The Nine-point Circle.

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The nine-point circle of a triangle touches the inscribed circle.

Figure 2.

I. Let ABC be a triangle, having $\angle C$ greater than $\angle B$, D, E, F the middle points of the sides, and AX perpendicular to BC.

Then the upper segment of the nine-point circle cut off by DX contains an angle $C - B$; and conversely.

Figure 3.

II. If AP bisect the angle A and meet the base BC in P, and AC′ be taken along AB equal to AC; then PC′ touches the inscribed circle. Also $\angle BPC' = \angle C - \angle B$.

For the triangles APC, APC′ are congruent; hence the perpendiculars IM, IM′ on PC, PC′ respectively are equal.

Figure 4.

III. $$DM^2 = DP \cdot DX$$

For $$\begin{aligned} HI^2 &= HC^2 \\ &= HD \cdot HK \\ &= HP \cdot HA; \end{aligned}$$

and the projections of HI, HP, HA on BC are DM, DP, DX.

Figure 5.

IV. Let O be a fixed point on the tangent at A to a fixed circle S, and points P, Q be taken (the one on OA and the other on OA produced) such that $OA^2 = OP \cdot OQ$, then the segment of a circle Σ, described through O, Q and containing an angle equal to the external angle between the tangents to S from P, touches the circle S.

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For if PR, the second tangent to S from P, be drawn, and OR produced to meet S in T, since

$$OR \cdot OT = OA^2 = OP \cdot OQ,$$

therefore $\angle OTQ = \angle OPR$;

therefore the point T lies on Σ.

Again, drawing the tangent TU to S at T to meet PR produced in U;

$$\angle UTR = \angle URT = \angle ORP = \angle OQT;$$

therefore TU touches Σ at T.

Thus the circles S, Σ touch each other in T.

V. The application of IV. is fairly obvious. Since in Figure 4,

$$DM^2 = DP \cdot DX \text{ (III.)},$$

the segmental circle upon DX, containing an angle

$$BPC' = C - B \text{ (II.)},$$

touches the inscribed circle (V.). But (I.) the former circle is none other than the nine-point circle.