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On the Diameter of Unitary Cayley Graphs of Rings

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Abstract. The unitary Cayley graph of a ring R, denoted $\Gamma(R)$, is the simple graph defined on all elements of R, and where two vertices x and y are adjacent if and only if $x - y$ is a unit in R. The largest distance between all pairs of vertices of a graph G is called the diameter of G and is denoted by $\mathrm{diam}(G)$. It is proved that for each integer $n \geq 1$, there exists a ring R such that $\mathrm{diam}(\Gamma(R)) = n$. We also show that $\mathrm{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$ for a ring R with $R/J(R)$ self-injective and classify all those rings with $\mathrm{diam}(\Gamma(R)) = 1, 2, 3$, and ∞, respectively.

1 Introduction

This paper concerns the diameter of unitary Cayley graphs of rings. Let R be a ring with nonzero identity. We use $U(R)$ to denote the group of units of R. The *unitary Cayley graph* of R, denoted by $\Gamma(R)$, is the simple graph whose vertices are the elements of R, and where two vertices x and y are adjacent if and only if $x - y \in U(R)$.

The earliest work on the unitary Cayley graph of a ring is for the ring $\mathbb{Z}_n$ by Dejter and Giudici [8]. Since then, many publications are devoted to this topic. The study of $\Gamma(\mathbb{Z}_n)$ was continued by Berrizbeitia and Giudici [6, 7], Fuchs [10], and Klotz and Sander [17]. The unitary Cayley graph $\Gamma(R)$ was studied for a finite ring R by Akhtar, et al. [2], and for an Artinian ring R by Lucchini and Maróti [19] and Lanski and Maróti [20]. Several other papers are devoted to the spectral properties and the energy of unitary Cayley graphs of $\mathbb{Z}_n$ or a finite commutative ring (see [14, 16, 21]). Recently, Kiani and Aghaei [15] investigated the isomorphism problem for unitary Cayley graphs associated with finite (commutative) rings.

Let us recall some needed notions in graph theory. Let G be a simple graph. A *walk* is a sequence of vertices and edges, where each edge's endpoints are the preceding and following vertices in the sequence. The length of a walk is the number of edges that it uses. A *path* in a graph is a walk that has all distinct vertices (except the endpoints). We use $x — y$ to denote two vertices x and y in a graph G are adjacent. A graph G is *connected* if there is a path between each pair of the vertices of G; otherwise, G is *disconnected*. The *distance* between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path in G beginning at x and ending at y. The largest distance between

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all pairs of vertices of G is called the *diameter* of G, and is denoted by $\operatorname{diam}(G)$. A *complete graph* is a graph where each vertex is adjacent to all other vertices. Obviously, G is a complete graph if and only if $\operatorname{diam}(G) = 1$. We use $K_{m,n}$ and K_n to denote the complete bipartite graph with partitions of size m and n, and the complete graph of n vertices, respectively.

The diameter of graphs associated with rings is an active research subject. For instance, Anderson and Livingston [3] and Anderson and Mulay [4] investigated the diameter of the zero-divisor graph of a commutative ring. It was proved that the zero-divisor graph of a commutative ring is always connected with diameter at most three. A similar version for the zero-divisor graph of a commutative semigroup was shown in [9] by DeMeyer, McKenzie, and Schneider. Anderson and Badawi [1] proved that for each integer $n \geq 1$, there exists a ring R such that its total graph has diameter n. Concerning the diameter of the unitary Cayley graph of a ring, Akhtar et al. [2, Theorem 3.1] proved that $\operatorname{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$ for a left Artinian ring R and classified all left Artinian rings according to diameters of their unitary Cayley graphs. In this paper, we generalize the results to rings R with $R/J(R)$ self-injective (Theorems 3.5 and 3.6). We also prove that for each integer $n \geq 1$, there exists a ring R such that $\operatorname{diam}(\Gamma(R)) = n$ (Theorem 2.5). The diameter of some extensions of rings are also investigated.

As usual, $\mathbb{Z}_n$ will denote the ring of integers modulo n. We use $J(R)$ to denote the Jacobson radical of R and write $\overline{R} = R/J(R)$ and $\bar{a} = a + J(R) \in \overline{R}$ for $a \in R$. The polynomial ring over a ring R in the indeterminate t is denoted by $R[t]$. The formal power series ring over a ring is denoted by $R[[t]]$. Recall that a ring R is called *right self-injective* if, for any (principal) right ideal I of R, every homomorphism from I_R to R_R extends to a homomorphism from R_R to R_R.

2 Unitary Cayley Graphs with Diameter n

As we will shortly see, the connectedness of $\Gamma(R)$ is closely related to whether the ring R is generated additively by its units. So let us first recall the following definitions. Let R be a ring and let k be a positive integer. An element $r \in R$ is said to be *k-good* if $r = u_1 + \cdots + u_k$ with $u_i \in U(R)$ for each $1 \leq i \leq k$. A ring is said to be k-good if every element of R is k-good. The *unit sum number* of a ring R, denoted by $\mathbf{u}(R)$, is defined to be

(1) $\min\{k \in \mathbb{N} \mid R \text{ is a } k\text{-good}\}$ if R is k-good for some $k \geq 1$;
(2) ω if R is not k-good for every $k \geq 1$, but each element of R is k-good for some k;
(3) ∞ if some element of R is not k-good for any $k \geq 1$.

For example, $\mathbf{u}(\mathbb{Z}_3) = 2$, $\mathbf{u}(\mathbb{Z}) = \omega$ and $\mathbf{u}(\mathbb{Z}[t]) = \infty$. It is clear that if $2 \in U(R)$, then $r \in R$ being k-good implies that r is l-good for all $l \geq k$. The investigation of rings generated additively by their units started in 1953–1954 when Wolfson [23] and Zelinsky [24] proved independently that every linear transformation of a vector space V over a division ring D is the sum of two nonsingular linear transformations, except when $\dim V = 1$ and $D = \mathbb{Z}_2$. For the unit sum number of rings, we refer the reader to [11, 18, 22].

We recall another slightly different definition introduced in [13]. Let $\operatorname{usn}(R)$ be the smallest number n such that every element can be written as the sum of at most n units. If some element of R is not k-good for any $k \geq 1$, then $\operatorname{usn}(R)$ is defined to be ∞. Note that $\operatorname{usn}(R)$ and $\mathbf{u}(R)$ are different. For example, $\mathbf{u}(\mathbb{Z}_4) = \omega$ and $\operatorname{usn}(\mathbb{Z}_4) = 2$.

Our first lemma characterizes the rings R with $\operatorname{diam}(\Gamma(R)) = 1$.

Lemma 2.1 *Let R be a ring. Then* $\operatorname{diam}(\Gamma(R)) = 1$ *if and only if R is a division ring.*

Proof If $\operatorname{diam}(\Gamma(R)) = 1$, then $\Gamma(R)$ is a complete graph. For any nonzero element r in R, the vertex 0 is adjacent to r, so r is a unit, and hence R is a division ring. Conversely, suppose that R is a division ring. Then for any two distinct vertices x and y, $0 \neq x - y \in R$ is a unit of R. So $d(x, y) = 1$, and hence $\operatorname{diam}(\Gamma(R)) = 1$. ∎

Lemma 2.2 *Let R be a ring and $r \in R$. Then the following statements hold:*

(i) *If r is k-good, then $d(r, 0) \leq k$ in $\Gamma(R)$.*
(ii) *If $r \neq 0$ and $d(r, 0) = k$ in $\Gamma(R)$, then r is k-good but not l-good for all $l < k$.*
(iii) *For any $x, y, z \in R$, $d(x, y) = k$ if and only if $d(x + z, y + z) = k$.*

Proof (i) Let $r = u_1 + u_2 + \cdots + u_k$ with each $u_i \in U(R)$ and let $x_i = u_1 + \cdots + u_i$, $i = 1, \ldots, k$. Then $0 — x_1 — x_2 — \cdots — x_{k-1} — x_k = r$ is a walk from 0 to r, so $d(r, 0) \leq k$.

(ii) Let $0 = x_0 — x_1 — x_2 — \cdots — x_k = r$ be a path from 0 to r. Then $u_i := x_i - x_{i-1} \in U(R)$ for $1 \leq i \leq k$. It is easy to check that $r = \sum_{i=1}^{k} u_i$. So, r is k-good. By part (i), we know that r is not l-good for all $l < k$.

(iii) Let $d(x, y) = k$. Suppose that $x = x_0 — x_1 — x_2 — \cdots — x_k = y$ is a path from x to y. Then $x + z = (x_0 + z) — (x_1 + z) — (x_2 + z) — \cdots — (x_{k-1} + z) — (x_k + z) = y + z$ is a path from $x + z$ to $y + z$. So $d(x + z, y + z) \leq k$. Similarly, $d(x + z, y + z) = k$ implies $d(x, y) \leq k$. Thus, $d(x, y) = k$ if and only if $d(x + z, y + z) = k$. ∎

Lemma 2.3 *Let R be a ring. Then* $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if* $\operatorname{usn}(R) = 2$ *and R is not a division ring.*

Proof Assume that $\operatorname{diam}(\Gamma(R)) = 2$. Then R is not a division ring by Lemma 2.1. For any nonzero nonunit r in R, as $\operatorname{diam}(\Gamma(R)) = 2$, we have $d(r, 0) = 2$. So r is 2-good by Lemma 2.2(ii), and thus $\operatorname{usn}(R) = 2$. Conversely, it is clear that $\operatorname{diam}(\Gamma(R)) \geq 2$. For any $x, y \in R$, if $x - y \in U(R)$, then $d(x, y) = 1$; if $x - y \notin U(R)$, then $x - y$ is 2-good. So $d(x - y, 0) = 2$, and hence $d(x, y) = 2$ by Lemma 2.2(i)(iii). Thus, $\operatorname{diam}(\Gamma(R)) = 2$. ∎

Lemma 2.4 *Let R be a ring and let $k \geq 3$ be an integer. Then* $\operatorname{usn}(R) = k$ *if and only if* $\operatorname{diam}(\Gamma(R)) = k$.

Proof ($\Rightarrow$) For $x \neq y \in R$, as $\operatorname{usn}(R) = k$, $x - y$ can be expressed as a sum of m $(\leq k)$ units. Let $x - y = u_1 + u_2 + \cdots + u_m$ with each $u_i \in U(R)$. Set $x_i = u_1 + \cdots + u_i + y$, $i = 1, \ldots, m$. Then $y — x_1 — x_2 — \cdots — x_m = x$ is a walk from y to x, so $d(x, y) \leq m \leq k$.

By assumption, there exists an element $r \in R$, such that r is a sum of k units but not a sum of m units for any $m < k$. Then $d(r, 0) \leq k$. We claim that $d(r, 0) = k$. If

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$d(r,0) = l < k$, then, by Lemma 2.2(ii), r is l-good, a contradiction. So $d(r,0) = k$, hence $\operatorname{diam}(\Gamma(R)) = k$.

($\Leftarrow$). It is clear that 0 is 2-good. For any $0 \neq r \in R$, as $\operatorname{diam}(\Gamma(R)) = k$, we have $d(r,0) = l \le k$. It follows that r is l-good by Lemma 2.2(ii). Again as $\operatorname{diam}(\Gamma(R)) = k$, there exist x and y with $d(x,y) = k$. Then $d(x-y,0) = k$. By Lemma 2.2, $x-y$ is k-good, but not l-good for any $l < k$, so $\operatorname{usn}(R) = k$. ∎

Theorem 2.5 *For each integer $n \ge 1$, there exists a ring R such that* $\operatorname{diam}(\Gamma(R)) = n$.

Proof In [13, Corollary 4], the authors proved that there exists a ring R such that $\operatorname{usn}(R) = n$ for each $n \ge 2$. So the theorem holds for $n \ge 3$ by Lemma 2.4. It is clear that $\operatorname{diam}(\Gamma(\mathbb{Z}_2)) = 1$ and $\operatorname{diam}(\Gamma(\mathbb{Z}_4)) = 2$. This completes the proof. ∎

Corollary 2.6 *Let R be a ring. Then $\Gamma(R)$ is connected if and only if $\mathbf{u}(R) \le \omega$.*

Proof Suppose that $\Gamma(R)$ is connected. Then for any $0 \neq r \in R$, $d(r,0) = k$ for some k. So r is k-good by Lemma 2.2(ii). Thus, $\mathbf{u}(R) \le \omega$. Conversely, if $\mathbf{u}(R) \le \omega$, then for any two vertices x and y in R, we have that x is k-good and y is l-good for some k and l. So $d(x,0) \le k$ and $d(y,0) \le l$ by Lemma 2.2(i). So $d(x,y) \le d(x,0)+d(y,0) = k+l$. Thus, $\Gamma(R)$ is connected. ∎

Note that $\mathbf{u}(R) = n$ implies $\operatorname{usn}(R) = n$, but $\operatorname{usn}(R) = n$ cannot imply $\mathbf{u}(R) = n$ in general. For example, $\operatorname{usn}(\mathbb{Z}_4) = 2$, but $\mathbf{u}(\mathbb{Z}_4) = \omega$. In fact, we can easily obtain the following proposition.

Proposition 2.7 *Let R be a ring and let $n > 1$ be an integer. Suppose that $2 \in U(R)$. Then $\mathbf{u}(R) = n$ if and only if* $\operatorname{usn}(R) = n$.

3 Self-injective Rings

In [2, Theorem 3.1], the authors proved that $\operatorname{diam}(\Gamma(R)) \in \{1,2,3,\infty\}$ for a left Artinian ring R and classified all left Artinian rings according to the diameter of their unitary Cayley graphs. Next, we generalize the results to the rings R for which $R/J(R)$ is self-injective. To do so, we first study the relationship between $\operatorname{diam}(\Gamma(\overline{R}))$ and $\operatorname{diam}(\Gamma(R))$. Note that r is a unit in R if and only if $\overline{r}$ is a unit in $\overline{R}$. Using the idea of [12, Remark 1], we have $\operatorname{diam}(\Gamma(\overline{R}) \le \operatorname{diam}(\Gamma(R))$. Indeed, suppose $\operatorname{diam}(\Gamma(R)) = m$. Then for any $\overline{x} \neq \overline{y} \in \overline{R}$, we have $d(x,y) \le m$. As a path from x to y gives a walk from $\overline{x}$ to $\overline{y}$, $d(\overline{x},\overline{y}) \le d(x,y) \le m$. Thus, $\operatorname{diam}(\Gamma(\overline{R})) \le m$.

Lemma 3.1 *Let R be a ring. If* $\operatorname{diam}(\Gamma(R)) \ge 3$, *then* $\operatorname{diam}(\Gamma(\overline{R})) = \operatorname{diam}(\Gamma(R))$.

Proof It suffices to show that $\operatorname{diam}(\Gamma(R)) \le \operatorname{diam}(\Gamma(\overline{R}))$.

Suppose $\operatorname{diam}(\Gamma(R)) = \infty$. We show that $\operatorname{diam}(\Gamma(\overline{R})) = \infty$. Assume to the contrary that $\operatorname{diam}(\Gamma(\overline{R})) = m < \infty$. For any $x, y \in R$, if $\overline{x} = \overline{y}$, then $x - y \in J(R)$, and hence $1 + x - y \in U(R)$. So we get a path $x — (y-1) — y$ from x to y, so $d(x,y) \le 2$. If $\overline{x} \neq \overline{y}$, then a path form $\overline{x}$ to $\overline{y}$ deduces a path from x to y. This implies that $d(x,y) \le d(\overline{x},\overline{y}) \le m$. So $\operatorname{diam}(\Gamma(R)) \le m$, a contradiction.

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Assume that $\operatorname{diam}(\Gamma(R))$ is finite and $k := \operatorname{diam}(\Gamma(R)) \geq 3$. There exist $x, y \in R$, such that $d(x,y) = k$. First, we claim that $\overline{x} \neq \overline{y}$. In fact, if $\overline{x} = \overline{y}$, then $x - y \in J(R)$, and hence $1+x-y \in U(R)$. So $x—(y-1)—y$ is a walk from x to y. Thus, $d(x,y) \leq 2$, a contradiction. Assume that $m := d(\overline{x}, \overline{y}) < k$ and $\overline{x}—\overline{x_1}—\overline{x_2}—\cdots—\overline{x_{m-1}}—\overline{y}$ is a path from $\overline{x}$ to $\overline{y}$. Then $x—x_1—x_2—\cdots—x_{m-1}—y$ is path of length m, so $d(x,y) \leq m < k$, a contradiction. Thus, $d(\overline{x}, \overline{y}) = k$. This proves $\operatorname{diam}(\Gamma(\overline{R})) \geq k$. Hence, $\operatorname{diam}(\Gamma(\overline{R})) = \operatorname{diam}(\Gamma(R))$. ∎

Theorem 3.2 *Let R be a ring. Then the following are equivalent:*

(i) $\operatorname{diam}(\Gamma(\overline{R})) < \operatorname{diam}(\Gamma(R))$.
(ii) *R is a local ring with $J(R) \neq 0$.*
(iii) $\operatorname{diam}(\Gamma(R)) = 2$ *and* $\operatorname{diam}(\Gamma(\overline{R})) = 1$.

Proof (i)⇒(ii). Suppose that $\operatorname{diam}(\Gamma(\overline{R})) < \operatorname{diam}(\Gamma(R))$. Then by Lemma 3.1, $\operatorname{diam}(\Gamma(R)) \leq 2$. By assumption, $\operatorname{diam}(\Gamma(\overline{R})) = 1$. So $\overline{R}$ is a division ring by Lemma 2.1. Therefore, R is a local ring with $J(R) \neq 0$.

(ii)⇒(iii). Suppose that R is a local ring with $J(R) \neq 0$. Then $\overline{R} = R/J(R)$ is a division ring. So $\operatorname{diam}(\Gamma(\overline{R})) = 1$ by Lemma 2.1. On the other hand, for any $r \in R$, either $r \in J(R)$ or $r \in U(R)$. For any two distinct elements $a, b \in R$, if $a—b \in U(R)$, then $d(a,b) = 1$. Suppose that $a—b \in J(R)$. If $a \in J(R)$, then $b \in J(R)$ as well. So we have a path $a—1—b$, and hence $d(a,b) = 2$ (note that since $J(R) \neq 0$, such a, b do exist). If $a \in U(R)$, then $b \in U(R)$, we have a path $a—(a+b)—b$, so $d(a,b) = 2$. Hence, $\operatorname{diam}(\Gamma(R)) = 2$.

(iii)⇒ (i). It is clear. ∎

Corollary 3.3 *Let R be a ring. Then* $\operatorname{diam}(\Gamma(\overline{R})) = \operatorname{diam}(\Gamma(R))$ *if and only if one of the following holds:*

(i) *R is not a local ring.*
(ii) *R is a division ring.*

In [18, Theorem 6], Khurana and Srivastava determined the unit sum number $\mathbf{u}(R)$ of a regular right self-injective ring R. We use the notion $\operatorname{usn}(R)$ to restate the theorem below.

Lemma 3.4 ([18]) *Let R be a regular self-injective ring. Then* $\operatorname{usn}(R) = 2$, *3, or* ∞. *Moreover,*

(i) $\operatorname{usn}(R) = 2$ *if and only if R has no nonzero Boolean ring as a ring direct summand or* $R \cong \mathbb{Z}_2$;
(ii) $\operatorname{usn}(R) = 3$ *if and only if* $R \ncong \mathbb{Z}_2$ *and R has* $\mathbb{Z}_2$, *but no Boolean ring with more than two elements, as a ring direct summand;*
(iii) $\operatorname{usn}(R) = \infty$ *if and only if R has a Boolean ring with more than two elements as a ring direct summand.*

Theorem 3.5 *Let R be a ring with $R/J(R)$ right self-injective (in particular, R is right self-injective). Then* $\operatorname{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$.

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Proof As $\overline{R} = R/J(R)$ is a right (regular) self-injective ring, we have $\operatorname{usn}(\overline{R}) = 2, 3$ or, ∞ by Lemma 3.4. Then $\operatorname{diam}(\overline{R}) \in \{1, 2, 3, \infty\}$ by Lemmas 2.1, 2.3, and 2.4. Now, by Lemma 3.1, we get $\operatorname{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$. ■

Theorem 3.6 *Let R be a ring with $R/J(R)$ right self-injective. Then the following hold:*

(i) $\operatorname{diam}(\Gamma(R)) = 1$ *if and only if R is a division ring.*
(ii) $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if R is not a division ring and one of following holds:*
  (a) $\overline{R}$ *has no nonzero Boolean ring as a ring direct summand.*
  (b) $\overline{R} \cong \mathbb{Z}_2$.
(iii) $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if* $\overline{R} \ncong \mathbb{Z}_2$ *and* $\overline{R}$ *has* $\mathbb{Z}_2$*, but no Boolean ring with more than two elements, as a ring direct summand.*
(iv) $\operatorname{diam}(\Gamma(R)) = \infty$ *if and only if* $\overline{R}$ *has a Boolean ring with more than two elements as a ring direct summand.*

Proof (i) This follows from Lemma 2.1.

Next, we assume that R is not a division ring and prove (ii), (iii), and (iv) together. Note that $\overline{R}$ is a regular right self-injective ring. So $\mathbf{u}(\overline{R}) = 2$, ω or ∞ by [18, Theorem 6]. To complete the proof, we determine the diameter in each case.

Case 1: $\mathbf{u}(\overline{R}) = 2$. In this case, $\overline{R}$ has no nonzero Boolean ring as a ring direct summand or $\overline{R} \cong \mathbb{Z}_2$ by Lemma 3.4. Note that $\operatorname{diam}(\Gamma(\overline{R})) \in \{1, 2\}$. So $\operatorname{diam}(\Gamma(R)) = 2$ by Lemma 3.1.

Case 2: $\mathbf{u}(\overline{R}) = \omega$. If $\overline{R} \cong \mathbb{Z}_2$, then $\Gamma(R)$ is a complete bipartite graph. So $\operatorname{diam}(\Gamma(R)) = 2$. If $\overline{R} \ncong \mathbb{Z}_2$, in this case, $\operatorname{usn}(\overline{R}) = 3$, so $\operatorname{diam}(\Gamma(\overline{R})) = 3$ by Lemma 2.4. Thus, $\operatorname{diam}(\Gamma(R)) = 3$ by Lemma 3.1.

Case 3: $\mathbf{u}(\overline{R}) = \infty$. Then $\Gamma(R)$ is disconnected by Corollary 2.6, so $\operatorname{diam}(\Gamma(R)) = \infty$. Thus, $\operatorname{diam}(\Gamma(R)) = \infty$ by Lemma 3.1. ■

4 Extensions of Rings

In this section, we consider the diameter of the unitary Cayley graphs of some extensions of rings.

Proposition 4.1 *Let R be a commutative ring. Then $\Gamma(R[t])$ is disconnected.*

Proof As $\mathbf{u}(R[t]) = \infty$, $\Gamma(R[t])$ is disconnected by Corollary 2.6. ■

Proposition 4.2 *Let R be a commutative ring. Then the following conditions are equivalent:*

(i) $\mathbf{u}(R) \le \omega$.
(ii) $\Gamma(R)$ *is connected.*
(iii) $\Gamma(R[[t]])$ *is connected.*

Proof (i)⇒(ii). This follows from Corollary 2.6.

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(ii)⇒(iii). Let $f(t), g(t) \in R[[t]]$. Since $\Gamma(R)$ is connected, there is a path from $f(0)$ to $g(0)$ in $\Gamma(R)$, say $f(0)—a_1—a_2—\cdots—a_k—g(0)$. Then $f(t)—a_1—a_2—\cdots—a_k—g(t)$ is a path from $f(t)$ to $g(t)$ in $\Gamma(R[[t]])$. So $\Gamma(R[[t]])$ is connected.

(iii)⇒ (i). Let $0 \neq a \in R$. As $\Gamma(R[[t]])$ is connected, $d(a,0) = k$ in $\Gamma(R[[t]])$ for some integer $k \geq 1$. Let $f_0(t) := a—f_1(t)—f_2(t)—\cdots—f_{k-1}(t)—f_k(t) := 0$ be a path from a to 0 in $\Gamma(R[[t]])$. Then $u_i := f_i(0) - f_{i+1}(0) \in U(R)$ for $0 \leq i \leq k-1$. So $a = \sum_{i=0}^{k-1} u_i$, which is k-good, so $\mathbf{u}(R) \leq \omega$. ∎

Proposition 4.3 *Let R be a commutative ring. Then the following statements hold:*

(i) *If R is a field, then* $\operatorname{diam}(\Gamma(R[[t]])) = 2$.
(ii) *If R is not a field, then* $\operatorname{diam}(\Gamma(R[[t]])) = \operatorname{diam}(\Gamma(R))$.

Proof (i) As $R[[t]]$ is not a field, $\operatorname{diam}(\Gamma(R[[t]])) \geq 2$ by Lemma 2.1. For any $f(t), g(t) \in R[[t]]$, if $f(0) = g(0)$, taking $a \neq f(0)$, then $f(t)—a—g(t)$ is a path from $f(t)$ to $g(t)$. So $\operatorname{diam}(\Gamma(R[[t]])) = 2$.

(ii) Note that in this case, both $\operatorname{diam}(\Gamma(R[[t]]))$ and $\operatorname{diam}(\Gamma(R))$ are at least two. We first prove that $\operatorname{diam}(\Gamma(R)) \leq \operatorname{diam}(\Gamma(R[[t]]))$. If $\operatorname{diam}(\Gamma(R[[t]])) = \infty$, there is nothing to prove. Suppose that $\operatorname{diam}(\Gamma(R[[t]])) = n < \infty$. Let $a, b \in R$. Then we have $k := d(a,b) \leq n$ in $\Gamma(R[[t]])$. Let

$$a—f_1(t)—f_2(t)—\cdots—f_k(t) = b$$

be a path from a to b. Then

$$a—f_1(0)—f_2(0)—\cdots—f_k(0) = b$$

is a walk from a to b in $\Gamma(R)$, so $d(a,b) \leq k \leq n$ in $\Gamma(R)$, and hence $\operatorname{diam}(\Gamma(R)) \leq n$.

Now we prove that $\operatorname{diam}(\Gamma(R)) \geq \operatorname{diam}(\Gamma(R[[t]]))$. If $\operatorname{diam}(\Gamma(R)) = \infty$, there is nothing to prove. Suppose that $\operatorname{diam}(\Gamma(R)) = n < \infty$. Let $f(t), g(t) \in R[[t]]$. Then we have $k := d(f(0), g(0)) \leq n$ in $\Gamma(R)$. Let

$$f(0)—a_1—a_2—\cdots—a_k—g(0)$$

be a path from $f(0)$ to $g(0)$ in $\Gamma(R)$. Then

$$f(t)—a_1—a_2—\cdots—a_k—g(t)$$

is a path from $f(t)$ to $g(t)$ in $\Gamma(R[[t]])$. So, $d(f(t), g(t)) = k \leq n$ in $\Gamma(R[[t]])$, and hence $\operatorname{diam}(\Gamma(R[[t]])) \leq n$. ∎

Proposition 4.4 *Let* $T := \mathbb{M}_n(R)$ *be the* $n \times n$ $(n \geq 2)$ *matrix ring over a ring R. Then* $2 \leq \operatorname{diam}(\Gamma(T)) \leq 3$. *Moreover,* $\operatorname{diam}(\Gamma(T)) = 2$ *if and only if* $usn(R) = 2$.

Proof We know that $\mathbf{u}(T) \leq 3$ by [11, Theorem 3]. So $\operatorname{usn}(R) \leq 3$. As T is not a division ring, $2 \leq \operatorname{diam}(\Gamma(T)) \leq 3$. If $\operatorname{usn}(R) = 2$, then $\operatorname{usn}(T) = 2$ as well, so $\operatorname{diam}(\Gamma(T)) = 2$. Conversely, if $\operatorname{diam}(\Gamma(T)) = 2$, then $\operatorname{usn}(T) = 2$, so $\operatorname{usn}(R) = 2$. ∎

The group ring of a group H over ring R is denoted by RH.

Proposition 4.5 *Let R be a ring and H be a nontrivial group. Then $\Gamma(RH)$ is connected if and only if $\Gamma(R)$ is connected.*

Proof This follows from Corollary 2.6 and [5, Proposition 9]. ■

Proposition 4.6 *Let F be a field and H be a locally finite group (that is, every finitely generated subgroup of H is finite). Then $\operatorname{diam}(\Gamma(\mathbb{Z}_2 H)) = \infty$ and $\operatorname{diam}(\Gamma(FH)) = 2$ if $F \ncong \mathbb{Z}_2$.*

Proof By [5, Proposition 9(v)], $\operatorname{diam}(\Gamma(FH)) = 2$ if $F \ncong \mathbb{Z}_2$. As $\mathbf{u}(\mathbb{Z}_2 H) = \omega$, we have $\operatorname{diam}(\Gamma(\mathbb{Z}_2 H)) = \infty$.

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