THE DEDEKIND PROPERTY FOR SEMIRINGS

BILL J. DULIN and JAMES R. MOSHER

(Received 6 November 1970)

Communicated by G. B. Preston

1. Introduction

In this article the concept of Dedekind semidomain is defined, and it is shown that certain structures of this kind are Noetherian, have integral closure, and have the property that their prime k-ideals are maximal. The second section provides the appropriate transportation theorems, while the third gives the main result concerning Dedekind semidomains. Examples are given throughout the paper to show that the hypothesies of certain theorems in the paper cannot be greatly weakened.

2. Transportation theorems

An additive identity of a semiring is the *zero* of the semiring if $x0 = 0x = 0$ for all x in the semiring. A *left semi-ideal* of a semiring S is a non-empty subset A of S such that, for $x, y \in A$ and $s \in S$, $x + y \in A$ and $sx \in A$. If further $x + s \in A$ or $s + x \in A$ implies $s \in A$, then A is a *left k-ideal*. Dropping "left" from these terms allows absorption from the right also. A semiring S is *left semisubtractive* if for each $x, y \in S$ there is $z \in S$ with $z + x = y$ or $x = z + y$.

A *hemiring* is a semiring with commutative addition and zero. A *commutative semiring* (*hemiring*) is a semiring (hemiring) with commutative multiplication. The *zeroid* of a semiring S is $\{x \in S \mid x + y = y \text{ for some } y \in S\}$.

A *halfring* is a hemiring with additive cancellation. It is known that a semiring can be imbedded in a ring called its *ring of differences* if and only if the semiring is a halfring. If the halfring is R, then $\bar{R}$ denotes its ring of differences. A *semifield* is a hemiring whose nonzero elements form an abelian group under multiplication. A hemiring is *Noetherian* if it contains 1, has commutative multiplication, and satisfies the ascending chain condition on k-ideals.

PROPOSITION 1. *Let S and S' be two semirings, S' having an additive identity 0. If S is left semisubtractive, if $\{0\}$ is the zeroid of S', and if f is a homomorphism of S onto S', then there is a one-to-one correspondence between the left k-ideals of S containing* $\ker(f)$ *and the left k-ideals of S'.*

PROOF. The correspondence we desire is $A \to f(A)$. If $f(x)+f(w)=f(y)$ where $x, y \in A$ and $w \in S$, then $z+w=y$ or $w=z+y$ for some $z \in S$. In the second case

$$f(x+z)+f(y)=f(y),$$

so that $x+z \in \ker(f) \subset A$. Hence $z \in A$ and consequently $w \in A$. In the first case $v+x=z$ or $x=v+z$ for some $v \in S$. If $v+x=z$, then

$$f(x+w)=f(v)+f(x+w)$$

and $v \in \ker(f) \subset A$, so that $z \in A$ and hence $w \in A$. If $x=v+z$, again $w \in A$. Similarly

$$f(w)+f(x)=f(y)$$

implies $f(w) \in f(A)$. Therefore $f(A)$ is a left k-ideal of S'.

Consider two left k-ideals A and B of S that contain $\ker(f)$ where $f(A)=f(B)$. For $b \in B$, $f(a)=f(b)$ for some $a \in A$ and hence $x+a=b$ or $a=x+b$ for some $x \in S$. If $x+a=b$, then

$$f(b)=f(x+a)=f(x)+f(a)=f(x)+f(b)$$

and $f(x)=0$. Hence $x \in \ker(f) \subset A$ and $b=x+a \in A$. If $a=x+b$, then again $x \in \ker(f) \subset A$, so that $x, x+b=a \in A$ imply $b \in A$. Hence $B \subset A$; similarly $A \subset B$ meaning $A=B$ and the mapping is one-to-one.

If C is a left k-ideal of S', then $f^{-1}(C)$ is clearly a left k-ideal of S that contains $\ker(f)$. Since

$$f^{-1}(C) \to f(f^{-1}(C))=C,$$

the mapping is onto, completing the proof.

Examples will be given now to show the hypothesis of Proposition 1 cannot be relaxed.

EXAMPLE 1. Let

$$A=\{(x,0,0) \mid x \in Z_0^+\},$$

$$B=\{(0,x,0) \mid x \in Z_0^+\},$$

and

$$C=\{(0,0,x) \mid x \in Z_0^+\}$$

where Z_0^+ is the set of non-negative integers. Denote $(x,0,0)$ by $x^{(1)}$, $(0,x,0)$ by $x^{(2)}$ and $(0,0,x)$ by $x^{(3)}$.

Define $\oplus$ on $D=A \cup B \cup C$ as follows:

$$x^{(i)} \oplus y^{(i)}=(x+y)^{(i)}, \quad i=1,2,3,$$

$$x^{(i)} \oplus y^{(j)}=(x+y)^{(3)}, \quad i \neq j.$$

https://doi.org/10.1017/S1446788700009678 Published online by Cambridge University Press

Define $\odot$ on D to be the zero multiplication. It is easy to show that under these operations D is a commutative hemiring that is not semisubtractive. If E is the non-negative integers under usual addition and zero multiplication, and if $f: D \to E$ is defined by $f(x^{(i)}) = x$, $i = 1,2,3$, then A and B are distinct k-ideals of D that contain $\ker(f)$ with $f(A) = E = f(B)$. Note that D and E have $\{0\}$ as their zeroids.

EXAMPLE 2. Let $S = Z_0^+$ and $S' = \{1\} \cup \{x \in Z_0^+ \mid x \text{ is even}\}$. In S and S' define $\oplus$ by $a \oplus b = \max\{a,b\}$ and $\odot$ by $a \odot b = \min\{a,b\}$. Under these operations both structures are commutative hemirings, S is semisubtractive, and S' is the zeroid of S'. Define $f: S \to S'$ as follows: $f(x) = x + 1$, if $x > 1$ and odd, $f(x) = x$ otherwise. Clearly f is a homomorphism of S onto S' with $\ker(f) = \{0\}$. As well $A = \{x \in S \mid x \leqq 3\}$ and $B = \{x \in S \mid x \leqq 4\}$ are k-ideals of S, $B \neq A$, and $f(A) = \{0,1,2,4\} = f(B)$.

The following is an easy extension to Proposition 1.

PROPOSITION 2. *If R is a semisubtractive halfring and $\bar{R}$ is its ring of differences, then there is a one-to-one correspondence between the ideals of $\bar{R}$ and the k-ideals of R. If A is a k-ideal of R, then* $\{x \in \bar{R} \mid x \in A$ or $x = -y$ for *some* $y \in A\}$ *is the ideal of $\bar{R}$ corresponding to A. Conversely, if B is an ideal of $\bar{R}$, then $\{x \in R \mid x \in \bar{A}\}$ is the k-ideal of R corresponding to B.*

PROPOSITION 3. *If S is a semisubtractive hemiring, and if A is a k-ideal of S, then A contains the zeroid Z of S if and only if S/A has additive cancellation.* (Bourne in [1] denoted S/A by $S - A$.)

We give an example to show the necessity of semisubtraction in Proposition 3.

EXAMPLE 3. Let $A = \{(x,0) \mid x \in Z_0^+\}$ and $B = \{(0,x) \mid x \in Z_0^+\}$. Denote $(x,0)$ by $x^{(1)}$ and $(0,x)$ by $x^{(2)}$. On $S = A \cup B$ define $\oplus$ as follows:

$$x^{(i)} \oplus y^{(i)} = (x+y)^{(i)}, \quad i = 1,2,$$

$$x^{(i)} \oplus y^{(j)} = (x+y)^{(2)}, \quad i \neq j,$$

and define $\odot$ on S to be zero multiplication. Under these operations S is a commutative hemiring that is not semisubtractive. Also A is a k-ideal of S, the zeroid of S is zero, and $S/A = \{A,B\}$, where A and B are now considered as the classes in S/A. Since $B + B = B = A + B$ and $A \neq B$, S/A does not have additive cancellation.

COROLLARY. *If S is a semisubtractive hemiring with A as a k-ideal of S containing Z, then $S/A = S[\,/\,]A$.* (See LaTorre [2].)

Proposition 3 states that the k-ideals that contain Z and the h-ideals of a semisubtractive semiring with commutative addition are the same. It is to be noted that there exist k-ideals properly contained in Z and semi-ideals that are not k-ideals properly contained in Z.

EXAMPLE 4. Considering the hemiring S as described in Example 2, the zeroid Z of S is S. If $n \in S$,

$$A_n = \{x \in S \mid x \leqq n\}$$

is a k-ideal which is obviously in the zeroid and which is not an h-ideal.

EXAMPLE 5. Let $S = \{(x,y) \in Z \times Z \mid x,\ y \geqq 0 \text{ or } y \leqq x \leqq 0\}$ and define $\oplus$ and $\odot$ on S as follows:

$$(0,0) \oplus (a,b) = (a,b) \oplus (0,0) = (a,b) \text{ for all } (a,b) \in S,$$

$$(x,y) \oplus (a,b) = (\max\{|x|,|a|\},\ \max\{|y|,|b|\}) \text{ if } x, y \geqq 0 \text{ or } a,\ b \geqq 0 \text{ where } (x,y) \text{ and } (a,b) \text{ are not } (0,0),$$

$$(x,y) \oplus (a,b) = (-\max\{|x|,|a|\},\ -\max\{|y|,|b|\}) \text{ otherwise where } (x,y) \text{ and } (a,b) \text{ are not } (0,0),$$

$$(x,y) \odot (a,b) = (\min\{|x|,|a\cdot\},\ \min\{|y|,|b|\}) \text{ if } x,y \geqq 0 \text{ or } a,\ b \geqq 0,$$

$$(x,y) \odot (a,b) = (-\min\{|x|,|a|\},\ -\min\{|y|,|b|\}) \text{ otherwise.}$$

With these operations S is a hemiring; if $A = \{(0,0),(0,1)\}$, then A is a semi-ideal of S that is in the zeroid of S which is S. Since $(0,-1) \oplus (0,1) = (0,1) \in A$ but $(0,-1) \notin A$, A is not a k-ideal.

PROPOSITION 4. *If P is a prime k-ideal in a semisubtractive hemiring with commutative multiplication and if $Z \subset P$, then S/P is multiplicatively cancellable.*

EXAMPLE 6. Letting A, B, and C be as in Example 1, on $D = A \cup B \cup C$ define $\oplus$ and $\odot$ as follows:

$$x^{(i)} \oplus y^{(i)} = (x+y)^{(i)}, \quad i = 1,2,3,$$

$$x^{(i)} \oplus y^{(j)} = y^{(j)} \oplus x^{(i)} = (x+y)^{(j)}, \quad i<j,$$

$$x^{(i)} \odot y^{(i)} = (xy)^{(i)}, \quad i = 1,2,3,$$

$$x^{(i)} \odot y^{(j)} = y^{(j)} \odot x^{(i)} = (xy)^{(i)}, \quad i<j.$$

Under these operations D is a hemiring with commutative multiplication, with zero for its zeroid, and with A as a prime k-ideal. Since $S/A = \{A,B,C\}$ and $BC = B^2$ but $B \neq C$, S/A does not have multiplicative cancellation, pointing up the necessity of semisubtraction in Proposition 4.

PROPOSITION 5. *If S is a semiring with commutative addition and multiplication and with additive identity, if P is a k-ideal of S, and if S/P is multiplicatively cancellable, then P is prime.*

PROPOSITION 6. *If M is a maximal semi-ideal of a semisubtractive semiring S with commutative addition and multiplication, then* $S[\,/\,]M$ *is zero or* $S[/]M$ *is a semifield.* (See Mosher [4], p. 61.)

PROPOSITION 7. *If M is a maximal semi-ideal of a hemiring with commutative multiplication and multiplicative identity* 1, *and if M is also a k-idea then S/M is a semifield.*

Let S be a commutative halfring and $\bar{S}$ its ring of differences. If $\bar{A}$ is an ideal in $\bar{S}$, $\bar{A}^c$ denotes the set of $x \in S$ with $x \in \bar{A}$ which is clearly a k-ideal of S and is called the *contraction* of $\bar{A}$. If A is a *k-ideal of* S, A^e denotes the ideal of $\bar{S}$ generated by A and is called the *extension* of A.

The following facts hold:

(1) $\bar{S}^c = S$, $S^e = \bar{S}$, $(0)^e = (0)$;

(2) if $\bar{A} \subset \bar{B}$, then $\bar{A}^c \subset B^c$; if $A \subset B$, then $A^e \subset B^e$;

(3) $\bar{A}^{ce} \subset \bar{A}$; $A^{ec} \supset A$;

(4) $\bar{A}^{cec} = \bar{A}^c$; $A^{ece} = A^e$;

(5) $(\bar{A} \cap \bar{B})^c = \bar{A}^c \cap \bar{B}^c$; $(A \cap B)^e \subset A^e \cap B^e$;

(6) $(AB)^e = A^e B^e$ provided AB is a k-ideal; $(\bar{A}\bar{B})^c \supset \bar{A}^c \bar{B}^c$.

If S is also semisubtractive, then we can characterize the ideals of $\bar{S}$ that have the form A^e, A a k-ideal of S. It is that

$$A^e = A \cup \{-x \in \bar{S} \mid x \in A\}.$$

In this setting we have

$$A^e \cap B^e = (A \cap B)^e, \ \bar{A}^{ce} = \bar{A}, \ A^{ec} = A, \text{ and } (\bar{A}\bar{B})^c = \bar{A}^c \bar{B}^c.$$

Also, if A and B are k-ideals of S, then $AB = A^{ec}B^{ec} = (A^e B^e)^c$ is a k-ideal.

In working with sums and products of k-ideals in semirings in general we encounter the following problem: the sum or product might not be a k-ideal.

EXAMPLE 7. The non-negative integers Z_0^+ under usual operations is a semisubtractive halfring. The multiples of 2 and the multiples of 3 are k-ideals but their sum is $Z_0^+ - \{1\}$ which is not a k-ideal.

EXAMPLE 8. Define $\oplus$ on Z_0^+ as follows: $a \oplus b = \max\{a,b\}$ if $a \leqq 6$ or $b \leqq 6$, $a \oplus b = a + b$ otherwise. Define $\odot$ on Z_0^+ as follows; $a \odot b = \min\{a,b\}$ if $a \leqq 6$ or $b \leqq 6$, $a \odot b = ab$ otherwise. With these operations Z_0^+ is a hemiring which is not semisubtractive. Also

$$A = \{x \in Z_0^+ \mid 0 \leqq x \leqq 6\} \cup \{8, 10, 12, \cdots\}$$

and

$$B = \{x \in Z_0^+ \mid 0 \leqq x \leqq 6\} \cup \{9, 12, 15, \cdots\}$$

are k-ideals but the semi-ideal AB contains 96 and $120 = 96 \oplus 24$ but not 24 which implies AB is not a k-ideal.

In the light of these examples it is difficult to work with extensions and contractions of sums and products without strong conditions being placed on the semirings.

3. Dedekind semidomains

A *semidomain* is a commutative halfring with multiplicative cancellation and with a multiplicative identity. A *Dedekind semidomain* is a semidomain in which every k-ideal is a product of prime k-ideals.

The non-negative integers under the usual operations is a Dedekind semidomain. In fact, any principal ideal semidomain (a semidomain in which every k-ideal is principal) is a Dedekind semidomain.

If R is a semisubtractive Dedekind semidomain, and if M is a multiplicative subsemigroup of R with $0 \notin M$, one can show R_M is a Dedekind semidomain. (For the construction of R_M, see [3].)

LEMMA. *If R is a semisubtractive commutative halfring, and if P is a k-ideal of R such that $P^{ec} = P$, where extension and contraction are relative to R_M, M a multiplicative subsemigroup of R with $0 \notin M$, then P^e is a k-ideal of R_M.*

For the proof see [4], Lemma 5.13, or the reader can easily supply the proof.

Continuing with the discussion just before the lemma, if A is a k-ideal of R_M, then $A^{ce} = A$ (see [3], Theorem 13). Since A^c is a k-ideal of R, $A^c = \pi P_i$ where P_i is a prime k-ideal of R. Thus

$$A = A^{ce} = (\pi P_i)^e = \pi(P_i^e),$$

where $P_i^e = R_M$ or is a prime k-ideal of R_M for each i (this second possibility follows from Theorem 16 of [3] and the lemma above). Thus R_M is a Dedekind semidomain.

Let R be a semidomain, Q its quotient semifield. Let $\bar{R}$ be the ring of differences of R. Letting R be semisubtractive, $\bar{R}$ becomes an integral domain. The converse is false, however, because with Q_0^+ denoting the non-negative rationals $Q_0^+[x]$ is a semidomain without semisubtraction whereas its ring of differences is $\bar{Q}_0^+[x]$, $\bar{Q}_0^+$ the rationals, which is an integral domain. If Q' is the field of quotients of $\bar{R}$, does $Q' = \bar{Q}$, the ring of differences of Q? If $z \in \bar{Q}$, then $z \in Q$ or $z = -y$, $y \in Q$, due to the fact that R is semisubtractive. In the first case $z = a/b$, where a, $b \in R \subset \bar{R}$. Thus $z \in Q'$ and $\bar{Q} \subset Q'$. In the second case $z = -(a/b)$, where a, $b \in R$. Thus

$$z = -(a/b) = (-a)/b \in Q'$$

since $-a$. $b \in \bar{R}$. Hence $\bar{Q} \subset Q'$. If $w \in Q'$, then $w = a/b$, where a, $b \in \bar{R}$. If a, $b \in R$, then $w = a/b \in \bar{Q}$. Suppose $-a, b \in R$; then

$$w = -((-a)/b) \in \bar{Q}$$

since $(-a)/b \in Q$. For a, $-b \in R$,

$$w = -(a/(-b)) \in \bar{Q}$$

since $a/(-b) \in Q$. Lastly for $-a$, $-b \in R$, $w = (-a)/(-b) \in \bar{Q}$. Thus $Q' = \bar{Q}$. We say $u \in Q$ is *integral* over R if there exist $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ in R such that

$$u^n + a_1 u^{n-1} + \cdots + a_n = b_1 u^{n-1} + \cdots + b_n.$$

We say R is *integrally closed* if R is the set of elements of Q that are integral over R.

PROPOSITION 8. *The semisubtractive semidomain R is integrally closed if and only if $\bar{R}$ is integrally closed.*

PROOF. Suppose $\bar{R}$ is integrally closed and let $u \in Q$ be intgeral over R. For some $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ in R,

$$u^n + a_1 u^{n-1} + \cdots + a_n = b_1 u^{n-1} + \cdots + b_n.$$

Hence $u^n + (\overline{a_1 - b_1}) u^{n-1} + \cdots + (\overline{a_n - b_n}) = 0$ meaning $u \in \bar{R}$ since $u \in Q \subset \bar{Q}$. Therefore $u \in \bar{R} \cap Q = R$ and R is integrally closed.

Suppose R is integrally closed and let $u \in Q'$ be integral over $\bar{R}$. Thus

$$u^n + (\overline{a_1 - b_1}) u^{n-1} + \cdots + (\overline{a_n - b_n}) = 0$$

where $a_i = 0$ or $b_i = 0$ for each i since R is semisubtractive. Hence

$$u^n + a_1 u^{n-1} + \cdots + a_n = b_1 u^{n-1} + \cdots + b_n.$$

Since Q is semisubtractive and $Q' = \bar{Q}$, $u = a/b \in Q$ or $u = -(a/b)$, $a/b \in Q$. If $u = a/b \in Q$, then u is integral over R and hence $u \in R \subset \bar{R}$. Suppose $u = -(a/b)$, $a/b \in Q$. If n is even, then

$$(a/b)^n + b_1(a/b)^{n-1} + a_2(a/b)^{n-2} + b_3(a/b)^{n-3} + \cdots + b_{n-1}(a/b) + a_n$$
$$= a_1(a/b)^{n-1} + b_2(a/b)^{n-2} + a_3(a/b)^{n-3} + \cdots + a_{n-1}(a/b) + b_n$$

and hence $a/b \in R$. If n is odd, then

$$a_1(a/b)^{n-1} + b_2(a/b)^{n-2} + a_3(a/b)^{n-3} + \cdots + b_{n-1}(a/b) + a_n$$
$$= (a/b)^n + b_1(a/b)^{n-1} + a_2(a/b)^{n-2} + b_3(a/b)^{n-3} + \cdots + a_{n-1}(a/b) + b_n$$

and hence $a/b \in R$. In either case $u = -(a/b) \in \bar{R}$. Therefore $\bar{R}$ is integrally closed and the proposition is proved.

Let R be Dedekind semisubtractive semidomain and $\bar{R}$ its ring of differences. Is $\bar{R}$ a Dedekind ring? We have already observed that $\bar{R}$ is an integral domain. Let $\bar{A}$ be an ideal in $\bar{R}$. Now $A = \bar{A}^c$ is a k-ideal in R. By hypothesis $A = \pi P_i$, a product of prime k-ideals of R. Thus

$$\bar{A} = \bar{A}^{ce} = A^e = (\pi P_i)^e = \pi(P_i^e).$$

Is P_i^e prime in $\bar{R}$? The answer is yes. Let $\bar{x}, \bar{y} \in \bar{R}$ such that $\overline{xy} \in P_i^e$. If $\bar{x} = x \in R$ and $\bar{y} = y \in R$, then $\overline{xy} = xy \in P_i$, so that

$$\bar{x} = x \in P_i^e \text{ or } \bar{y} = y \in P_i^e.$$

If $\bar{x} = x \in R$ and $\bar{y} = -y$, $y \in R$, then

$$\overline{xy} = x(-y) = -xy \in P_i^e,$$

so that $xy \in P_i$. Thus $x \in P_i$ or $y \in P_i$; i.e., $\bar{x} = x \in P^e$ or $\bar{y} = -y \in P_i^e$. If $\bar{x} = -x$ and $\bar{y} = -y$ where $x, y \in R$, then

$$\overline{xy} = (-x)(-y) = (\overline{0-x})(\overline{0-y}) = (\overline{0+xy) - (0+0}) = xy \in P_i^e,$$

so that $xy \in P_i$ and hence $x \in P_i$ or $y \in P_i$; i.e., $\bar{x} = -x \in P_i^e$ or $-y \in P_i^e$. Thus P_i^e is a prime ideal of $\bar{R}$. Thus $\bar{R}$ is a Dedekind ring.

By [6, Theorem 13, page 275], $\bar{R}$ is Noetherian, integrally closed, and every proper prime ideal of $\bar{R}$ is maximal. By Proposition 2, there is a one-to-one correspondence between the prime k-ideals of R and the prime ideals of $\bar{R}$. Thus each prime k-ideal of R is maximal. By [5] R is Noetherian since $\bar{R}$ is. By Proposition 8, R is integrally closed. Therefore, if R is Dedekind, it is Noetherian, integrally closed, and each of its prime k-ideals is maximal.

Suppose now that R is Noetherian, integrally closed, and that each of its prime k-ideals is maximal. By [5] $\bar{R}$ is Noetherian since R is semisubtractive. Let $\bar{A}$ be a prime ideal in $\bar{R}$. By Proposition 2, $\bar{A}$ corresponds to a unique prime k-ideal A of R. Since A is then maximal, so is $\bar{A}$. Thus in $\bar{R}$ prime ideals are maximal. By Proposition 8, $\bar{R}$ is also integrally closed. Therefore by [6] again, $\bar{R}$ is Dedefind. If A is a k-ideal of R, then A corresponds uniquely to an ideal A' of $\bar{R}$ by Proposition 2. Since $A' = \pi P_i'$, each P_i' a prime ideal of $\bar{R}$, and since each P_i' corresponds to a unique prime k-ideal P_i of R by Proposition 2,

$$A = A^{ec} = (A')^c = (\pi P_i')^c = \pi((P_i')^c) = \pi P_i.$$

Thus R is Dedekind, and the following theorem has been proved.

THEOREM 1. *If R is a semisubtractive semidomain, then R is Dedekind if and only if R is Noetherian, integrally closed, and each of its prime k-ideals is maximal.*

EXAMPLE 9. Let A be the set of subsets of finite set B with more than 2 elements. Let addition on A be union of sets and multiplication intersection of sets; A is a commutative hemiring with $\emptyset$ as its zero and B as its multiplicative identity. Let K be a semi-ideal of A, and let D be the union of the elements of K. We claim K is the set F of subsets of D. Clearly $K \subset F$; if $M \in F$, then for each $x \in M$ we have

$$\{x\} = H \cap \{x\} \in K$$

for any $H \in K$ and since B is finite

$$M = \cup \{\{x\} \mid x \in M\} \in K.$$

Hence $K = F$, so that every semi-ideal of A is the set of subsets of some subset of B. Easily then every semi-ideal is a k-ideal. A prime k-ideal of A is any set of all subsets of a set of the form $B - \{x\}$, $x \in B$. Clearly any k-ideal of A is the product of prime k-ideals. Therefore a hemiring can have the Dedekind property though it is not a semidomain.

EXAMPLE 10. Let Q_0^+ be the non-negative rationals under the usual operations and Q the rationals under the usual operations. Letting $Q_0^+[x]$ and $Q[x]$ be the set of polynomials over Q_0^+ and Q, respectively, it is true that $Q_0^+[x]$ is a Dedekind semidomain that is not semisubtractive, yet $Q[x] = \overline{Q_0^+[x]}$ is a Dedekind ring.

EXAMPLE 11. (Stone, [5].) Let S be the set of all positive rational sequences and the constant sequence $\bar{0} = \{0, 0, \cdots\}$. Under componentwise addition and multiplication, S is a semifield without semisubtraction. Its ring of differences is a countable direct product of copies of the rationals which is not an integral domain. Thus this example shows the necessity of semisubtraction in Theorem 1, since S is Dedekind and $\bar{S}$ is not Dedekind.

References

[1] S. Bourne, 'The Jacobson radical of a semiring', *Proc. Nat. Acad. Sci. U.S.A.* 37 (1951), 163–170.

[2] D. LaTorre, 'On h-ideals and k-ideals in hemirings', *Publ. Math. Debrecen*, 12 (1965), 219–226.

[3] J. Mosher, 'Generalized quotients of hemirings', *Comp. Math.* 22 (1970), 275–281.

[4] J. Mosher, *Generalized semirings of quotients*, Dissertation (Texas Christian University, 1968).

[5] H. E. Stone, *Structure theory of semirings*, Dissertation, (Texas Christian University, 1969).

[6] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. 1 (Princeton, D. Van Nostrand, 1959).

Texas A&M University
College Station, Texas 77843
U.S.A.