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BIHARMONIC SUBMANIFOLDS IN NONFLAT LORENTZ 3-SPACE FORMS

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Abstract

The purpose of this paper is to classify nonharmonic biharmonic curves and surfaces in de Sitter 3-space and anti-de Sitter 3-space.

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1. Introduction

In [8], Eells and Sampson defined *biharmonic maps* between Riemannian manifolds as an extension of harmonic maps, and Jiang [10] obtained their first and second variational formulas. Since harmonic maps are always biharmonic, it is natural and interesting to investigate nonharmonic biharmonic maps, which are called *proper* biharmonic maps. A submanifold is called a *biharmonic submanifold* if the isometric immersion that defines the submanifold is a biharmonic map. During this last decade, many interesting results on proper biharmonic maps and submanifolds have been obtained (see, for example, [3]). In particular, proper biharmonic curves and surfaces in real 3-space forms have been classified (see [4, 5]).

In [12], the author introduced the notion of biharmonic maps between *pseudo-Riemannian* manifolds. If the ambient space is the pseudo-Euclidean space, the notion of biharmonic submanifolds introduced in [12] coincides with Chen's notion of biharmonic submanifolds, that is, submanifolds with harmonic mean curvature vector field (see [6]). Proper biharmonic curves and surfaces in the pseudo-Euclidean 3-space have been classified (see [6, 7]). In particular, it was proved that there exists no proper biharmonic surface in pseudo-Euclidean 3-space.

In this paper, we classify proper biharmonic curves and surfaces in de Sitter 3-space and anti-de Sitter 3-space (Theorems 4.4, 4.5 and 5.4). Contrary to the case of pseudo-Euclidean 3-space, there exist proper biharmonic surfaces in those spaces. A further important point is that an example of a proper biharmonic surface in de Sitter space having no Riemannian counterpart is obtained.

2. Preliminaries

Let E^n_s be pseudo-Euclidean n-space with metric given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{n} dx_j^2,$$

where $\{x_1, \ldots, x_n\}$ is the natural coordinate system of E^n_s. Then E^n_s is a flat pseudo-Riemannian manifold with index s. We put

$$S^n_s(c) = \left\{(x_1, \ldots, x_{n+1}) \in E^{n+1}_s \;\middle|\; -\sum_{i=1}^{s} x_i^2 + \sum_{j=s+1}^{n+1} x_j^2 = \frac{1}{c}\right\},$$

$$H^n_s(c) = \left\{(x_1, \ldots, x_{n+1}) \in E^{n+1}_{s+1} \;\middle|\; -\sum_{i=1}^{s+1} x_i^2 + \sum_{j=s+2}^{n+1} x_j^2 = \frac{1}{c} < 0\right\}.$$

These spaces are complete pseudo-Riemannian manifolds with index s of constant curvature c. The pseudo-Riemannian manifolds E^n_1, $S^n_1(c)$ and $H^n_1(c)$ are called *Minkowski space*, *de Sitter space* and *anti-de Sitter space*, respectively. These spaces with index 1 are called *Lorentz space forms*.

Denote n-dimensional Lorentz space forms of constant curvature c by $M^n_1(c)$. The curvature tensor $\tilde{R}$ of $M^n_1(c)$ is given by

$$\tilde{R}(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y), \tag{2.1}$$

where $\langle\ ,\ \rangle$ is the metric tensor of $M^n_1(c)$.

Let M^2 be a pseudo-Riemannian surface in $M^3_1(c)$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M^2 and $M^3_1(c)$, respectively. Let X and Y be vector fields tangent to M^2 and let ξ be a normal vector field. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

respectively, where h, A and D are the second fundamental form, the shape operator and the normal connection, respectively. The mean curvature vector field H is defined by $H = \frac{1}{2}\operatorname{trace} h$.

Denote by R the curvature tensor of M^2. Then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = c(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)\\ + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle,\end{aligned} \tag{2.2}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.3}$$

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where X, Y, Z, W are vectors tangent to M, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Laplace operator which acts on the sections of the normal bundle $T^{\perp}M^2$ is defined by $\Delta^D = -\sum_{i=1}^{2}\langle e_i, e_i\rangle(D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$, where $\{e_i\}$ is a local orthonormal frame of M^2. The gradient of a function f on M^2 is defined by grad $f = -\sum_{i=1}^{2}\langle e_i, e_i\rangle(e_i f)e_i$.

3. Biharmonic maps

Let M^m and N^n be pseudo-Riemannian manifolds of dimensions m and n, respectively, and $\phi : M^m \to N^n$ a smooth map. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M^m and N^n, respectively. Then the *tension field* $\tau_1(\phi)$ is a section of the vector bundle $\phi^* TN^n$ defined by

$$\tau_1(\phi) := \text{trace}(\nabla^{\phi} d\phi) = \sum_{i=1}^{m}\langle e_i, e_i\rangle(\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i}e_i)).$$

Here ∇^{ϕ} and $\{e_i\}$ denote the induced connection by ϕ on the bundle $\phi^* TN^n$, which is the pull-back of $\tilde{\nabla}$, and a local orthonormal frame field of M^m, respectively.

A smooth map ϕ is called a *harmonic map* if its tension field vanishes. A map ϕ is harmonic if and only if it is a critical point of the *energy*

$$E(\phi) = \int \sum_{i=1}^{m}\langle d\phi(e_i), d\phi(e_i)\rangle \, dv$$

under compactly supported infinitesimal variations, where dv is the volume form of M^m.

We define the *bitension field* as follows:

$$\tau_2(\phi) := \sum_{i=1}^{m}\langle e_i, e_i\rangle((\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i}e_i})\tau_1(\phi) + R^N(\tau_1(\phi), d\phi(e_i))\, d\phi(e_i)), \tag{3.1}$$

where R^N is the curvature tensor of N^n. We say that a smooth map ϕ is a *biharmonic map* (or *2-harmonic map*) if its bitension field vanishes (see [10, 12, 13]). If ϕ is a biharmonic isometric immersion, then M^m is called a *biharmonic submanifold* in N^n. Harmonic maps are clearly biharmonic. Nonharmonic biharmonic maps are called *proper* biharmonic maps. A biharmonic map ϕ is characterised as a critical point of the *bienergy*

$$E_2(\phi) = \int \langle \tau_1(\phi), \tau_1(\phi)\rangle \, dv$$

under compactly supported infinitesimal variations (see [10]). For recent information on biharmonic maps, we refer to [3, 11].

In the case in which N^n is pseudo-Euclidean space and $\phi = (\phi_1, \ldots, \phi_n)$ is an isometric immersion, then

$$\tau_2(\phi) = \Delta_M \Delta_M(\phi_1, \ldots, \phi_n),$$

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where $\Delta_M = -\sum_{i=1}^{m} \langle e_i, e_i \rangle (e_i e_i - (\nabla_{e_i} e_i))$. Thus, biharmonicity for an isometric immersion into pseudo-Euclidean space is equivalent to biharmonicity in the sense of Chen (see [6]).

As for proper biharmonic submanifolds in real or Lorentz 3-space forms, the following classification results have been obtained.

THEOREM 3.1 [5]. *Let M be a submanifold of real 3-space forms of nonpositive constant sectional curvature. Then M is biharmonic if and only if it is harmonic.*

THEOREM 3.2 [4]. *Let $\gamma : I \to S^3(1)$ be a proper biharmonic unit speed curve and let $x = \mathbf{i} \circ \phi$, where $\mathbf{i} : S^3(1) \to E^4$ is the canonical inclusion. Then x is congruent to one of the following two families.*

(1) $x(t) = (\cos \sqrt{2}t/\sqrt{2}, \sin \sqrt{2}t/\sqrt{2}, d_1, d_2)$, *where* $d_1^2 + d_2^2 = 1$.

(2)
$$x(t) = \left(\frac{\cos \sqrt{1+k}t}{\sqrt{2}}, \frac{\sin \sqrt{1+k}t}{\sqrt{2}}, \frac{\cos \sqrt{1-k}t}{\sqrt{2}}, \frac{\sin \sqrt{1-k}t}{\sqrt{2}}\right),$$

where $0 < k < 1$.

THEOREM 3.3 [4]. *Let M^2 be a proper biharmonic surface in $S^3(1)$. Then M^2 is locally a piece of $S^2(2) \subset S^3(1)$.*

PROPOSITION 3.4 [6, 7]. *Let x be a unit speed curve in E_1^3. Then x is proper biharmonic if and only if x is congruent to one of the following:*

(1) *a spacelike curve such that $\langle x'', x'' \rangle = 0$, which is given by*
$$x(s) = (as^3 + bs^2, as^3 + bs^2, s)$$
for some constants a and b satisfying $a^2 + b^2 \neq 0$;

(2) *a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 = \tau^2 = a^2$;*
$$x(s) = \left(\frac{a^2}{6}s^3, \frac{a}{2}s^2, -\frac{a^2}{6}s^3 + s\right)$$
for some nonzero constant a;

(3) *a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$;*
$$x(s) = \left(\frac{a^2}{6}s^3 + s, \frac{a^2}{6}s^3, \frac{a}{2}s^2\right)$$
for some nonzero constant a.

THEOREM 3.5 [6]. *Let $x : M^2 \to E_1^3$ be a biharmonic isometric immersion of a pseudo-Riemannian surface M^2 into E_1^3. Then x is harmonic.*

The purpose of this paper is to classify proper biharmonic submanifolds in de Sitter 3-space and anti-de Sitter 3-space.

4. Biharmonic curves in nonflat Lorentz 3-space forms

Let $(M_1^3, g, \tilde{\nabla})$ be a Lorentz 3-manifold and let $\gamma : I \to M_1^3$ be a unit speed curve, that is, a curve satisfying $g(\gamma', \gamma') = \pm 1$. A unit speed curve γ is called *spacelike* (respectively, *timelike*) if $g(\gamma', \gamma') = 1$ (respectively, $g(\gamma', \gamma') = -1$). A unit speed curve γ is said to be a *Frenet curve* if $g(\tilde{\nabla}_{\gamma'}\gamma', \tilde{\nabla}_{\gamma'}\gamma') \neq 0$. A unit speed curve is said to be a geodesic if $\tilde{\nabla}_{\gamma'}\gamma' = 0$.

Let $P = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ be an orthonormal frame field along a Frenet curve γ such that $\mathbf{p}_1 = \gamma'$ and P satisfies the following Frenet–Serret equations (see [9]):

$$\tilde{\nabla}_{\gamma'} P = P \begin{pmatrix} 0 & -\epsilon_1\kappa & 0 \\ \epsilon_2\kappa & 0 & \epsilon_2\tau \\ 0 & -\epsilon_3\tau & 0 \end{pmatrix},$$

where κ and τ are called the *curvature* and *torsion* of γ, respectively, and $\epsilon_i = g(\mathbf{p}_i, \mathbf{p}_i)$. Note that $\epsilon_1\epsilon_2\epsilon_3 = -1$. A unit speed curve is a geodesic if and only if $\kappa = 0$ at any point.

The vectors $\mathbf{p}_2$ and $\mathbf{p}_3$ are called the *principal normal vector field* and the *binormal vector field* of γ, respectively. A Frenet curve with constant curvature and torsion is called a *helix*. In particular, a helix with zero torsion is called a *circle*. The mean curvature vector field H_γ of γ is given by $H_\gamma = -\epsilon_3\kappa\mathbf{p}_2$. We remark that $H_\gamma = \tau_1(\gamma)$.

Let $\gamma : I \to M_1^3(c)$ be a Frenet curve. Then, using (2.1) and (3.1), we see that γ is biharmonic if and only if the mean curvature vector H_γ satisfies $\tilde{\Delta}H_\gamma = cH_\gamma$, where $\tilde{\Delta}$ is the Laplace operator acting on the sections of $\gamma^* T M_1^3(c)$. Applying [9, Theorem 3.2], we have the following proposition.

Proposition 4.1. *Let $\gamma : I \to M_1^3(c)$ be a Frenet curve. Then γ is proper biharmonic if and only if γ is a helix with*

$$c = -\epsilon_3(\epsilon_1\kappa^2 + \epsilon_3\tau^2), \qquad \kappa \neq 0.$$

Let E_t^{n+1} be the corresponding pseudo-Euclidean space where $M_1^n(c)$ is lying. By a similar computation to [5, Proof of Proposition 4.1], we obtain the following proposition.

Proposition 4.2. *Let $\phi : M^m \to M_1^n(c)$ be an isometric immersion and let $x = \mathbf{i} \circ \phi$, where $\mathbf{i} : M_1^n(c) \to E_t^{n+1}$ is the canonical inclusion. Then*

$$\tau_2(\phi) = \tau_2(x) + 2cm\tau_1(x) + (2m^2 - c\langle\tau_1(x), \tau_1(x)\rangle)x.$$

By Proposition 4.2, we have the following corollary (see [5, Corollary 4.2]).

Corollary 4.3. *Let $\gamma : I \to M_1^3(c)$ be a unit speed curve. Then γ is biharmonic if and only if $x(s) = \mathbf{i} \circ \gamma(s)$ satisfies*

$$x^{(iv)} + 2c\langle x', x'\rangle x'' + (1 - c\langle\tau_1(\gamma), \tau_1(\gamma)\rangle)x = 0. \tag{4.1}$$

By applying Corollary 4.3, we classify proper biharmonic curves in nonflat Lorentz space forms as follows.

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THEOREM 4.4. *Let $\gamma : I \to S^3_1(1)$ be a unit speed curve in de Sitter 3-space. Then γ is proper biharmonic if and only if $x = \mathbf{i} \circ \gamma$ is congruent to one of the following five families.*

(1) *A spacelike curve with $\langle \tau_1(\gamma), \tau_1(\gamma)\rangle = 0$;*

$$x(s) = (c_1 + c_2 s)\cos s + (c_3 + c_4 s)\sin s,$$

where c_1, c_2, c_3 and c_4 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1\rangle = \langle c_3, c_3\rangle = 1$ and $\langle c_2, c_2\rangle = \langle c_4, c_4\rangle = 0$; however, $\langle c_2, c_2\rangle^2 + \langle c_4, c_4\rangle^2 \neq 0$.

(2) *A spacelike circle with a spacelike principal normal vector field satisfying $\kappa = 1$;*

$$x(s) = c_1 + c_2 \cos \sqrt{2}s + c_3 \sin \sqrt{2}s,$$

where c_1, c_2 and c_3 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = \langle c_3, c_3\rangle = \frac{1}{2}$.

(3) *A spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 - \tau^2 = 1$;*

$$x(s) = c_1 \cosh(\sqrt{\kappa - 1}s) + c_2 \sinh(\sqrt{k - 1}s) + c_3 \cos(\sqrt{\kappa + 1}s) + c_4 \sin(\sqrt{\kappa + 1}s),$$

where c_1, c_2, c_3 and c_4 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1\rangle = -\langle c_2, c_2\rangle = \langle c_3, c_3\rangle = \langle c_4, c_4\rangle = \frac{1}{2}$.

(4) *A timelike circle satisfying $\kappa = 1$;*

$$x(s) = c_1 + c_2 \cosh \sqrt{2}s + c_3 \sinh \sqrt{2}s,$$

where c_1, c_2 and c_3 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = -\langle c_3, c_3\rangle = \frac{1}{2}$.

(5) *A timelike helix satisfying $\kappa^2 - \tau^2 = 1$;*

$$x(s) = c_1 \cos \sqrt{\kappa - 1}s + c_2 \sin \sqrt{\kappa - 1}s + c_3 \cosh \sqrt{\kappa + 1}s + c_4 \sinh \sqrt{\kappa + 1}s,$$

where c_1, c_2, c_3 and c_4 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = \langle c_3, c_3\rangle = -\langle c_4, c_4\rangle = \frac{1}{2}$.

PROOF. We solve (4.1) under the condition that $\langle x, x\rangle = 1$, $\langle x', x'\rangle = \pm 1$ and $\tau_1(\gamma) \neq 0$, that is, $\langle x', x'\rangle x'' + x \neq 0$ for all s. We divide (4.1) into three types.

Case (i). x is spacelike and $\langle \tau_1(\gamma), \tau_1(\gamma)\rangle = 0$: (4.1) is rewritten as

$$x^{(iv)} + 2x'' + x = 0,$$

whose solutions are congruent to (1).

Case (ii). x is a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 - \tau^2 = 1$: (4.1) is rewritten as

$$x^{(iv)} + 2x'' + (1 - \kappa^2)x = 0.$$

If $\kappa = 1$ (respectively, $\kappa > 1$), then x is congruent to (2) (respectively, (3)).

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Case (iii). x is a timelike helix satisfying $\kappa^2 - \tau^2 = 1$: (4.1) is rewritten as

$$x^{(iv)} - 2x'' + (1 - \kappa^2)x = 0.$$

If $\kappa = 1$ (respectively, $\kappa > 1$), then x is congruent to (4) (respectively, (5)). □

THEOREM 4.5. *Let $\gamma : I \to H_1^3(-1)$ be a unit speed curve in anti-de Sitter 3-space. Then γ is proper biharmonic if and only if $x = \mathbf{i} \circ \gamma$ is congruent to one of the following five families.*

(1) *A spacelike curve with $\langle \tau_1(\gamma), \tau_1(\gamma) \rangle = 0$;*

$$x(s) = (c_1 + c_2 s)e^s + (c_3 + c_4 s)e^{-s},$$

where c_1, c_2, c_3 and c_4 are constant vectors satisfying $\langle c_1, c_3 \rangle = -\frac{1}{2}$ and $\langle c_i, c_j \rangle = 0$ unless $(i, j) = (1, 3)$ or $(3, 1)$; however, $\langle c_2, c_2 \rangle^2 + \langle c_4, c_4 \rangle^2 \neq 0$.

(2) *A spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 - \tau^2 = -1$ and $\kappa \neq 0$;*

$$x(s) = e^{\alpha s}(c_1 \cos \beta s + c_2 \sin \beta s) + e^{-\alpha s}(c_3 \cos \beta s + c_4 \sin \beta s),$$

where α and β are constants satisfying $\alpha^2 - \beta^2 = 1$, $2\alpha\beta = \kappa$, and moreover c_1, c_2, c_3 and c_4 are constant vectors satisfying $\langle c_1, c_3 \rangle = \langle c_2, c_4 \rangle = -\frac{1}{2}$ and $\langle c_i, c_j \rangle = 0$ unless $(i, j) = (1, 3), (3, 1), (2, 4)$ or $(4, 2)$.

(3) *A spacelike circle with a timelike principal normal vector field satisfying $\kappa = 1$;*

$$\gamma(s) = c_1 + c_2 \cosh \sqrt{2}s + c_3 \sinh \sqrt{2}s,$$

where c_1, c_2 and c_3 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = -\langle c_3, c_3 \rangle = -\frac{1}{2}$.

(4) *A spacelike helix with a timelike principal normal vector field satisfying $\kappa^2 + \tau^2 = 1$ and $\kappa \neq 0$;*

$$x(s) = c_1 \cos \sqrt{1 - \kappa}s + c_2 \sin \sqrt{1 - \kappa}s + c_3 \cosh \sqrt{1 + \kappa}s + c_4 \sinh \sqrt{1 + \kappa}s,$$

where c_1, c_2, c_3 and c_4 are constant vectors orthogonal to each other satisfying $\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = \langle c_3, c_3 \rangle = -\langle c_4, c_4 \rangle = -\frac{1}{2}$.

(5) *A timelike helix satisfying $\kappa^2 - \tau^2 = -1$ and $\kappa \neq 0$;*

$$x(s) = e^{\alpha s}(c_1 \cos \beta s + c_2 \sin \beta s) + e^{-\alpha s}(c_3 \cos \beta s + c_4 \sin \beta s),$$

where α and β are constants satisfying $\alpha^2 - \beta^2 = -1$, $2\alpha\beta = \kappa$, and moreover c_1, c_2, c_3 and c_4 are constant vectors satisfying $\langle c_1, c_3 \rangle = \langle c_2, c_4 \rangle = -\frac{1}{2}$ and $\langle c_i, c_j \rangle = 0$ unless $(i, j) = (1, 3), (3, 1), (2, 4)$ or $(4, 2)$.

PROOF. We solve (4.1) under the condition that $\langle x, x \rangle = 1$, $\langle x', x' \rangle = \pm 1$ and $\tau_1(\gamma) \neq 0$, that is, $\langle x', x' \rangle x'' - x \neq 0$ for all s. We divide (4.1) into four types.

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Case (i). x is spacelike and $\langle \tau_1(\gamma), \tau_1(\gamma) \rangle = 0$: (4.1) is rewritten as

$$x^{(iv)} - 2x'' + x = 0,$$

whose solutions are congruent to (1).

Case (ii). x is a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 - \tau^2 = -1$ and $\kappa \neq 0$: (4.1) is rewritten as

$$x^{(iv)} - 2x'' + (1 + \kappa^2)x = 0,$$

whose solutions are congruent to (2).

Case (iii). x is a spacelike helix with a timelike principal normal vector field satisfying $\kappa^2 + \tau^2 = 1$ and $\kappa \neq 0$: (4.1) is rewritten as

$$x^{(iv)} - 2x'' + (1 - \kappa^2)x = 0.$$

If $\kappa = 1$ (respectively, $0 < \kappa < 1$), then x is congruent to (3) (respectively, (4)).

Case (iv). x is a timelike helix satisfying $\kappa^2 - \tau^2 = -1$: (4.1) is rewritten as

$$x^{(iv)} + 2x'' + (1 + \kappa^2)x = 0,$$

whose solutions are congruent to (5). □

5. Biharmonic surfaces in nonflat Lorentz 3-space forms

By a similar computation to [4, Proof of Theorem 4.1] (see [6, Lemma 2.1]), we obtain the following lemma.

LEMMA 5.1. *Let M^2 be a pseudo-Riemannian surface in a Lorentz 3-space form of constant sectional curvature c. Then M^2 is biharmonic if and only if*

$$\begin{cases} \Delta^D H = (2c - \langle N, N\rangle \operatorname{trace} A_N^2)H, \\ 2 \operatorname{trace} A_{D_{(\cdot)}H}(\cdot) + \operatorname{grad}(\langle H, H\rangle) = 0, \end{cases}$$

where N is a unit normal vector field.

We need the following two lemmas to classify proper biharmonic pseudo-Riemannian surfaces in nonflat Lorentz 3-space forms.

LEMMA 5.2. *Let M^2 be a pseudo-Riemannian surface in a Lorentz 3-space form of constant sectional curvature c. Then M^2 is proper biharmonic if and only if it has nonzero constant mean curvature and A_N satisfies*

$$2c - \langle N, N\rangle \operatorname{trace} A_N^2 = 0. \tag{5.1}$$

PROOF. Let M^2 be a proper biharmonic pseudo-Riemannian surface in $M_1^3(c)$ and let $\{e_1, e_2, e_3\}$ be a local orthonormal frame field such that $\{e_1, e_2\}$ are tangent to M^2 and e_3 is normal to M^2. We put $\epsilon_i = \langle e_i, e_i\rangle$ and $H = fe_3$. Note that $\epsilon_1\epsilon_2\epsilon_3 = -1$.

By Lemma 5.1,

$$\Delta_M f = (2c - \epsilon_3 \operatorname{trace} A^2) f, \qquad (5.2)$$
$$A(\operatorname{grad} f) = -\epsilon_3 f (\operatorname{grad} f),$$

where $\Delta_M f = -\sum_{i=1}^{2}\langle e_i, e_i\rangle\{e_i(e_i f) - (\nabla_{e_i} e_i) f\}$ and $A = A_{e_3}$.

Let $U = \{p \in M^2 \mid (\operatorname{grad} f^2)(p) \neq 0\}$. Assume that U is nonempty. Since $-\epsilon_3 f$ is an eigenvector of A by (5.2) and $\operatorname{trace} A = 2\epsilon_3 f$, the shape operator A is diagonalisable and we can choose a local orthonormal frame field $\{e_1, e_2\}$ such that e_1 is parallel to $\operatorname{grad} f$. Then

$$e_2 f = 0, \qquad (5.3)$$
$$h(e_1, e_1) = -\epsilon_1 f e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = 3\epsilon_2 f e_3, \qquad (5.4)$$
$$\operatorname{trace} A^2 = 10 f^2. \qquad (5.5)$$

We put $\nabla e_i = \omega_i^j e_j$. Then $\omega_1^2 = \omega_2^1$. From (5.3) and equation (2.3) of Codazzi,

$$\omega_1^2(e_1) = 0, \quad 3e_1 f = -4 f \omega_1^2(e_2). \qquad (5.6)$$

By the first equation of (5.6), we can choose a local coordinate system $\{u, v\}$ such that $e_1 = \partial/\partial u$ and e_2 is parallel to $\partial/\partial v$. Then, from (5.3), we have $f = f(u)$.

We denote by f' and f'' the first and the second derivatives of f with respect to x. Using (5.6),

$$4 f \Delta_M f = 3\epsilon_1 (f')^2 - 4\epsilon_1 f f''. \qquad (5.7)$$

By combining (5.2), (5.5) and (5.7), we obtain

$$4 f f'' - 3(f')^2 + 40\epsilon_2 f^4 + 8c\epsilon_1 f^2 = 0. \qquad (5.8)$$

If we put $\theta = (f')^2$, then (5.8) can be rewritten as

$$\frac{d\theta}{df} - \frac{3}{2f}\theta = -20\epsilon_2 f^3 - 4c\epsilon_1 f,$$

which implies that

$$\theta = -8\epsilon_2 f^4 - 8\epsilon_1 c f^2 + C f^{\frac{3}{2}}, \qquad (5.9)$$

where C is a constant.

On the other hand, by (5.4), (5.6) and equation (2.2) of Gauss with $X = W = e_1$ and $Y = Z = e_2$,

$$4 f f'' - 7(f')^2 - 16\epsilon_2 f^4 - \frac{16}{3} c\epsilon_1 f^2 = 0. \qquad (5.10)$$

Combining (5.8) and (5.10) implies that

$$\theta = -14\epsilon_2 f^4 - \frac{10}{3} c\epsilon_1 f^2. \qquad (5.11)$$

By (5.9) and (5.11), f is constant on U, which is a contradiction. Therefore, U is empty, that is, M^2 has constant mean curvature. Hence, by (5.2), we get (5.1).

The converse is clear from Lemma 5.1. □

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LEMMA 5.3 [2]. *Let M^2 be a surface with index 1 in $M_1^3(c)$ and let $(t-\lambda)^2$, λ being a nonzero constant, be the minimal polynomial of its shape operator A_N. Then, in a neighbourhood of any point, M^2 is a B-scroll over a null curve given by*

$$x(s,u) = \gamma(s) + uY(s), \tag{5.12}$$

where $\gamma(s)$ is a null curve in $M_1^3(c) \subset E_t^4$ with an associated Cartan frame $\{X, Y, Z\}$. That is, $\{X, Y, Z\}$ is a pseudo-orthonormal frame field along $\gamma(s)$:

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \quad \langle X, Y\rangle = -1,$$
$$\langle X, Z\rangle = \langle Y, Z\rangle = 0, \quad \langle Z, Z\rangle = 1,$$

such that

$$\gamma'(s) = X(s),$$
$$Z'(s) = -\lambda X(s) - k(s)Y(s),$$

where $k(s) \neq 0$ for all s. Moreover, the shape operator A_N of (5.12) is represented as

$$A_N = \begin{pmatrix} \lambda & 0 \\ k(s) & \lambda \end{pmatrix} \tag{5.13}$$

with respect to the usual frame $\{\partial/\partial s, \partial/\partial u\}$.

Contrary to Theorem 3.5, there exist proper biharmonic pseudo-Riemannian surfaces in nonflat Lorentz 3-space forms as follows.

THEOREM 5.4. *Let M^2 be a pseudo-Riemannian surface in a Lorentz 3-space form of constant sectional curvature c, where $c \in \{-1, 1\}$. Then M^2 is proper biharmonic if and only if it is congruent to one of the following:*

(1) $S_1^2(2) \subset S_1^3(1)$;
(2) $H^2(-2) \subset H_1^3(-1)$;
(3) *a B-scroll over a null curve of constant Gauss curvature 2 in $S_1^3(1)$.*

PROOF. Let M^2 be a proper biharmonic pseudo-Riemannian surface in $M_1^3(c)$, where $c \in \{-1, 1\}$. By Lemma 5.2, the eigenvalues of the shape operator A_N are constant. For simplicity, we put $A = A_N$.

Case (i). A is diagonalisable: if A has exactly two mutually distinct eigenvalues λ and μ, by [1, Theorem 3.5] we have that $c + \langle N, N\rangle\lambda\mu = 0$. Combining this and (5.1) yields that $\lambda + \mu = 0$, which is a contradiction with the assumption that M^2 is proper biharmonic. Therefore, $A = \lambda I$, where I is the identity operator. Since A satisfies (5.1), we obtain that $\lambda^2 = 1$. By virtue of [1, Theorem 5.1], M^2 is congruent to (1) or (2).

Case (ii). A is not diagonalisable: in this case, the index of M^2 is equal to 1 and hence $\langle N, N\rangle = 1$. If A has a double real eigenvalue λ, then, by Lemma 5.3, M^2 is congruent to a B-scroll over a null curve (5.12) with (5.13). It follows from (5.1) and (5.13)

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that $\lambda^2 = c = 1$. Therefore, M^2 is congruent to (3). If A has complex eigenvalues $\alpha \pm \beta i$, then A is represented as

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{5.14}$$

with respect to an orthonormal frame. By equation (2.3) of Codazzi and (5.14), we find that M^2 is a flat surface. Hence, equation (2.2) of Gauss and (5.14) show that $c = -1$ and $\alpha^2 + \beta^2 = 1$. In this case, the shape operator A does not satisfy (5.1) because M^2 is nonharmonic.

The converse is verified by using Lemma 5.1. □

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