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WEYL'S THEOREM FOR p-HYPONORMAL OR M-HYPONORMAL OPERATORS

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Abstract. In 1997, M. Cho, M. Ito and S. Oshiro showed that Weyl's theorem holds for p-hyponormal operators, for any $p > 0$. In this note we give another proof of this result. Also, it is shown that Weyl's theorem holds for M-hyponormal operators. Further, in 1962, Stampfli showed that if T is hyponormal and its Weyl spectrum is $\{0\}$ then T is compact and normal. We show that this result remains true if the hypothesis of hyponormality is replaced by either (a) p-hyponormality or (b) M-hyponormality.

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1. Introduction. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. For a $T \in \mathcal{B}(\mathcal{H})$ and for some $p > 0$, if $(T^*T)^p \geq (TT^*)^p$, then T is said to be *p-hyponormal*. $T \in \mathcal{B}(\mathcal{H})$ is called *M-hyponormal* if there exists a positive constant M for each $z \in \mathbb{C}$ such that $(T-zI)(T-zI)^* \leq M^2(T-zI)^*(T-zI)$. If $p=1$, then T is called simply *hyponormal* and it is equivalent to the case where $M=1$.

$T \in \mathcal{B}(\mathcal{H})$ is called a *Fredholm operator* if $T\mathcal{H}$ is closed and both $\mathrm{Ker}T = \{x \in \mathcal{H} : Tx = 0\}$ and $\mathrm{Ker}T^*$ are finite-dimensional. To any Fredholm operator T there corresponds an integer $i(T) = \dim\mathrm{Ker}T - \dim\mathrm{Ker}T^*$, called the *index* of T. Let $\mathcal{F}_0$ denotes the class of all Fredholm operators in $\mathcal{B}(\mathcal{H})$ of index 0. Then $w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{F}_0\}$ is called the *Weyl spectrum* of T. It is known that, for a $T \in \mathcal{B}(\mathcal{H})$, $w(T)$ is non-empty and $w(T) = \bigcap_{K \in \mathcal{C}(\mathcal{H})} \sigma(T+K)$, where $\sigma(T)$ and $\mathcal{C}(\mathcal{H})$ denote the spectrum of T and the set of all compact operators in $\mathcal{B}(\mathcal{H})$ respectively.

For a $T \in \mathcal{B}(\mathcal{H})$, let $\sigma_p(T)$ and $\pi_{00}(T)$ denote the point spectrum and the set of all isolated eigenvalues of finite multiplicity of T respectively. According to Coburn [**3**], we say that *Weyl's theorem holds for* T if $\sigma(T)\backslash w(T) = \pi_{00}(T)$. He showed that Weyl's theorem holds for hyponormal operators and this result was generalized to p-hyponormal operators by Cho-Ito-Oshiro [**4**]. Also, by Stampfli [**8**], it is known that if T is hyponormal and if $w(T) = \{0\}$, then T is compact and normal. In this paper, we shall give another proof of the result of Cho-Ito-Oshiro and prove that Weyl's theorem holds for M-hyponormal operators and that Stampfli's result above is also applicable to p-hyponormal or M-hyponormal operators.

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2. Preliminaries. According to Berberian [**2**], we say that T is *isoloid* if every isolated point of $\sigma(T)$ is in the point spectrum or T. Also if every restriction $T|_{\mathcal{M}}$ to its reducing subspace $\mathcal{M}$ is isoloid, then we say that T satisfies the condition (α''').

The following result was given by Berberian [**2**].

Theorem A. *If $T \in \mathcal{B}(\mathcal{H})$ satisfies the condition (α''') and if every finite-dimensional eigenspace of T reduces T, then Weyl's theorem holds for T.*

Definition 1. If $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$, for all $x \in \mathcal{H}$, then we say that T is *paranormal.*

The following results are well known.

Proposition 1. *If T is paranormal, then $\|T\| = \sup\{|\lambda| \ ; \ \lambda \in \sigma(T)\}$.*

Proposition 2. (Heinz's, inequality [**6**]) *If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$, for all $\alpha \in (0, 1]$.*

Proposition 3. (Hansen's inequality [**5**]) *If $A \geq 0$ and $\|B\| \leq 1$ then $(B^*AB)^\delta \geq B^* A^\delta B$, for all $\delta \in (0, 1]$.*

Proposition 4. (Hölder-McCarthy inequality [**7**]) *If $A \geq 0$, then for each $x \in \mathcal{H}$ we have*

$$\langle A^r x, x\rangle \begin{cases} \leq \langle Ax, x\rangle^r \|x\|^{2(1-r)} & (0 < r \leq 1), \\ \geq \langle Ax, x\rangle^r \|x\|^{2(1-r)} & (r \geq 1). \end{cases}$$

Proposition 5. [**12**] *Let T be p-hyponormal with its polar decompositon $T = U|T|$. Then, for any s and t such that $s \geq 0$ and $t \geq 0$,*

$$|T|^s U|U|^t \ \textit{is} \ \begin{cases} 1\textit{-hyponormal} & (\max(s, t) \leq p), \\ \frac{p+\min(s,t)}{s+t}\textit{-hyponormal} & (\max(s, t) > p). \end{cases}$$

Proposition 6. ([**11**]) *The restriction $T|_{\mathcal{M}}$ of an M-hyponormal operator T to its invariant subspace $\mathcal{M}$ is also M-hyponormal.*

Definition 2. For a $T \in \mathcal{B}(\mathcal{H})$, we say that T belongs to the class $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$ if there is a positive number K_α such that $|T^*T - TT^*|^\alpha \leq K_\alpha^2 (T - zI)^* (T - zI)$, for all $z \in \mathbb{C}$. Also, let $\mathcal{Y} = \bigcup_{\alpha \geq 1} \mathcal{Y}_\alpha$.

The following results are known.

Proposition 7. [**10**] *If T is M-hyponormal, then $T \in \mathcal{Y}_2 \subseteq \mathcal{Y}$.*

Proposition 8. [**10**] *If $T \in \mathcal{Y}$, then $Tx = \lambda x$* implies *$T^*x = \bar{\lambda}x$.*

Proposition 9. [**10**] *If $T \in \mathcal{Y}$ and if $\sigma(T) = \{0\}$, then $T = 0$.*

COROLLARY 1. *If T is M-hyponormal and $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

Proof. Let T be a M-hyponormal operator such that $\sigma(T) = \{\lambda\}$, then $T - \lambda I$ is also M-hyponormal and $\sigma(T - \lambda I) = \{0\}$. Therefore $T - \lambda I = 0$ by Propositions 7 and 9.

3. Main theorems.

LEMMA 1. *If T is invertible and p-hyponormal, then T^{-1} is also p-hyponormal.*

Proof. Since T is an invertible p-hyponormal operator, we have $|T|^{2p} \geq |T^*|^{2p}$ and $|T|^{-2p} \leq |T^*|^{-2p}$. It is easy to verify the equalities $\left|T^{-1^*}\right|^{2p} = |T|^{-2p}$ and $|T^*|^{-2p} = \left|T^{-1}\right|^{2p}$, and the assertion of Lemma 1 holds.

REMARK. It is well known that the inverse operator T^{-1} of an invertible paranormal operator T is also paranormal.

LEMMA 2. *If T is p-hyponormal, then T is paranormal.*

Proof. Let $x \in \mathcal{H}$ be an arbitrary non-zero vector. Then we have

$$\begin{aligned}
\|T^2x\|^2 &= \left\langle (T^*T)^{p\cdot\frac{1}{p}}Tx, Tx\right\rangle \\
&\geq \|Tx\|^{2\left(1-\frac{1}{p}\right)}\langle (T^*T)^pTx, Tx\rangle^{\frac{1}{p}} \qquad \text{(by Proposition 4)} \\
&\geq \|Tx\|^{2\left(1-\frac{1}{p}\right)}\langle T^*(TT^*)^pTx, Tx\rangle^{\frac{1}{p}} \qquad \text{(since } T \text{ is } p\text{-hyponormal)} \\
&= \|Tx\|^{2\left(1-\frac{1}{p}\right)}\left\langle (T^*T)^{1+p}x, x\right\rangle^{\frac{1}{p}} \\
&\geq \|Tx\|^{2\left(1-\frac{1}{p}\right)}\|x\|^{-2}\|Tx\|^{2\left(1+\frac{1}{p}\right)} \qquad \text{(by Proposition 4)} \\
&= \|Tx\|^4\|x\|^{-2}.
\end{aligned}$$

Hence, we have $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$, and the proof of Lemma 2 is complete.

COROLLARY 2. *If T is p-hyponormal and if $\sigma(T) = \{0\}$, then $T = 0$.*

Proof. By Lemma 2 and by Proposition 1, we have the conclusion.

LEMMA 3. **[9]** *If T is p-hyponormal for a p such that $0 < p \leq 1$, then the restriction $T|_{\mathcal{M}}$ to its invariant subspace $\mathcal{M}$ is also p-hyponormal.*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Then $T|_{\mathcal{M}} = TP$ on $\mathcal{M}$. Thus we obtain

$$\{(T|_{\mathcal{M}})^*(T|_{\mathcal{M}})^p\} = (PT^*TP)^p \geq P(T^*T)^pP \quad \text{(by Proposition 3)},$$

and

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$$\{(T|_{\mathcal{M}})(T|_{\mathcal{M}})^*\}^p = (TPT^*)^p = P(TPT^*)^p P \leq P(TT^*)^p P \quad \text{(by Proposition 2)}.$$

Since T is p-hyponormal

$$\{(T|_{\mathcal{M}})(T|_{\mathcal{M}})^*\}^p \leq P(TT^*)^p P \leq P(T^*T)^p P \leq \{(T|_{\mathcal{M}})^*(T|_{\mathcal{M}})\}^p$$

and this inequality shows that $T|_{\mathcal{M}}$ is p-hyponormal.

THEOREM 1. *If T is p-hyponormal or M-hyponormal, then T is isoloid and satisfies the condition (α''') by Lemma 3 or Proposition 6 respectively.*

Proof. Let λ be an isolated point of $\sigma(T)$. Then the range of the Riesz projection $E = \frac{1}{2\pi i}\int_{\partial D}(zI - T)^{-1}dz$ is a closed invariant subspace for T and $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$. Here D is a closed ball with center λ that satisfies $\sigma(T) \cap D = \{\lambda\}$, and ∂D is the boundary of D described once counterclockwise.

First, we prove that every M-hyponormal operator is isoloid.

If T is M-hyponormal, then $T|_{E\mathcal{H}}$ is also M-hyponormal, by Proposition 6. Since $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$, we have $T|_{E\mathcal{H}} = \lambda E$, by Corollary 1. Hence, the assertion of Theorem 1 holds for M-hyponormal operators.

Next, we prove that every p-hyponormal operator is isoloid.

If $\lambda = 0$, then $\sigma(T|_{E\mathcal{H}}) = \{0\}$ and $T|_{E\mathcal{H}}$ is paranormal, by Lemma 3. $T|_{E\mathcal{H}} = 0$ by Corollary 2. Therefore 0 is in the point spectrum of T.

If $\lambda \neq 0$, $T|_{E\mathcal{H}}$ is an invertible paranormal operator and hence $(T|_{E\mathcal{H}})^{-1}$ is also paranormal by Lemmas 1 and 2. By Proposition 1, we see that $\|T|_{E\mathcal{H}}\| = |\lambda|$ and $\|(T|_{E\mathcal{H}})^{-1}\| = \frac{1}{|\lambda|}$. Let $x \in E\mathcal{H}$ be an arbitrary vector. Then

$$\|x\| \leq \|(T|_{E\mathcal{H}})^{-1}\|\|T|_{E\mathcal{H}}x\| = \tfrac{1}{|\lambda|}\|T|_{E\mathcal{H}}x\| \leq \tfrac{1}{\lambda}|\lambda|\|x\| = \|x\|.$$

This implies that $\frac{1}{\lambda}T|_{E\mathcal{H}}$ is unitary. Therefore $T|_{E\mathcal{H}}$ is normal and equal to λE. Hence, the assertion of Theorem 1 holds for p-hyponormal operators. This completes the proof of Theorem 1.

LEMMA 4. *For any $T \in \mathcal{B}(\mathcal{H})$ with its polar decomposition $T = U|T|$ and $\lambda = re^{i\theta} \neq 0$, $(T - \lambda I)x = (T - \lambda I)^*x = 0$ if and only if $(|T| - rI)x = (U - e^{i\theta}I)x = 0$.*

Proof. If $(T-\lambda I)x = (T-\lambda I)^*x = 0$, then $|T|^2x = T^*T = r^2x$ and $|T|x = rx$ because $|T| + rI$ is invertible. By the assumption, $re^{i\theta}x = Tx = U|T|x = rUx$ and we have $(U-e^{i\theta}I)x = 0$ because $r \neq 0$.

Conversely, if $(|T| - rI)x = (U - e^{i\theta}I)x = 0$, then $(U-e^{i\theta}I)^*x = 0$ by the general theory and we obtain that $Tx = U|T|x = Urx = re^{i\theta}x = \lambda x$ and $T^*x = |T|U^*x = |T|e^{-i\theta}x = re^{-i\theta}x = \lambda x$.

LEMMA 5. *If T is p-hyponormal for $p = \frac{1}{2}$, then $(T - \lambda I)x = 0$ implies that $(T - \lambda I)^*x = 0$.*

Proof. If $\lambda = 0$, then the assertion is trivial since $\operatorname{Ker} T \subseteq \operatorname{Ker} T^*$, for every p-hyponormal operator T.

For a p-hyponormal operator T with $p = \frac{1}{2}$ and a $\lambda = re^{-i\theta} \neq 0$, we have

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$$(T-\lambda I)^*(T-\lambda I) = (|T|-|\lambda|I)^2 + |\lambda|\big(U-e^{i\theta}I\big)|T|\big(U-e^{i\theta}I\big)^* + |\lambda|(|T|-|T^*|)$$
$$\geq (|T|-|\lambda|I)^2.$$

Therefore, if $(T-\lambda I)x=0$, then $(|T|-|\lambda|I)x=0$ and hence $(U-e^{i\theta}I)x=0$ because $r\neq 0$. We have $(T-\lambda I)^*x=0$ by Lemma 4.

LEMMA 6. *If T is p-hyponormal with the polar decomposition $T=U|T|$, then every eigenspace of U is invariant under $|T|$ and U^*.*

Proof. If $\lambda\in\mathbb{C}$ is an eigenvalue of U and x is a non-zero eigenvector with respect to λ, then $\lambda=0$ or $|\lambda|=1$, since the range of U is a subspace of the domain space of U.

If $\lambda=0$, then the assertion follows from $\operatorname{Ker}U=\operatorname{Ker}|T|\subseteq\operatorname{Ker}|T^*|=\operatorname{Ker}U^*$.

If $\lambda=e^{i\theta}$ and $(U-e^{i\theta}I)x=0$, then $(U-e^{i\theta}I)^*x=0$ by the general theory and, since

$$\left\|\left(|T|^{2p}-|T^*|^{2p}\right)^{\frac{1}{2}}x\right\|^2 = \left\langle\left(|T|^{2p}-|T^*|^{2p}\right)x, x\right\rangle = \left\langle(|T|^{2p}-U|T|^{2p}U^*)x, x\right\rangle$$
$$= \left\langle(|T|^{2p}-Ue^{-i\theta}|T|^{2p})x, x\right\rangle = -e^{-i\theta}\left\langle(|T|^{2p}x, \big(U-e^{i\theta}I)\big)^*x\right\rangle = 0,$$

it follows that $\big(U-e^{i\theta}I\big)|T|^{2p}x = -e^{i\theta}\big(|T|^{2p}-|T^*|^{2p}\big)x=0$. Therefore $\operatorname{Ker}(U-e^{i\theta}I)$ is invariant under $|T|^{2p}$ and hence invariant under $|T|$.

COROLLARY 3. *If T is p-hyponormal, then $(T-\lambda I)x=0$ implies that $(T-\lambda I)^*x=0$.*

Proof. Since $\operatorname{Ker}T\subseteq\operatorname{Ker}T^*$ for every p-hyponormal operator T, the assertion holds for $\lambda=0$.

By Propositions 5 and 2, if T is a p-hyponormal operator and $\tilde{T}=|T|^{1/2}U|T|^{1/2}$, then $\tilde{T}$ is always p-hyponormal with $p=\frac{1}{2}$ and $|\tilde{T}^*|\leq|T|\leq|\tilde{T}|$. Hence if $(T-\lambda I)x=0$, for some $\lambda=re^{i\theta}\neq 0$, then $\big(\tilde{T}-\lambda I\big)|T|^{\frac{1}{2}}x=0$ and $|\tilde{T}||T|^{\frac{1}{2}}x=|\tilde{T}^*||T|^{\frac{1}{2}}x=r|T|^{\frac{1}{2}}x$, by Lemmas 5 and 4. From the inequality $|\tilde{T}^*|\leq|T|\leq|\tilde{T}|$ it is easy to show that $|T||T|^{\frac{1}{2}}x=r|T|^{\frac{1}{2}}x$. Hence we have $(|T|-rI)x\in\operatorname{Ker}|T|=\operatorname{Ker}U$ and $Ux=\frac{1}{r}U|T|x=\frac{1}{r}Tx=\frac{1}{r}\lambda x=e^{i\theta}x$. Also $x=U^*Ux\in[|T|\mathcal{H}]^{\sim}$. Since $(|T|-rI)x\in\operatorname{Ker}|T|\cap[|T|\mathcal{H}]^{\sim}=\{0\}$, $|T|x=rx$ and $T^*x=|T|U^*x=e^{-i\theta}|T|x=re^{-i\theta}x=\bar{\lambda}x$.

REMARK. In the proof of Corollary 3 (the case $\lambda\neq 0$), we only used the fact that T was w-hyponormal (i.e., T satisfies the condition $|\tilde{T}^*|\leq|T|\leq|\tilde{T}|$). In **[1]**, Aluthge-Wang proved that every w-hyponormal operator is paranormal and it is easy to show that every w-hyponormal operator T with $\operatorname{Ker}T\subseteq\operatorname{Ker}T^*$ satisfies the condition (α'''). Hence Weyl's theorem holds also for a w-hyponormal operator T which satisfies the property $\operatorname{Ker}T\subseteq\operatorname{Ker}T^*$, by Berberian's result (Theorem A).

THEOREM 2. *Weyl's theorem holds for p-hyponormal or M-hyponormal operators.*

Proof. If T is p-hyponormal or M-hyponormal, then every eigenspace of T is a reducing subspace of T, by Corollary 3 or Propositions 7 and 8, respectively. Also T

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satisfies the condition α''', by Theorem 1, and therefore Berberian's result (Theorem A) shows that Weyl's theorem holds for T.

THEOREM 3. *For a* w-*hyponormal operator* T, $\sigma(T)\backslash w(T) \subseteq \pi_{00}(T)$. *Moreover, Weyl's theorem holds for* T *if* $\mathrm{Ker}T|_{[T\mathcal{H}]^{\sim}} = \{0\}$.

Proof. Firstly, we shall show that $\sigma(T)\backslash w(T) \subseteq \pi_{00}(T)$, for every w-hyponormal operator T. It follows from the Remark after Corollary 3 that we have $\sigma(T)\backslash\{w(T)\bigcup\{0\}\} = \pi_{00}(T)\backslash\{0\}$. Also it suffices to show that if $0 \in \sigma(T)\backslash w(T)$, then $0 \in \pi_{00}(T)$. Assume that $0 \notin \pi_{00}(T)$. Since $0 < \dim\mathrm{Ker}T < \infty$ and $T\mathcal{H}$ is closed because $0 \in \sigma(T)\backslash w(T)$, our assumption implies that 0 is a cluster point of $\sigma(T)$. Since $T \in \mathcal{F}_0$, there is $s > 0$ with $\{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T) \subseteq \sigma(T)\backslash\{w(T)\bigcup\{0\}\}$ $\subseteq \pi_{00}(T)$ and therefore $\{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T)$ is a countable infinite set whose only cluster point is 0. Put $\{\lambda_n : n \in \mathbb{N}\} = \{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T)$. Then each λ_n is an eigenvalue of T, satisfying $(T - \lambda_n I)x = 0$ implies $(T - \lambda I)^*x = 0$, with finite multiplicity and $\lambda_n \to 0$ as $n \to \infty$. Let $\mathcal{M} = \oplus_n \mathrm{Ker}(T - \lambda_n I)$ and let E_n be the orthogonal projection onto $\mathrm{Ker}(T - \lambda_n I)$. Then $\mathcal{M}$ is an infinite dimensional reducing subspace of T and the restriction $T|_{\mathcal{M}} = \oplus_n \lambda_n E_n$ is a compact normal operator with $\mathrm{Ker}T|_{\mathcal{M}} = \{0\}$. Hence $T\mathcal{M}$ is not closed and this contradicts the fact that $T\mathcal{H}$ is closed. Thus we have $0 \in \pi_{00}(T)$ and this completes the proof of the first part of this theorem.

Next, we shall show that Weyl's theorem holds for w-hyponormal operators which satisfy the condition $\mathrm{Ker}T|_{[T\mathcal{H}]^{\sim}} = \{0\}$.

Since $\sigma(T)\backslash w(T) \subseteq \pi_{00}(T) \subseteq \{\sigma(T)\backslash w(T)\}\bigcup\{0\}$, it suffices to show that if $0 \in \pi_{00}(T)$, then $0 \in \sigma(T)\backslash w(T)$.

Let

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{H} = [T\mathcal{H}]^{\sim} \oplus \mathrm{Ker}T^*,$$

be w-hyponormal with $\mathrm{Ker}A = \{0\}$. If $0 \in \pi_{00}(T)$, then $0 \notin \sigma(A)$ or 0 is an isolated point of $\sigma(A)$ because $\sigma(A) \subseteq \sigma(T) \subseteq \sigma(A)\bigcup\{0\}$. We see that A is isoloid because A is paranormal. Hence if 0 is an isolated point of $\sigma(A)$, then $0 \in \sigma_p(A)$ and this contradicts $\mathrm{Ker}A = \{0\}$. Also we have $0 \notin \sigma(A)$. It is easy to see that $\mathrm{Ker}T = \{-A^{-1}Su \oplus u : 0 \oplus u \in \mathrm{Ker}T^*\}$ and this implies $\dim\mathrm{Ker}T^* = \dim\mathrm{Ker}T < \infty$. Since the closedness of $T\mathcal{H}$ follows from the invertibility of A, we have $0 \in \sigma(T)\backslash w(T)$, and this completes the proof of the second part of this theorem.

THEOREM 4. *If* T *is* p-*hyponormal or* M-*hyponormal and if* $w(T) = \{0\}$, *then* T *is compact and normal.*

Proof. Since Weyl's theorem holds for T, by Theorem 2, and $w(T) = \{0\}$, by the assumption, every non-zero point of $\sigma(T)$ is an isolated point of $\sigma(T)$ with finite dimensional eigenspace which reduces T, by Corollary 3 or Propositions 7 and 8, respectively. Hence $\sigma(T)\backslash w(T)$ is a finite set or a countably infinite set whose only cluster point is 0. Let $\sigma(T)\backslash w(T) = \{\lambda_n\}$ with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots > 0$ and let E_n be the orthogonal projection onto $\mathrm{Ker}(T - \lambda_n I)$. Then $TE_n = E_n T = \lambda_n E_n$ and $E_n E_m = 0$ if $n \neq m$. Put $E = \oplus_n E_n$. Then $T = \oplus_n \lambda_n E_n \oplus T|_{(I-E)\mathcal{H}}$ with the property $\sigma(T|_{(I-E)\mathcal{H}}) = \{0\}$. Since $T|_{(I-E)\mathcal{H}}$ is also p-hyponormal or M-hyponormal by Lemma 3 or by

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Proposition 6, respectively, $T|_{(I-E)\mathcal{H}} = 0$, by Corollary 2 or by Corollary 1 respectively. Hence $T = \oplus_n \lambda_n E_n$ is normal. The compactness of T follows from the finiteness or the countability of $\{\lambda_n\}_n$ satisfying $|\lambda_n| \downarrow 0$. Also each E_n is a finite rank projection.

COROLLARY 4. *If T is w-hyponormal and if $w(T) = \{0\}$, then T is compact and normal.*

Proof. For every w-hyponormal operator T, $\sigma(T)\backslash w(T) \subseteq \pi_{00}(T)$, by Theorem 3. The proof is similar to that of Theorem 4.

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