

§1. Preliminaries

In this section we survey briefly the notion of Kählerian submanifold in P_N (for the detail, see e.g. [2]). Let M be an n-dimensional Kählerian submanifold in P_{n+q}. We use the following convention on the range of indices unless otherwise stated: $A, B, \cdots = 1, \cdots, n, n+1, \cdots, n+q$; $i, j, \cdots = 1, \cdots, n$; $\alpha, \beta, \cdots = n+1, \cdots, n+q$. Let $\{e_1, \cdots, e_{n+q}\}$

Received July 16, 1983.

https://doi.org/10.1017/S0027763000021139 Published online by Cambridge University Press

be a local field of unitary frames in P_{n+q} such that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. Denote its dual frame field by $\omega^1, \cdots, \omega^{n+q}$. The connection forms ω_B^A with respect to ω_A and the connection ∇ on P_{n+q} are related by

$$\nabla_{e_A} e_B = \sum_C \omega_B^C(e_A) e_C . \tag{1.1}$$

Restrict the forms under consideration to M. Then, since $\omega^\alpha = 0$, the forms ω_i^α can be written as

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j , \qquad h_{ij}^\alpha = h_{ji}^\alpha . \tag{1.2}$$

The quadratic form $\sum_{i,j} h_{ij}^\alpha \omega^i \cdot \omega^j$ is called the second fundamental form of M in the direction of e_α. The curvature form Ω_j^i of M is defined by

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k . \tag{1.3}$$

It can be expressed as

$$\Omega_j^i = \sum_{k,\ell} R_{jk\bar{\ell}}^i \, \omega^k \wedge \bar{\omega}^\ell . \tag{1.4}$$

The equation of Gauss is given by

$$R_{jk\bar{\ell}}^i = c(\delta_j^i \delta_{k\ell} + \delta_k^i \delta_{j\ell}) - \sum_\alpha h_{jk}^\alpha \bar{h}_{i\ell}^\alpha , \tag{1.5}$$

where $2c$ denotes the constant holomorphic sectional curvature of P_{n+q}. The value c itself is not important in this paper. The Ricci tensor $S = (S_{i\bar{j}})$ of M is defined by

$$S_{i\bar{j}} = \sum_k R_{ik\bar{j}}^k = (n+1)c\delta_{ij} - \sum_{\alpha,k} h_{ik}^\alpha \bar{h}_{kj}^\alpha . \tag{1.6}$$

We define the higher covariant derivatives $h_{i_1 \cdots i_m j}^\alpha$ and $h_{i_1 \cdots i_m \bar{j}}^\alpha$ of h_{ij}^α inductively as follows.

$$\begin{aligned} &\sum_j h_{i_1 \cdots i_m j}^\alpha \, \omega^j + \sum_j h_{i_1 \cdots i_m \bar{j}}^\alpha \, \bar{\omega}^j \\ &\quad = dh_{i_1 \cdots i_m}^\alpha - \sum_{r=1}^m \sum_j h_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_m}^\alpha \, \omega_{i_r}^j \\ &\qquad + \sum_\beta h_{i_1 \cdots i_m}^\beta \, \omega_\beta^\alpha . \end{aligned} \tag{1.7}$$

Then the component of the tensor $\overset{+}{\nabla}{}^m H$ used in the introduction is nothing but $h_{i_1 \cdots i_{m+2}}^\alpha$.

LEMMA 1.1 ([2]). *The following relation holds.*

$$h^{\alpha}_{i_1\cdots i_m\bar{j}} = \frac{m-2}{2}c\sum_{r=1}^{m} h^{\alpha}_{i_1\cdots\hat{i}_r\cdots i_m}\delta_{i_r j} - \sum_{r=1}^{m-2}\frac{1}{r!\,(m-r)!}\sum_{\alpha,\beta,\ell,\sigma} h^{\alpha}_{\ell i_{\sigma(1)}\cdots i_{\sigma(r)}} h^{\beta}_{i_{\sigma(r+1)}\cdots i_{\sigma(m)}} \bar{h}^{\beta}_{\ell j}\,,$$

where the summation on σ *is taken over all permutations of* $\{1, \cdots, m\}$. *In particular,* $h^{\alpha}_{i_1\cdots i_m}$ *is symmetric with respect to* $i_1, \cdots, i_m$, *and* $h^{\alpha}_{ij\bar{k}} = 0$.

§ 2. Results and proofs

In this section we denote by M a Kählerian submanifold in P_{n+q} and keep the notation in Section 1.

DEFINITION. Denote the tangent space of a manifold N at a point p by $T_p(N)$. For a point p of M we denote by N_p the normal space of $T_p(M)$ in $T_p(P_{n+q})$, and by N_p^C the complexification of N_p. Let $m(\geqq 2)$ be an integer. To each point p of M we assign the complexification of the subspace of N_p spanned by the vectors $\sum_\alpha h^{\alpha}_{i_1\cdots i_m}(p)(e_\alpha)_p$ over C, which we denote by $H_m(p)$.

Remark that Lemma 1.1 implies

$$\sum_\alpha h^{\alpha}_{i_1\cdots i_m\bar{j}}e_\alpha \in H_2 + \cdots + H_{m-1}\,. \tag{2.1}$$

LEMMA 2.1. *Assume there exist two integers* r *and* ℓ *such that* $r > \ell \geqq 2$ *and* $H_r \perp (H_2 + \cdots + H_\ell)$. *Then* (1) $H_s \perp (H_2 + \cdots + H_\ell)$ *for any integer* s *with* $s \geqq r$, *and* (2) $H_{2r-2} \perp (H_2 + \cdots + H_{\ell+1})$.

Proof. Let a be any integer such that $2 \leqq a \leqq \ell$. Then the assumption can be rewritten as

$$\sum_\alpha h^{\alpha}_{i_1\cdots i_r}\bar{h}^{\alpha}_{j_1\cdots j_a} = 0\,. \tag{2.2}$$

In order to show (1) it suffices to show $H_{r+1} \perp (H_2 + \cdots + H_\ell)$. Taking the covariant derivative of (2.2) with respect to e_k, we have

$$\sum_\alpha h^{\alpha}_{i_1\cdots i_r k}\bar{h}^{\alpha}_{j_1\cdots j_a} + \sum_\alpha h^{\alpha}_{i_1\cdots i_r}\bar{h}^{\alpha}_{j_1\cdots j_a\bar{k}} = 0\,.$$

The second term of the left hand side of this equation vanishes by (2.1) and (2.2), which shows (1). Now by (1) we have

$$\sum_\alpha h^{\alpha}_{i_1\cdots i_{2r-2}}\bar{h}^{\alpha}_{j_1\cdots j_a} = 0\,. \tag{2.3}$$

Taking the covariant derivative of (2.3) with respect to $\bar{e}_k$, we have

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$$\sum_\alpha h^\alpha_{i_1\cdots i_{2r-2}} \bar{h}^\alpha_{j_1\cdots j_a k} + \sum_\alpha h^\alpha_{i_1\cdots i_{2r-2}\bar{k}} \bar{h}^\alpha_{j_1\cdots j_a} = 0 \, .$$

It follows from Lemma 1.1 that the second term of the left hand side of this equation is equal to

$$\begin{aligned} &2(r-2)c \sum_{b=1}^{2r-2} h^\alpha_{i_1\cdots \hat{i}_b \cdots i_{2r-2}} \delta_{i_b k} \bar{h}^\alpha_{j_1\cdots j_a} \\ &\quad - \sum_{b=1}^{2r-4} \sum_{\beta,\ell,\sigma} \frac{1}{b!\,(2r-4-b)!} h^\alpha_{\ell i_{\sigma(1)}\cdots i_{\sigma(b)}} h^\beta_{i_{\sigma(b+1)}\cdots i_{\sigma(2r-2)}} \bar{h}^\beta_{\ell k} \bar{h}^\alpha_{j_1\cdots j_a} \, . \end{aligned}$$

But the first term vanishes by (1), and the second also vanishes by (1) since $b+1 \geqq r$ or $2r-2-b \geqq r$. q.e.d.

DEFINITION. Let d be an integer with $d \geqq 3$. Define a sequence $\{d_i\}_{i=1,2,\cdots}$ of integers inductively as follows. First put $d_1 = 2$ and $d_2 = d$. Assume d_k was defined for $k = 1, \cdots, i$. Let $\{c_m\}$ be a sequence of integers defined by $c_1 = d_i$ and $c_{m+1} = 2c_m - 2$. Then put $d_{i+1} = c_m$ where $m = d_i - d_{i-1}$. The sequence $\{d_i\}$ shall be said to be associated with an integer d.

LEMMA 2.2. *Assume there exists an integer* $d \geqq 3$ *such that* $H_d \perp H_2$. *Let* $\{d_i\}$ *be the sequence of integers associated with* d. *Then the vector spaces* $H_{d_1}, H_{d_2}, \cdots$ *are mutually orthogonal.*

Proof. Since $H_d \perp H_2$, applying Lemma 2.1(2) $d_2 - d_1$ times, we find $H_{d_3} \perp (H_{d_1} + \cdots + H_{d_2})$. Repeat this argument to obtain

$$H_{d_i} \perp (H_{d_1} + \cdots + H_{d_2} + \cdots + H_{d_{i-1}})$$

for each positive integer i. q.e.d.

The following Theorem gives our problem a geometric meaning.

THEOREM 2.3. *Let* M *be an* n*-dimensional Kählerian submanifold in* P_{n+q}. *Let* R *be the curvature tensor of* M, H *be the second fundamental tensor of* M, *and* $\overset{+}{\nabla}$ *be the covariant derivative of type* (1, 0) *on* M. *Then the following two conditions are equivalent.*

(A) *There exists a positive integer* d *such that* $\overset{+}{\nabla}{}^d R = 0$.

(B) *There exists a positive integer* m *such that* $\overset{+}{\nabla}{}^m H = 0$.

Proof. By (1.5) the condition (A) is equivalent to

$$\text{(C)} \qquad H_{d+2} \perp H_2 \, .$$

Thus clearly (B) implies (A). Now assume (C). If $H_m \neq \{0\}$ for all integers

m ($\geqq 2$), then Lemma 2.2 implies that for each point p of M there exists a sequence $H_{d_1}(p)$, $H_{d_2}(p)$, $\cdots$ of infinitely many mutually orthogonal nonzero vector subspaces of N_p^C, which is a contradiction. q.e.d.

Now we state a relation between two integers d and m in Theorem 2.3.

THEOREM 2.4. *Let M, P_{n+q}, R, H and $\overset{+}{\nabla}$ be as in Theorem 2.3. Assume that M is neither flat nor totally geodesic, and that there exists a positive integer d such that $\overset{+}{\nabla}{}^d R = 0$ and $\overset{+}{\nabla}{}^{d-1} R \neq 0$. Let m be the positive integer determined by $\overset{+}{\nabla}{}^m H = 0$ and $\overset{+}{\nabla}{}^{m-1} H \neq 0$. Let $\{d_i\}$ be the sequence of integers associated with $d+2$. Then $m \leqq d_{q+1} - 2$.*

Proof. By Lemma 2.2 we see that there exist a positive integer i and a point p of M such that the subspaces $H_{d_1}(p)$, $H_{d_2}(p)$, $\cdots$, $H_{d_i}(p)$ of N_p^C are mutually orthogonal and $H_{d_i}(p) \neq \{0\}$ and $H_{d_{i+1}}(p) = \{0\}$. Since $\dim_C N_p^C = q$, we have $i \leqq q$. This and the definition of m give $m + 2 \leqq d_{i+1} \leqq d_{q+1}$. q.e.d.

Here we consider our problem in the case of codimension 1.

THEOREM 2.5. *Let M be a Kählerian hypersurface in P_{n+1}. Let H be the second fundamental tensor of M and $\overset{+}{\nabla}$ be the covariant derivative of type* (1, 0) *on M. Assume there exists a positive integer m such that $\overset{+}{\nabla}{}^m H = 0$. Then M is totally geodesic or a part of a complex quadric.*

Proof. Since $q = 1$, we may omit the index α. In the case where $m = 1$, our theorem has been already proved by B. Smyth [3]. So assume $m \geqq 2$. Let an index a (resp. r) stand for any index i such that $h_{\overbrace{i \cdots i}^{m+1}} \neq 0$ (resp. $h_{\overbrace{i \cdots i}^{m+1}} = 0$). The set of such indices a's is not empty. In fact, if empty, we have $h_{\overbrace{i \cdots i}^{m+1}} = 0$ for each i, which implies $H_{m+1} = 0$. In this proof, let the index ℓ run from 1 to $m - 1$, and the index u run from 0 to $\ell - 1$. By Lemma 1.1, we can rewrite

$$h_{\underbrace{a \cdots a \overbrace{r \cdots r}^{u} \bar{i}}_{m+2+\ell}} = 0$$

as follows.

$$E_{\ell,u} \cdots \sum_{w=0}^{u} \sum_{v=\ell+2}^{m+1} \binom{m+2+\ell-u}{m+2+\ell-v-w} \binom{u}{w} \sum_j h_{\underbrace{ja \cdots a \overbrace{r \cdots r}^{w}}_{m+2+\ell-v}} h_{\underbrace{a \cdots a \overbrace{r \cdots r}^{u-w}}_{v}} \bar{h}_{ji} = 0 .$$

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Then $E_{m-1,0}$ is given by

$$\sum_j h_{j\overbrace{a\cdots a}^{m}} h_{\overbrace{a\cdots a}^{m+1}} \bar{h}_{ji} = 0 ,$$

which yields

$$\sum_j h_{j\overbrace{a\cdots a}^{m}} h_{ji} = 0 , \tag{2.1}$$

since $h_{\overbrace{a\cdots a}^{m+1}} \neq 0$.

Moreover $E_{m-2,0}$ is given by

$$\binom{2m}{m} \sum_j h_{j\overbrace{a\cdots a}^{m}} h_{\overbrace{a\cdots a}^{m}} \bar{h}_{ji} + \binom{2m}{m-1} \sum_j h_{j\overbrace{a\cdots a}^{m-1}} h_{\overbrace{a\cdots a}^{m+1}} \bar{h}_{ji} = 0 ,$$

which, together with (2.1), implies

$$\sum_j h_{j\overbrace{a\cdots a}^{m-1}} \bar{h}_{ji} = 0 .$$

Repeat this argument $m-3$ more times to obtain

$$\sum_j h_{j\overbrace{a\cdots a}^{\ell}} \bar{h}_{ji} = 0 \qquad \text{for } \ell \geqq 2 . \tag{2.2}$$

Next $E_{m-1,1}$ is given by

$$\binom{2m}{m} \sum_j h_{j\overbrace{a\cdots a}^{m}} h_{\overbrace{a\cdots ar}^{m+1}} \bar{h}_{ji} + \binom{2m}{m-1} \sum_j h_{j\overbrace{a\cdots ar}^{m}} h_{\overbrace{a\cdots a}^{m+1}} \bar{h}_{ji} = 0 ,$$

which, together with (2.1), yields

$$\sum_j h_{j\overbrace{a\cdots ar}^{m}} \bar{h}_{ji} = 0 .$$

Just as we obtained (2.2), we have from $E_{m-2,1}, \cdots, E_{1,1}$ and (2.2)

$$\sum_j h_{j\overbrace{a\cdots ar}^{\ell}} \bar{h}_{ji} = 0 \qquad \text{for } \ell \geqq 2 . \tag{2.3}$$

Similarly, from $E_{m-1,2}, \cdots, E_{1,2}$, (2.2) and (2.3) we have

$$\sum_j h_{j\overbrace{a\cdots arr}^{m}} \bar{h}_{ji} = 0 \qquad \text{for } \ell \geqq 2 . \tag{2.4}$$

In particular, from (2.2), (2.3) and (2.4) we have

$$\sum h_{jk\ell} \bar{h}_{ji} = 0 .$$

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This and (1.6) mean that the Ricci tensor of M is parallel. Now our theorem is reduced to Takahashi's one [6]. q.e.d.

§ 3. Examples of $\overset{+}{\nabla}{}^2 H = 0$ but $\overset{+}{\nabla} H \neq 0$

In this section we give three examples of a Kählerian submanifold in P_n satisfying $\overset{+}{\nabla}{}^2 H = 0$ but $\overset{+}{\nabla} H \neq 0$. They are given as orbits in P_n under certain Lie subgroups of the special unitary group $SU(n+1)$. We fix a flat Hermitian metric on C^{n+1}. Let S be a hypersphere in C^{n+1} centered at the origin. Let π be the canonical projection of S onto P_n. For a point p of S we denote by H_p the linear subspace of $T_p(S)$ orthogonal to the 1-dimensional linear subspace $RI(p)$, where I denotes the complex structure of C^{n+1}. The restriction $\pi_{*|H_p}$ of the differential map π_* of π at p to H_p is an isometric isomorphism of H_p onto $T_{\pi(p)}(P_n)$. For $v \in T_p(C^{n+1})$ (resp. $v \in T_p(S)$) we denote by v_S (resp. v_H) the orthogonal projection of v to $T_p(S)$ (resp. H_p). Let X be any element of the Lie algebra $\mathfrak{su}(n+1)$ of $SU(n+1)$. Then the 1-parameter subgroup $\exp tX$ of $SU(n+1)$ induces Killing vector fields both on C^{n+1} and P_n, which are denoted by X^* and $\tilde{X}^*$ respectively. The restriction $X^*|_S$ is a Killing vector field of S, which is also denoted by X^* for simplicity. Clearly $\pi_* X^* = \tilde{X}_*$. Let ∇ (resp. $\tilde{\nabla}$) denote the connection on S (resp. P_n). Then we have

$$\nabla_x Y^* = (YX(p))_S \qquad \text{for } X, Y \in \mathfrak{su}(n+1)\,, \tag{3.1}$$

where we put $x = X_p^*$. In fact, if we denote by $\mathring{\nabla}$ the flat connection on C^{n+1}, then

$$\begin{aligned}
\nabla_x Y^* = (\mathring{\nabla}_x Y^*)_S &= \left(\frac{d}{dt}\Big|_o Y^*_{(\exp tX((p))}\right)_S \\
&= \left(\frac{d}{dt}\Big|_o Y^*(\exp tX)(p)\right)_S \\
&= \left(\frac{d}{dt}\Big|_o Y((\exp tX)(p))\right)_S \\
&= (YX(p))_S\,.
\end{aligned}$$

Moreover the following formula is fundamental.

$$\tilde{\nabla}_{\pi_*(x)} \tilde{Y}^* = \pi_*((\nabla_x Y^*)_H) \qquad \text{for } X, Y \in \mathfrak{su}(n+1)\,. \tag{3.2}$$

Let G be a Lie subgroup of $SU(n+1)$. We consider an orbit $\tilde{M} = G(\tilde{p}) = \pi(G(p))$, where $\tilde{p} = \pi(p)$. Denote the normal space of $T_{\tilde{p}}(\tilde{M})$ (resp.

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$T_p(S)$) in $T_{\tilde{p}}(P_n)$ (resp. $T_p(M)$) by $\tilde{N}$ (resp. N). Let $\tilde{x}, \tilde{y} \in T_{\tilde{p}}(\tilde{M})$, and Y be any element of the Lie algebra $\mathfrak{g}$ of G such that $\tilde{y} = Y^*_p$. Then the $\tilde{N}$-component of a vector $\tilde{\nabla}_{\tilde{x}}\tilde{Y}^*$ is not independent of a choice of Y, which is denoted by $\alpha(\tilde{x}, \tilde{y})$. α is just the second fundamental form of $\tilde{M}$ at $\tilde{p}$. The image of α is called the first normal space of $\tilde{M}$ at $\tilde{p}$. Similarly we can define the first normal space of M in S at p. From (3.1) and (3.2) we have

LEMMA 3.1. *Let the notation be as above. If the vectors* $(XY(p))_N$ *where* $X, Y \in \mathfrak{g}$ *span the normal space* N, *then the first normal space of* $\tilde{M}$ *at* $\tilde{p}$ *coincides with the normal space* $\tilde{N}$.

In the following we shall give a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{su}(n+1)$ and a point p satisfying the assumption of Lemma 3.1. Let $\ell(\geqq 3)$ be an integer, and let the indices $A, B, \cdots$ stand for $2\ell+1$ values $\bar{1}, \cdots, \bar{\ell}, 0, 1, \cdots, \ell$. Denote by E_{AB} the matrix $(\delta_{CA}\delta_{DB})$. Define the elements H_i, X_{AB} of the Lie algebra $\mathfrak{sl}(n+1)$ of the special linear group by

$$
\begin{cases} H_i = E_{ii} - E_{\bar{i}\bar{i}} & (i = 1, \cdots, \ell) \\ X_{AB} = E_{A\bar{B}} - E_{B\bar{A}}\,, & \text{where} \quad \bar{\bar{A}} = A\,. \end{cases} \tag{3.3}
$$

Let $\mathfrak{h}$ be the complex vector space generated by the vectors $H_1, \cdots, H_\ell$, and $\lambda_1, \cdots, \lambda_\ell$ be the dual forms of $H_1, \cdots, H_\ell$. Then the vectors H_i and X_{AB} generate a complex simple Lie algebra $\mathfrak{g}_1$ of type B_ℓ in the sence of E. Cartan in such a way that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_1$ and a vector X_{AB} is a root vector belonging to a root $\lambda_A + \lambda_B$ with respect to $\mathfrak{h}$, where $\lambda_o = 0$ and $\lambda_{\bar{i}} = -\lambda_i$ $(i = 1, \cdots, \ell)$ (cf. [1]). It is easily seen that, with respect to an ordering $\lambda_1 > \cdots > \lambda_\ell$, the set $\{\lambda_1 - \lambda_2, \cdots, \lambda_{\ell-1} - \lambda_\ell, \lambda_\ell\}$ is a fundamental root system. Let $\{\Lambda_1, \cdots, \Lambda_\ell\}$ be the corresponding fundamental weight system. Then the above description (3.3) of $\mathfrak{g}_1$ is nothing but the one of the irreducible representation ρ_1 of $\mathfrak{g}_1$ with the highest weight $\Lambda_1 = \lambda_1$. Define a representation ρ_2 of $\mathfrak{g}_1$ on $\overset{2}{\wedge} C^{2\ell+1}$ by

$$
\rho_2(X)(e_A \wedge e_B) = Xe_A \wedge e_B + e_A \wedge Xe_B\,, \qquad X \in \mathfrak{g}_1\,. \tag{3.4}
$$

Then ρ_2 is irreducible and the highest weight is equal to $\Lambda_2 = \lambda_1 + \lambda_2$. Let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}_1$ such that $\mathfrak{g}_u \subset \mathfrak{sl}(2\ell+1)$, and G_u be the Lie subgroup of $SU(2\ell+1)$ with the Lie algebra $\mathfrak{g}_u$. We want to show that $\mathfrak{g} = \mathfrak{g}_u$ and $p = e_1 \wedge e_2$ satisfy the assumption of Lemma 3.1. For this it suffices to show that the vectors

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$$\rho_2(X)\rho_2(Y)(p)\,, \qquad X, Y \in \mathfrak{g}_1 \tag{3.5}$$

span the complexification N^C of the normal space of an orbit $G_u(p)$ in $T_p(S)$ over C. Hereafter we abbreviate $e_A \wedge e_B$ to $A \wedge B$. Let the indices i, j run from 3 to ℓ. Since $E_{AB}(e_C) = \delta_{BC}e_A$, it follows from (3.3) and (3.4) that the complexification $\mathfrak{g}_1(p)$ of $T_p(G_u(p))$ is spanned by the $4\ell - 5$ vectors

$$\begin{aligned}
&H_1(p) = 1 \wedge 2\,, \qquad H_2(p) = 1 \wedge 2\,,\\
&X_{\bar 1}(p) = (E_{\bar 10} - E_{01})1 \wedge 2 = 2 \wedge 0\,, \quad X_{\bar 2}(p) = (E_{\bar 20} - E_{02})1 \wedge 2 = -1 \wedge 0\,,\\
&X_{i\bar 1}(p) = (E_{i1} - E_{\bar 1\bar i})1 \wedge 2 = -2 \wedge i\,, \quad X_{i\bar 2}(p) = (E_{i2} - E_{\bar 2\bar 1})1 \wedge 2 = 1 \wedge i\,,\\
&X_{\bar 1\bar i}(p) = (E_{\bar i2} - E_{\bar 2i})1 \wedge 2 = 2 \wedge \bar i\,, \quad X_{\bar 2\bar i}(p) = (E_{\bar i2} - E_{\bar 2i})1 \wedge 2 = -1 \wedge \bar j\,,\\
&X_{\bar 1\bar 2}(p) = 1 \wedge \bar 1 + 2 \wedge \bar 2\,.
\end{aligned}$$

Therefore the space N^C is spanned by the vectors

$$\begin{aligned}
&1 \wedge \bar 1 - 2 \wedge \bar 2,\ 1 \wedge \bar 2,\ 2 \wedge \bar 1,\ i \wedge 0,\ i \wedge \bar 1,\ i \wedge \bar 2,\ i \wedge \bar j,\ 0 \wedge \bar 1,\ 0 \wedge \bar 2\,,\\
&0 \wedge \bar i,\ \bar 1 \wedge \bar 2,\ \bar 1 \wedge \bar i,\ \bar 2 \wedge \bar i,\ \bar i \wedge \bar j,\ i \wedge j\,.
\end{aligned}$$

On the other hand, the following vectors are of the form (3.5)

$$\begin{aligned}
&X_{\bar 1}X_{\bar 1}(p) = 2 \wedge \bar 1,\ \ X_{\bar 2}X_{\bar 1}(p) = 2 \wedge \bar 2,\ \ X_{i\bar 2}X_{\bar 1}(p) = i \wedge 0,\ \ X_{\bar 2\bar 1}X_{\bar 1}(p) = -0 \wedge \bar i\,,\\
&X_{\bar 1\bar 2}X_{\bar 1}(p) = -0 \wedge \bar 1,\ \ X_{\bar 2}X_{\bar 2}(p) = -1 \wedge \bar 2,\ \ X_{\bar 1\bar 2}X_{\bar 2}(p) = -0 \wedge \bar 2\,,\\
&X_{j\bar 2}X_{i\bar 1}(p) = i \wedge j,\ \ X_{\bar 2\bar j}X_{i\bar 1}(p) = i \wedge \bar j + \delta_{ij}2 \wedge \bar 2,\ \ X_{\bar 1\bar 2}X_{i\bar 1}(p) = i \wedge \bar 1\,,\\
&X_{\bar 1\bar 2}X_{i\bar 2}(p) = -i \wedge \bar 2,\ \ X_{\bar 2\bar j}X_{\bar 1\bar i}(p) = \bar i \wedge \bar j,\ \ X_{\bar 1\bar 2}X_{\bar 1\bar i}(p) = \bar i \wedge \bar 1\,,\\
&X_{\bar 1\bar 2}X_{\bar 2\bar i}(p) = -\bar 2 \wedge \bar i,\ \ X_{\bar 1\bar 2}X_{\bar 1\bar 2}(p) = \bar 1 \wedge \bar 2 + \bar 1 \wedge \bar 2\,.
\end{aligned}$$

Thus we have proved that $G = G_u$ and $p = e_1 \wedge e_2$ satisfy the assumption of Lemma 3.1.

Now we assert that the second fundamental tensor H of our orbit $\tilde M = G_u(\pi(p))$ in P_N, where $N = 2\ell^2 + \ell$, satisfies $\overset{+}{\nabla}{}^2 H = 0$ but $\overset{+}{\nabla} H \neq 0$. Indeed, let R be the curvature tensor of M. Then we proved in [4] that $\overset{+}{\nabla}{}^2 R = 0$ but $\overset{+}{\nabla} R \neq 0$. This and (1.5) imply

$$\sum_\alpha h^\alpha_{ijk\ell}\bar h^\alpha_{mr} = 0\,, \qquad \sum_\alpha h^\alpha_{ijk}\bar h^\alpha_{\ell m} \neq 0\,.$$

Hence every normal vector $h_{ijk\ell} = (h^\alpha_{ijk\ell})$ is orthogonal to the complexification of the first normal space of M at every point. Thus, owing to

Lemma 3.1, we have $h^{\alpha}_{ijk\ell}(p)=0$. By homogeneity of $\tilde{M}$ we find $h^{\alpha}_{ijk\ell}=0$, $h^{\alpha}_{ijk}\neq 0$, which proves our assertion.

We have two more examples of Kählerian submanifold in P_n such that $\overset{+}{\nabla}{}^2 H=0$ but $\overset{+}{\nabla} H\neq 0$. But we omit to descrive them since their constructions are essentially the same as above. We only mention that they are given as C-spaces $M_1=M(A_\ell, \alpha_1, \alpha_\ell)$ and $M_2=M(D_\ell, \alpha_2)$ holomorphically embedded in P_n (see [4] for the notation). Under the same notation, the previous example is a C-space $M(B_\ell, \alpha_2)$. We remark that $\dim_C M=2\ell-1$ $(\ell\geqq 2)$, $\dim_C M_2=4\ell-7$ $(\ell\geqq 4)$, $\mathrm{codim}_C M_1=\ell^2$ and $\mathrm{codim}_C M_2=2\ell^2+3\ell+6$.

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