SOME CHARACTERIZATIONS OF THE ONE-PARAMETER FAMILY OF PROBABILITY DISTRIBUTIONS

A. M. Mathai

(received September 14, 1965)

1. Summary. We consider one-parameter families of distributions which are of the general exponential type, the general series distribution type, or which can be transformed into the exponential type by a one-to-one transformation. In this paper we establish theorems to the effect that such distributions may be characterized by a simple differential equation involving the mean function. It is illustrated that almost all the classical one-parameter families of distributions are characterized by these theorems. Multivariate generalizations are given, and it is also noticed that the functional form of the normalizing factor determines the specific distribution in each general family.

2. Introduction. Characterization of series and exponential type distributions, from the moment and cumulant relations, have been considered by Kosambi [1949] and Patil [1961]. These involve a number of basic relations for the characterization. Characterization of the individual members, in the exponential type families, from the forms of the mean function is discussed by Patil and Shorrock [1965].

3. Some Theorems.

THEOREM 1. Among all one-parameter families of probability distributions the exponential type family is completely characterized by

$$f'(w) - \phi(w)f(w) = 0 , \tag{1}$$

where $\phi(w)$ is the mean function, w is the parameter, $f(w+t)$ is expansible in Taylor series at w for some $t \neq 0$ and $f'(w) = \frac{d}{dw} f(w)$. A probability distribution is said to be

Canad. Math. Bull. vol. 9, no. 1, 1966

of the general exponential type with the parameter w if the probability function can be written in the form

$$dF = a(x)\, e^{wx} / g(w), \tag{2}$$

$a(x) > 0$, $w \in \Omega$ where Ω is the parameter space, $g(w)$ is the normalizing factor which is finite and differentibale, $x \in T$ where T is a discrete or continuous subset of the set of real numbers R. We will denote summation or integration by $\int_T$.

Hence $g(w) = \int_T a(x)\, e^{wx}$.

Proof. $f(w+t) = f(w) + t f'(w) + t^2 f''(w)/2! + \dots$.

Let $M(t) = f(w+t)/f(w) = 1 + t f'(w)/f(w) + t^2 f''(w)/2! f(w) + \dots$

$$= 1 + t\,\Phi(w) + \frac{t^2}{2!}\,\frac{1}{f(w)}\,\frac{d}{dw}[\Phi(w) f(w)] + \dots$$

$$+ \dots + \frac{t^{n+1}}{(n+1)!}\,\frac{1}{f(w)}\,\frac{d^n}{dw^n}[\Phi(w) f(w)] + \dots .$$

$M(t)$ uniquely determines a distribution whose moment generating function is $M(t)$. But for the exponential type family of distributions,

$$\Phi(w) = \int_T a(x) . x . e^{xw} / g(w) \tag{3}$$

$$g(w) . \Phi(w) = \int_T a(x)\, x\, e^{xw} \tag{4}$$

$$\frac{1}{g(w)}\,\frac{d^n}{dw^n}[\Phi(w)\, g(w)] = m'_{n+1}, \quad n = 1, 2, \dots, \tag{5}$$

where m'_{r+1} is the $(r+1)$st moment about the origin of the exponential type distribution. Therefore the moment generating function of the exponential type distribution is $M(t)$ where $g(w) = f(w)$.
Hence the result.

If the functional form of $f(w)$ is specified then the different member distributions in the exponential type family are obtained. Some of the examples are given below.

https://doi.org/10.4153/CMB-1966-013-8 Published online by Cambridge University Press

1. Normal: $dF = (2\pi\beta^2)^{-1/2} \cdot \exp -(x-\alpha)^2/2\beta^2$;

$-\infty < \alpha < \infty$; $\beta > 0$ known; $-\infty < x < \infty$ and

$f(w) = (2\pi\beta^2)^{-1/2} \cdot \exp(\alpha^2/2\beta^2)$ where $w = \alpha/\beta^2$.

2. Exponential: $dF = \theta e^{-x\theta}$, $\theta > 0$, $0 < x < \infty$ and

$f(w) = -1/w$ where $w = -\theta$.

3. Gamma: $dF = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, α known, $\beta > 0$,

$x > 0$, $f(w) = (-1/w)^{\alpha}$ where $w = -1/\beta$.

4. Binomial: $dF = \binom{n}{x} p^x(1-p)^{n-x}$, $0 < p < 1$, n known, $x = 0$,

$1, \ldots, n$. $f(w) = (1+e^w)^n$ where $w = \log\theta$ and

$\theta = p/(1-p)$.

5. Poisson: $dF = \beta^x e^{-\beta}/x!$, $\beta > 0$, $x = 0, 1, \ldots,$

$f(w) = e^{e^w}$ where $W = \log\beta$.

THEOREM 2. Among all one-parameter families of probability distributions the series distribution is completely characterized by the equation

$$\phi(\theta)\, h(\theta) = \theta\, h'(\theta) , \tag{6}$$

where $\phi(\theta)$ is the mean function and $h(\theta+t)$ is expansible in Taylor series at θ for some $t \neq 0$.

The series distribution is given by the probability function

$$dF = b(x)\ \theta^x/m(\theta) , \tag{7}$$

$b(x) > 0$, $\theta > 0$, $\theta \in \Omega'$ (parameter space), $m(\theta)$ is the normalizing factor which is finite and differentiable, $x \in T'$ and T' is a countable subset of the set of real numbers.

Hence $m(\theta) = \sum_{T'} b(x)\theta^x$. This can be proved as a corollary to theorem 1, since,

$$(8) \qquad b(x)\, \theta^{x}/m(\theta) = b(x)\, e^{xw}/m(e^{w}),$$

where $w = \log \theta$. (8) can be identified with the probability function in (2). Therefore the equation (6) reduces to the equation (1). Hence the result. The power series distribution, the classical discrete distributions such as the Binomial, Negative Binomial, Geometric, Poisson and Logarithmic Series, all can be considered to be special cases of the series distribution. Some illustrations are given below and more may be found in Noack [1950].

a. Power Series: $dF = b(x)\, \theta^{x}/m(\theta)$, $b(x) > 0$, $\theta > 0$, $x \in T''$ and T'' is the set of positive integers.

b. Geometric: $dF = p(1-p)^{x-1}$, $0 < p < 1$, $x = 1, 2, \ldots,$ $m(\theta) = \theta/(1-\theta)$ where $\theta = 1-p$.

c. Negative Binomial: $dF = \binom{x-1}{k-1} p^{k}(1-p)^{x-k}$, $0 < p < 1$, k known, $x = k, k+1, \ldots$, $m(\theta) = \theta^{k}/(1-\theta)^{k}$ and $\theta = (1-p)$.

d. Logarithmic Series: $dF = \dfrac{\theta^{x}}{-\log(1-\theta).x}$, $0 < \theta < 1$, $x = 1, 2, \ldots$, $m(\theta) = -\log(1-\theta)$.

THEOREM 3. Among all k-variate k-parameter families of probability distributions, the k-variate exponential type family is completely characterized by the equations

$$(9) \qquad \frac{\partial}{\partial w_i} f(w_1, \ldots, w_k) = \phi(w_i)\, f(w_1, w_2, \ldots, w_k) \text{ for } i = 1, \ldots, k,$$

where $\phi(w_i)$ is the mean function for the ith variate X_i (that is, $\phi(w_i) = E(X_i)$) and $f(w_1+t_1, \ldots, w_k+t_k)$ is expansible in Taylor series at $(w_1, \ldots, w_k)$ for some $(t_1, \ldots, t_k) \neq 0$.

A k-variate exponential type family is given by the probability function,

$$(10) \qquad dF = a(x_1, \ldots, x_k)(\exp.\Sigma w_i x_i)/g(w_1, \ldots, w_k),$$

$a(x_1, \ldots, x_k) > 0$, $(w_1, \ldots, w_k) \in \Omega''$ (parameter space),

$g(w_1, \ldots, w_k)$ is the normalizing factor which is finite and differentiable, $x_i \in T_i$ for all i where T_i is a continuous or discrete subset of the set of real numbers.

Proof. $f(w_1+t_1, \ldots, w_k+t_k) = f(w_1, \ldots, w_k) + \Sigma\, t_i\, \partial f/\partial w_i$

$$+ \Sigma \frac{t_i t_j}{2!} \frac{\partial^2 f}{\partial w_i \partial w_j} + \ldots .$$

Let $M(t_1, \ldots, t_k) = f(w_1+t_1, \ldots, w_k+t_k)/f(w_1, \ldots, w_k)$

$$= 1 + \Sigma\, t_1 \frac{\partial f}{\partial w_i} / f + \Sigma\, t_i t_j \frac{\partial^2 f}{\partial w_i \partial w_j} / f + \ldots$$

$$= 1 + \Sigma\, t_i \phi(w_i) + \Sigma \frac{t_i t_j}{2!} \frac{1}{f} \frac{\partial}{\partial w_j} \phi(w_i) f + \ldots .$$

$M(t_1, \ldots, t_k)$ uniquely determines a k-variate probability distribution whose moment generating function is $M(t_1, \ldots, t_k)$. But for the k-variate exponential type distribution

(11) $\phi(w_i) = \int_T a(x_1, \ldots, x_k)\, x_i\, (\exp \Sigma w_i x_i)/g(w_1, \ldots, w_k)$,

where $T = T_1 \times T_2 \times \ldots T_k$ and $\int_T$ stands for the multiple integral or sum as the case may be.

(12) $\frac{1}{g} \frac{\partial}{\partial w_j} g . \phi(w_i) = \frac{1}{g} \int_T a(x_1, \ldots, x_k)\, x_i x_j\, e^{\Sigma x_i w_i}$

$$= E(X_i X_j) \text{ etc.}$$

The moment generating function of the k-variate exponential type is $M(t_1, \ldots, t_k)$ where $g = f$. Hence the result.

THEOREM 4. Among all k-variate k-parameter families of probability distributions, the k-variate series type family is completely characterized by the equations

https://doi.org/10.4153/CMB-1966-013-8 Published online by Cambridge University Press

(13) $h(\theta_1, \ldots, \theta_k)\, \phi(\theta_i) = \theta_i \frac{\partial}{\partial \theta_i} h(\theta_1, \ldots, \theta_k)$ for $i = 1, \ldots, k$,

where $\phi(\theta_i) = E(X_i)$ is the mean function of the ith variate and $h(\theta_1 + t_1, \ldots, \theta_k + t_k)$ is expansible in Taylor series at $(\theta_1, \ldots, \theta_k)$ for some $(t_1, \ldots, t_k) \neq 0$.

A k-variate distribution is said to be of the power series type if the probability function is given by

$$(14) \qquad dF = b(x_1, \ldots, x_k)\, \theta_1^{x_1} \ldots \theta_k^{x_k} / m(\theta_1, \ldots, \theta_k),$$

$b(x_1, \ldots, x_k) > 0$, $\theta_i > 0$ for all i, $m(\theta_1, \ldots, \theta_k)$ is the normalizing factor, $(x_1, \ldots, x_k) \in I_1 \times I_2 \times \ldots \times I_k$ where I_i is a countable subset of the set of real numbers for all i, and $(\theta_1, \ldots, \theta_k) \in \Omega'''$ (parameter space). The proof follows from theorem 2 and theorem 3 by the transformation $w_i = \log \theta_i$ for all i.

THEOREM 5. If there exists a one-to-one transformation $Y = Q(X)$ between X and Y such that Y designates the general exponential type family of distributions then among all the one-parameter families of distributions the family designated by X is completely characterized by the equation

$$(15) \qquad f'(w) - f(w)\, \phi(w) = 0,$$

where w is the parameter, $\phi(w) = E[Q(X)]$ and $f(w+t)$ is expansible in Taylor series at w and for some $t \neq 0$.

$\phi(w) = E[Q(X)] = E(Y)$ is the mean function of Y and therefore the equation (15) can be identified with equation (1) which in turn completely characterizes the general exponential family by theorem 1. Since the transformation $Y = Q(X)$ is one-to-one the result follows.

Example 1: Consider the Gamma distribution

$$dF = \frac{1}{\Gamma(\alpha)} x^{\alpha - 1} e^{-x}\, dx,$$

https://doi.org/10.4153/CMB-1966-013-8 Published online by Cambridge University Press

$$= \frac{\exp(-x+\alpha \log x)}{\Gamma(\alpha)} \frac{dx}{x}, \quad x>0, \ \alpha>0 .$$

Put $y = \log x$, to obtain

$$dF(y) = \frac{e^{-e^y}}{\Gamma(\alpha)} \cdot e^{\alpha y} \, dy .$$

Clearly, $dF(y)$ is of the general exponential type, and

$$(16) \qquad \frac{d}{d\alpha} \Gamma(\alpha) - \Gamma(\alpha) . \phi(\alpha) = 0 ,$$

where $\phi(\alpha) = E(Y) = E(\log X)$.

Example 2: Consider the Beta distribution

$$dF(x) = \frac{x^{\alpha-1}}{B(\alpha,\beta)} \cdot (1-x)^{\beta-1} . dx , \quad 0 < x < 1, \ \alpha > 0, \beta > 0 \text{ known},$$

$$= \frac{e^{\alpha y}}{B(\alpha,\beta)} \cdot (1-e^y)^{\beta-1} dy, \text{ where } y = \log x .$$

This is of the exponential type. Similarly, if α is known then $dF(x)$ can be transformed to the exponential type by a similar transformation. Theorem 5 may be generalized to a k-variate k-parameter family of distributions under the existence of a similar one-to-one transformation.

4. Discussion. It is seen in the theorems discussed in this article that the differential equation (1) characterizes the exponential type families of probability distributions. Equation (1) may be written as

$$(17) \qquad \phi(w) = f'(w)/f(w) .$$

If w is the only parameter in a probability distribution and if $\phi(w)$ is the mean function where $w \in \Omega$ (parameter space) and if $\phi(w)$ is analytic in Ω then $\exp(\int \phi(w) \, dw + c)$ is analytic, where c is a constant. Let

$$(18) \qquad f(w) = \exp(\int \phi(w) \, dw + c).$$

Then $\phi(w) = f'(w)/f(w)$ and $f(w+t) = f(w) + t f'(w) + \ldots$ for some $t \neq 0$. Hence the condition in theorem 1 may be modified as follows: if the mean function $\phi(w)$ is analytic in Ω then $\emptyset(w)$ will determine the exponential type family among the one-parameter families of distributions.

REFERENCES

1. C.G. Khatri, On certain properties of the power series distribution. Biometrika, vol. 46, (1959) pp. 486-490.

2. D.D. Kosambi, Characterization properties of series distributions. Proc. Indian Academy of Sciences, vol. 15 (1949) pp. 109-113.

3. A. Noack, A class of random variables with discrete distributions. Ann. Math. Statist. vol. 21, (1950) pp. 127-132.

4. G.P. Patil, A characterization of the exponential type distribution. Biometrika, vol. 50, (1961) pp. 205-207.

5. G.P. Patil and R. Shorrock, On certain properties of the exponential type families. J. Roy Statist. Soc. Series B, vol. 27, (1965) part 1.

McGill University

https://doi.org/10.4153/CMB-1966-013-8 Published online by Cambridge University Press