J. Austral. Math. Soc. (*Series A*) **26** (1978), 65–69

A UNIQUENESS SET FOR ALL $H^p(B_n)$ WITH $p>0$

P. S. CHEE

(Received 8 September; revised 5 December 1977)

Communicated by E. Strezelecki

Abstract

For $n \geqslant 2$, a hypersurface in the open unit ball B_n in $\mathfrak{C}^n$ is constructed which satisfies the generalized Blaschke condition and is a uniqueness set for all $H^p(B_n)$ with $p>0$. If $n \geqslant 3$, the hypersurface can be chosen to have finite area.

Subject classification (*Amer. Math. Soc.* (*MOS*) *1970*): primary 32 A 10.

Key words and phrases: Balls, Nevanlinna class, Hardy spaces, zero sets, uniqueness sets, Blaschke condition, hypersurfaces.

1. Introduction

Let B_n be the open unit ball in $\mathfrak{C}^n$. Denote its boundary S^{2n-1} by ∂B_n. (Note that $B_1 = U$, the open unit disc in $\mathfrak{C}$.) For f holomorphic in B_n, define

$$\text{(1)} \qquad \|f\|_p = \sup_{0<r<1} \left\{ \int_{\partial B_n} |f(rw)|^p \, dm(w) \right\}^{1/p}, \quad 0<p<\infty,$$

$$\|f\|_\infty = \sup_{z \in B_n} |f(z)|,$$

where m is the Lebesgue measure on ∂B_n normalized so that $m(\partial B_n) = 1$. The *Hardy space* $H^p(B_n)$ is the set of all holomorphic functions f in B_n with $\|f\|_p < \infty$. The *Nevanlinna class* $N(B_n)$ is defined analogously (with $\log^+|f|$ in place of $|f|^p$ in (1)). We have the inclusion

$$H^\infty(Bn) \subseteq H^q(Bn) \subseteq H^p(Bn) \subseteq N(Bn) \quad \text{if} \quad 0<p<q<\infty.$$

If $f \in N(B_n)$, its zero set is denoted by $Z(f) = f^{-1}(0)$. A subset E of B_n is a *uniqueness set* (or determining set) for a family $\mathscr{F} \subseteq N(B_n)$ if $f \in \mathscr{F}$ and $f = 0$ on E imply $f \equiv 0$ in B_n.

When $n = 1$, $B_n = U$ and the zero sets of $N(U)$ are completely characterized by the Blaschke condition.

When $n > 1$, it was shown in Chee (1970a, b) that the Blaschke condition can be generalized and is a necessary condition for the zero sets of $N(B_n)$: For any $f \in N(B_n), f \not\equiv 0$,

$$\int_0^1 dr \int_{\bar{B}(r)} \mu(z)\, dH_{2n-2}(z) < \infty \tag{2}$$

where $\bar{B}(r)$ is the closed ball in $\mathfrak{C}^n$ of radius r, $\mu(z)$ is the zero multiplicity of f at z and H_k denotes the k-dimensional Hausdorff measure in $\mathfrak{C}^n$. However, it was also shown in Chee (1970a) that (2) is not a sufficient condition for the zero sets of $H^\infty(B_n)$. In Rudin (1976), Rudin showed that if $0 < p < q < \infty$, then there exists an $f \in H^p(B_n)$ whose zero set is a uniqueness set for $H^q(B_n)$. The analogous result for the polydisc was proved by Miles (1973).

The purpose of this paper is to prove the following.

THEOREM. *Let $n \geq 2$. Then there exists a hypersurface V in B_n such that V is a uniqueness set for all $H^p(B_n)$ with $p > 0$, and satisfies the Blaschke condition*

$$\int_0^1 H_{2n-2}(V(r))\, dr < \infty,$$

where $V(r) = \{(z_1, \ldots, z_n) \in V: |z_1|^2 + \ldots + |z_n|^2 \leq r\}$.

When $n \geq 3$, V can be chosen so that $H_{2n-2}(V) < \infty$.

It follows that the generalized Blaschke condition is not a sufficient condition for the zero sets of $H^p(B_n)$, $p > 0$. (This also follows from Rudin's result and (2).)

2. Proof of the theorem

We construct explicitly the hypersurface with the required properties. Consider first the case $n \geq 3$.

PROPOSITION 1. *Fix $n \geq 3$, $1/(n-1) < \alpha < 1$. For each integer $k \geq 2$, let $r_k = 1 - k^{-\alpha}$,*

$$V_k = \{(z_1, \ldots, z_n) \in B_n: z_n = r_k\} \quad \textit{and} \quad V = \bigcup_{k=2}^{\infty} V_k.$$

Then V is a uniqueness set for $H^p(B_n)$, $p > 0$ and $H_{2n-2}(V) < \infty$.

PROOF. For each k,

$$H_{2n-2}(V_k) = \pi^{n-1}(1 - r_k^2)^{n-1}/(n-1)! < (2\pi)^{n-1} k^{-(n-1)\alpha}/(n-1)!$$

Hence $H_{2n-2}(V) = \sum H_{2n-2}(V_k) < \infty$, since $(n-1)\alpha > 1$.

Now for each $z = (z_1, \ldots, z_{n-1}) \in B_{n-1}$, let $a > 0$ be given by $a^2 = 1 - \|z\|^2$, where $\|z\|$ is the Euclidean norm of z. Suppose $z \neq 0$ and consider $\lambda = \lambda_k(z)$ such that $(\lambda z, \lambda a) \in V_k$. Then $\lambda_k(z) = r_k/a$. Let $N = N(z) = \max\{k: r_k < a\}$. Finally, let b_z be the finite Blaschke product with simple zeros at λ_k, $2 \leqslant k \leqslant N$. Then

$$b_z(0) = \prod_2^N (r_k/a). \tag{3}$$

We claim that

$$|b_z(0)|^{-1} > c \exp\left\{\frac{\alpha}{1-\alpha}\left(\frac{1}{1-a}\right)^{(1-\alpha)/\alpha}\right\} \tag{4}$$

where c is a positive constant.

Assuming (4) for the moment, we show that V is a uniqueness set for $H^p(B_n)$, $p > 0$. Suppose $f \in H^p(B_n)$ and $f = 0$ on V. Write

$$f = f_s + f_{s+1} + f_{s+2} + \cdots,$$

where f_j is a homogeneous polynomial of degree j. To show that $f \equiv 0$ in B_n, it suffices to show that $f_s \equiv 0$.

For each $z \in B_{n-1}$, let $f_z(\lambda) = f(\lambda z, \lambda a)$, $\lambda \in U$, the unit disc. Then an argument similar to that given in Chee (1970b, p. 230) shows that $f \in H^p(U)$ for almost all $z \in B_{n-1}$. By the circular invariance of the Lebesgue measure m on B_{n-1} (see Chee, 1970a, p. 256) and Corollary 4.2 of Chee (1970b), we get

$$\begin{aligned} \int_{B_{n-1}} dm(z) \int_{-\pi}^{\pi} |f^*(ze^{i\theta}, ae^{i\theta})|^p \, d\theta &= \int_{B_{n-1}} dm(z) \int_{-\pi}^{\pi} |f^*(z, ae^{i\theta})|^p \, d\theta \\ &= \int_{\partial B_n} |f^*|^p \, dm < \infty, \end{aligned} \tag{5}$$

where the star denotes radial limit.

Since $f = 0$ on V, the construction of b_z shows that there exists a holomorphic function g_z, which is in $H^p(U)$ for almost all $z \in B_{n-1}$, such that

$$f_z(\lambda) = \lambda^s b_z(\lambda) g_z(\lambda), \quad \lambda \in U.$$

Hence, for almost all $z \in B_{n-1}$,

$$\begin{aligned} \int_{-\pi}^{\pi} |f_z^*(e^{i\theta})|^p \, d\theta = \int_{-\pi}^{\pi} |g_z^*(e^{i\theta})|^p \, d\theta &\geqslant |g_z(0)|^p \\ &= \left|\frac{f_s(z,a)}{b_z(0)}\right|^p. \end{aligned}$$

It follows from (5) that

$$\int_{B_{n-1}} \left|\frac{f_s(z,a)}{b_z(0)}\right|^p dm(z) < \infty. \tag{6}$$

By (4), $|b_z(0)|^{-1}$ has an exponential singularity at $z = 0$. Since $f_s(z, a)$ is a polynomial in z and a, (6) cannot hold unless $f_s \equiv 0$.

It remains to prove (4). We note that

$$(N-1)\log\left(1-\frac{1}{N^\alpha}\right) = -(N-1)\left(\frac{1}{N^\alpha}+\frac{1}{2N^{2\alpha}}+\dots\right)$$
$$> -(N^{1-\alpha}+\varepsilon_N),$$

where $\varepsilon_N \to 0$ as $N\to\infty$. So by the definition of $N = N(z)$,

(7) $$a^{N-1} > (1-N^{-\alpha})^{N-1} > \exp(-\varepsilon_N)\exp(-N^{1-\alpha})$$
$$\geqslant c_1 \exp\left\{-\left(\frac{1}{1-a}\right)^{(1-\alpha)/\alpha}\right\}, \quad c_1>0.$$

Next

$$\sum_2^N k^{-\alpha} > (N+1)^{1-\alpha}/(1-\alpha) - 1/(1-\alpha) - 1.$$

Since $(1-k^{-\alpha})^{-1} > \exp(k^{-\alpha})$, we obtain

(8) $$\prod_2^N (1-k^{-\alpha})^{-1} > c_2 \exp\left\{\frac{(N+1)^{1-\alpha}}{1-\alpha}\right\}$$
$$\geqslant c_2 \exp\left\{\frac{1}{1-\alpha}\left(\frac{1}{1-a}\right)^{(1-\alpha)/\alpha}\right\},$$

where $c_2 > 0$. Hence by (3), (7) and (8),

$$|b_z(0)|^{-1} > c\exp\left\{\frac{\alpha}{1-\alpha}\left(\frac{1}{1-a}\right)^{(1-\alpha)/\alpha}\right\}, \quad c>0.$$

This completes the proof of Proposition 1.

For $n = 2$, we have the following.

PROPOSITION 2. *Choose* α *such that* $\frac{1}{2} < \alpha < 1$. *Construct* V *as in Proposition* 1. *Then* V *is a uniqueness set for all* $H^p(B_2)$, $p > 0$, *and*

$$\int_0^1 H_2(V(r))\,dr < \infty.$$

PROOF. For $0 < r < 1$ and $r_k \leqslant r$,

$$H_2(V_k(r)) = \pi(r^2 - r_k^2) \leqslant 2\pi(r-r_k).$$

Hence

$$\int_0^1 H_2(V(r))\,dr \leqslant 2\pi \sum_2^\infty \int_{r_k}^1 (r-r_k)\,dr$$
$$= \pi \sum_2^\infty k^{-2\alpha} < \infty \quad \text{since } 2\alpha > 1.$$

https://doi.org/10.1017/S1446788700011526 Published online by Cambridge University Press

The proof that V is a uniqueness set for $H^p(B_2)$ is similar to that given in Proposition 1 above and is omitted.

The theorem follows from Propositions 1 and 2.

Acknowledgement

I wish to thank Professor W. Rudin for very helpful correspondence.

References

P. S. Chee (1970a), "The Blaschke condition for bounded holomorphic functions", *Trans. Amer. Math. Soc.* **148**, 249–263.

P. S. Chee (1970b), "On the generalized Blaschke condition", *Trans. Amer. Math. Soc.* **152**, 227–231.

J. Miles (1973), "Zero sets in $H^p(U^n)$", *Illinois J. Math.* **17**, 458–464.

W. Rudin (1976), "Zeros of holomorphic functions in balls", *Indag. Math.* **38**, 57–65.

Department of Mathematics
University of Malaya
Kuala Lumpur
Malaysia