THE ZEROS OF LINEAR COMBINATIONS OF TRANSLATES OF POLYNOMIALS

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Abstract

We investigate the location and separation of zeros of certain three-term linear combination of translates of polynomials. In particular, we find an interval of the form $I = [-1, 1 + h]$, $h > 0$ such that for a polynomial f, all of whose zeros are real, and $\lambda \in I$, all zeros of $f(x + 2ic) + 2\lambda f(x) + f(x - 2ic)$ are also real.

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1. Introduction

Given the location of the zeros of a polynomial or entire function, it is important to have corresponding information about the zeros of other functions which may be derived from the given one. For instance Rolle's theorem says that if all the zeros of a polynomial lie on the real axis, then the zeros of the derivative also lie on the axis, and lie one in each interval between successive zeros. We showed in [3] that the separation, measured by

$$\delta(f) := \min_{j}(a_j - a_{j-1}), \quad \text{where} \quad f(x) = \prod_{j=1}^{n}(x - a_j), \ a_1 < a_2 < \cdots < a_n,$$

is increased by differentiation, and more generally in [4] that for real k there is some explicit constant $c_n(k) > 1$, for which

$$\delta(f' - kf) \geq c_n(k)\delta(f).$$

Another operation which has been considered in this context is translation. For instance Pólya showed in [2, Lemma II, page 316], that for $c > 0$ and $\theta \in C$, $|\theta| = 1$, $f_\theta(x) := f(x + ic) + \theta f(x - ic)$ has only real zeros when f does, and we showed in [5] that the separation is also increased:

$$\delta(f_\theta) > \delta(f),$$

again with explicit estimates for the amount of the increase. If we apply this result twice, using $\theta, \bar\theta = e^{\pm i\alpha}$, $\alpha \in \mathbb{R}$, we find that when f has only real zeros then the same is true for $f(x + 2ic) + 2\cos\alpha f(x) + f(x - 2ic)$. Equivalently the map

(1) $f(x) \rightarrow T_\lambda f(x) := f(x + 2ic) + 2\lambda f(x) + f(x - 2ic),$

where $-1 \le \lambda \le 1$, $c > 0$, preserves the reality of the zeros of f and increases their separation. For convenience we shall replace $2c$ by c in (1) from now on.

In this paper we extend the range of these results to values of λ which lie outside the interval $[-1, 1]$. The motivation for doing this comes from the study of Mellin transforms in which given a function

$$F(x) = \int_0^\infty t^{ix}\phi(t)\,dt$$

we are interested in applying multipliers of the form $(t + r^2/t)$ and $(t + 1/r^2 t)$ with real r to the integrand. This gives

$$\int_0^\infty t^{ix}(t + r^2/t)(t + 1/r^2 t)\phi(t)\,dt = F(x + 2i) + (r^2 + r^{-2})F(x) + F(x - 2i)$$

$$= F(x + 2i) + 2\lambda F(x) + F(x - 2i),$$

where necessarily $\lambda > 1$.

More precisely we shall find neighbourhoods of 1 and ∞ such that for these values of λ, all the zeros of T_λ are real. (It is not possible to do this near -1 since two zeros are lost at $-\infty$.) The size of the neighbourhood of 1 turns out to depend only on the degree of the polynomial, and not on the values of c or δ, as Theorem 5 shows. We shall also estimate the separation of the zeros for $\lambda > 1$.

Our results also relate to Pólya's generalisation of the Hermite-Poulain theorem— see [1], particularly Lemma I*, page 228. This lemma states that if ϕ is in the Laguerre-Pólya class LP, and g is a polynomial with real coefficients, then, with D denoting the operation of differentiation, $\phi(D)g$ has at least as many real zeros as does g. In our context we have $T_\lambda f = 2(\lambda + \cos(cD))f$, and so $\phi(z) = \lambda + \cos(cz) \in LP$ if and only if $-1 \le \lambda \le 1$. In contrast, our Theorem 5 states that for suitable f, T_λ also preserves the zeros of f in some neighbourhood to the right of 1. In addition, our Theorem 7 constructs, for any given $\lambda > 1$, a polynomial with real zeros such that $T_\lambda f$ has some non-real zeros; this gives further information about the size of the neighbourhood of 1, and shows that the restriction on f cannot be omitted entirely.

2. Location of zeros

Let $f(x) := \prod_1^n (x - a_j)$, where $a_1 < a_2 < \cdots < a_n \in \mathbb{R}$. For given $c > 0$ and $x \in \mathbb{R}$, let

$$a_j + ic - x := r_j e^{i\theta_j}, \quad r_j > 0, \quad 0 < \theta_j = \cot^{-1}\left(\frac{a_j - x}{c}\right) < \pi.$$

Note that (i) each θ_j is a continuous strictly increasing function of x with $\theta_j(-\infty) = 0$, $\theta_j(a_j) = \pi/2$, $\theta_j(+\infty) = \pi$ and (ii) that $\pi > \theta_1 > \theta_2 > \cdots > \theta_n > 0$. Then

$$T_\lambda f(x) = f(x + ic) + 2\lambda f(x) + f(x - ic)$$

$$= \prod_1^n (x + ic - a_j) + \prod_1^n (x - ic - a_j) + 2\lambda \prod_1^n (x - a_j)$$

$$= \prod_1^n (-r_j e^{-i\theta_j}) + \prod_1^n (-r_j e^{i\theta_j}) + 2\lambda \prod_1^n (-r_j \cos\theta_j)$$

$$= 2 \prod_1^n (-r_j) \left[\cos \sum_1^n \theta_j + \lambda \prod_1^n \cos\theta_j \right].$$

The following result follows at once from this expression:

LEMMA 1. *Real zeros of $T_\lambda f$ occur where $\phi_1 + \lambda\phi_2 = 0$, where*

$$\phi_1(x) := \cos \sum_1^n \theta_j(x), \qquad \phi_2(x) := \prod_1^n \cos\theta_j(x),$$

and since $T_\lambda f$ has degree n (or $n - 2$ in case $\lambda = -1$) all zeros of $T_\lambda f$ will be real if n (or $n - 2$) real zeros can be identified.

Let $b_j, 1 \leq j \leq n$, be the points at which $\sum_1^n \theta_i = (j - 1/2)\pi$, that is, the zeros of ϕ_1. Note that $\phi_1 \to 1$ as $x \to -\infty$ and $\phi_1 \to (-1)^n$ as $x \to +\infty$. Note also that $b_1 < a_1 < a_n < b_n$, since for instance $\theta_1 = \pi/2$ at a_1 and so $\sum_1^n \theta_j = \pi/2$ at some point less than a_1. Let $c_1, c_2, \ldots, c_{n-1}$ be the points at which $\sum_1^n \theta_j = \pi, 2\pi, \ldots, (n-1)\pi$, that is, the successive maxima and minima of ϕ_1. Let $c_0 = -\infty, c_n = +\infty$. Note that ϕ_1 is strictly monotone from each (c_{j-1}, c_j) onto $(-1, 1)$ since each θ_j is an increasing function of x. More detailed information on the relation between b_j, c_j and the roots a_j of f is contained in Lemma 3 below.

For ϕ_2, observe that $\phi_2 \to 1$ as $x \to -\infty$, that $\phi_2 \to (-1)^n$ as $x \to \infty$, that ϕ_2 changes sign at each a_j, and that $|\phi_2| < 1$ for all real x. Let $h_j = \sup\{|\phi_2(x)| : a_j < x < a_{j+1}\}, 1 \leq j \leq n - 1$ with $h_j = |\phi_2(d_j)|$. Also let $h_0 = \phi_2(b_1)$, $h_n = |\phi_2(b_n)|$. Note that $h_0, h_1, \ldots, h_n$ are all in the interval $(0, 1)$. Let $h_* = \min_{0 \leq j \leq n} h_j$, $h^* = \max_{0 \leq j \leq n} h_j$.

LEMMA 2. $T_\lambda f$ has only real zeros if either (i) $-1 \le \lambda \le 1/h^*$ or (ii) $|\lambda| \ge 1/h_*$.

PROOF. (i) For $-1 < \lambda < 1$, observe that $|\lambda \phi_2| < |\phi_2| < 1$ for all x. Hence since the graph of ϕ_1 goes from -1 to 1 on each interval (c_{j-1}, c_j), $1 \le j \le n$, it follows from the the intermediate value theorem that there is at least one root of $\phi_1 + \lambda \phi_2$ in each of these intervals. Since there are n such intervals and at most n zeros, the result follows in this case. If $\lambda \to -1$ the result remains true since the roots vary continuously with λ, though the two extreme zeros tend to $\pm \infty$.

For positive λ, ϕ_1 and $-\lambda \phi_2$ have opposite sign outside $[b_1, b_n]$ (recall the earlier observation that $b_1 < a_1 < a_n < b_n$.) For $x \in [b_1, b_n]$ and $1 \le \lambda < 1/h^*$ we still have $|\lambda \phi_2(x)| < 1$ and hence there is exactly one root in each interval (c_{j-1}, c_j) as before. The result when $\lambda = 1/h^*$ again follows as a limiting case.

(ii) If $|\lambda| > 1/h_*$, then we reverse the roles of ϕ_1, $\lambda \phi_2$ in the above argument. The function $\lambda \phi_2$ now goes from a value less than -1 to a value greater than 1 on each interval (d_{j-1}, d_j) (with $d_0 = b_1$ and $d_n = b_n$), and so must intersect ϕ_1 there. Since there are n such intervals we have located the required number of real roots. The result when $|\lambda| = 1/h_*$ follows once more as a limiting case.   □

Lemma 2 shows the existence of some range of values of $\lambda > 1$ for which the zeros of $T_\lambda f$ are real. (The condition is that $\lambda |\phi_2| \gtrsim 1$ on $[b_1, b_n]$; compare Lemma 4 (i) below.) However as it stands, it does not give useful quantitative information, since estimation of h_j may involve the lengths of any of the intervals (a_j, a_{j+1}). To make further progress we need more detailed information about the location of zeros of $T_\lambda f$ for which the following definition is required:

DEFINITION. For each j, $1 \le j \le n$, let $I_j := [a_j - \mu_j, a_j + v_j]$, where μ_j, v_j are non-negative numbers, which are defined by the requirement that the angles θ_j to the right of a_j sum to $\pi/2$ at $a_j - \mu_j$, and the complementary angles $\pi - \theta_j$ to the left of a_j sum to $\pi/2$ at $a_j + v_j$. More formally, we require that

$$\sum_{k=j}^{n} \theta_k = \frac{\pi}{2}, \text{ at } x = a_j - \mu_j \quad \text{and} \quad \sum_{k=1}^{j}(\pi - \theta_k) = \frac{\pi}{2}, \text{ at } x = a_j + v_j.$$

Since as has been noted, each θ_k is an increasing function of x from R onto $(0, \pi)$, and $\theta_j = \pi/2$ at a_j, each of μ_j, v_j is well defined. (Note that in particular that $a_1 - \mu_1 = b_1$, $a_n + v_n = b_n$ and that $v_1 = \mu_n = 0$.) Let $I := \bigcup_1^n I_j$.

The following result describes the arrangement of these intervals, and their relation to the points b_j where $\phi_1 = 0$.

LEMMA 3. (i) *The end points of the intervals I_j form strictly increasing sequences*:

$$a_j - \mu_j < a_{j+1} - \mu_{j+1} \quad \text{and} \quad a_j + v_j < a_{j+1} + v_{j+1} \quad \text{for } 1 \le j \le n-1.$$

(ii) *For each j, $1 \leq j \leq n$, we have $b_j \in I_j$; more precisely*

$$a_j - \mu_j \leq b_j, \text{ with strict inequality when } j > 1, \text{ and}$$
$$a_j + \nu_j \geq b_j, \text{ with strict inequality when } j < n.$$

Moreover, if for some k, $a_k \leq b_j \leq a_{k+1}$, then $b_j \in I_k \cup I_{k+1}$. (This includes both of the extreme cases $k = 0$, and $k = n$; if $k = 0$ there is no a_k to the left of b_j and the claim is simply that $b_j \in I_1 = [b_1, a_1]$, similarly if $k = n$, then $b_j \in I_n = [a_n, b_n]$.)

(iii) *At any point not in I, ϕ_1 and ϕ_2 have strictly the same sign.*

PROOF. (i) follows since each $\theta_j > 0$ and $\sum_j^n \theta_i = \pi/2$ at $a_j - \mu_j$ while $\sum_{j+1}^n \theta_i = \pi/2$ at $a_{j+1} - \mu_{j+1}$.

For (ii) we see that $\sum_1^n \theta_i = \pi/2$ at b_1 and at $a_1 - \mu_1$, so $b_1 = a_1 - \mu_1$ as already noted. In general, for $j \geq 2$ we have $\sum_j^n \theta_i = \pi/2$ at $a_j - \mu_j$ and so $\sum_1^n \theta_i = \pi/2 + \sum_1^{j-1} \theta_i < \pi/2 + (j-1)\pi$. But $\sum_1^n \theta_i = (j - 1/2)\pi$ at b_j, so $a_j - \mu_j < b_j$. The inequality $a_j + \nu_j \geq b_j$ is proved similarly, and so $b_j \in I_j$.

If $a_k \leq b_j \leq a_{k+1}$, then either $j \leq k$, when since $a_j + \nu_j \leq a_k + \nu_k$ and $b_j \in I_j$ we have also $b_j \in I_k$, or $j \geq k+1$ when $b_j \in I_{k+1}$ follows similarly.

(iii) For $x \notin I$, suppose that $a_k < x < a_{k+1}$ (which includes the extreme cases mentioned in the statement of the lemma). Since $x \notin I$, we must have $a_k + \nu_k < x < a_{k+1} - \mu_{k+1}$. Then from the definition of ν_k, μ_{k+1} we see that at x,

$$0 < p := \sum_{i=k+1}^n \theta_i < \pi/2 \quad \text{and} \quad 0 < q := \sum_{i=1}^k (\pi - \theta_i) < \pi/2.$$

Hence $\sum_{i=1}^n \theta_i = k\pi + (p - q)$ must lie strictly between $(k-1/2)\pi$ and $(k+1/2)\pi$ and so $\phi_1(x) = \cos(\sum_{i=1}^n \theta_i)$ has the sign $(-1)^k$. But this is also the sign of ϕ_2 on the interval (a_k, a_{k+1}), so ϕ_1 and ϕ_2 have the same sign. □

LEMMA 4.   (i) *If $0 \leq \lambda|\phi_2| \leq 1$ on I, then all zeros $(p_j)_1^n$ of $T_\lambda f$ are real and lie in I.*

(ii) *For each zero p_j of $T_\lambda f$ there is some a_k with*

$$|a_k - p_j| \leq 2c \max(k, n + 1 - k)/\pi.$$

(iii) $\sup\{|\phi_2(x)| : x \in I\} \leq 1/\sqrt{1 + \pi^2/4n^2}.$

PROOF. (i) We know from Lemma 2 that there is one zero p_j in each interval (c_{j-1}, c_j) and we shall assume without loss of generality that ϕ_1 is increasing on this interval. Suppose that $a_k \leq b_j \leq a_{k+1}$, (including, as above, the extreme cases when $k = 0$ or $k = n$). We consider separately the two cases when (a) $I_k \cap I_{k+1} \neq \emptyset$, and (b) $I_k \cap I_{k+1} = \emptyset$.

In case (a) we have $[a_k, a_{k+1}] \subset I_k \cup I_{k+1}$, and suppose first that $c_{j-1} \le a_k$. Then $\phi_1(a_k) \le \phi_1(b_j) = 0 = -\lambda\phi_2(a_k)$. Otherwise if $c_{j-1} \ge a_k$, then $\phi_1(c_{j-1}) = -1 \le -\lambda\phi_2(c_{j-1})$, since c_{j-1} is now in I_k. Hence if $r := \max(a_k, c_{j-1})$, then $\phi_1(r) \le -\lambda\phi_2(r)$. Similarly if $s := \min(a_{k+1}, c_j)$, then $\phi_1(s) \ge -\lambda\phi_2(s)$. Hence from the intermediate value theorem, there is a zero of $\phi_1 = -\lambda\phi_2$ on $[r, s] \subset [a_k, a_{k+1}] \subset I$ as required.

In case (b) when $I_k \cap I_{k+1} = \emptyset$ we can assume without loss of generality that $b_j \in I_k$. Now if $c_j \le a_k + \nu_k$ then we can argue as in case (a) that there is a zero in $[r, c_j] \subset I_k$, where $r := \max(a_k, c_{j-1})$. But if $c_j > a_k + \nu_k$, then we know from Lemma 3 (iii) that $\phi_1, -\phi_2$ have opposite signs off I, and so $\phi_1(a_k + \nu_k) \ge 0 \ge -\lambda\phi_2(a_k + \nu_k)$ and there is a zero in $[r, a_k + \nu_k] \subset I_k$. This proves (i).

(ii) We have shown in (i) that each p_j lies in some I_k, so it is enough to prove the inequalities

$$\mu_j < 2c(n - j + 1)/\pi, \quad \nu_j < 2cj/\pi.$$

But at $x = a_j - \mu_j$, we have

$$\frac{\pi}{2} = \sum_{k=j}^{n} \theta_k < \sum_{k=j}^{n} \tan\theta_k \le (n - j + 1)\tan\theta_j = \frac{(n - j + 1)c}{\mu_j}$$

which gives the result for μ_j, and the result for ν_j is similar.

(iii) For each point $x \in I$, we have $x \in I_k$ for some k and so

$$|x - a_k| \le \max(\mu_k, \nu_k) \le 2cn/\pi.$$

Thus at x we have

$$|\cos\theta_k| = \frac{|x - a_k|}{\sqrt{(x - a_k)^2 + c^2}} \le \frac{1}{\sqrt{1 + \pi^2/4n^2}},$$

and so

(2) $$\sup\{|\phi_2(x)| : x \in I\} \le \frac{1}{\sqrt{1 + \pi^2/4n^2}}.$$ $\square$

We now come to our main result on the zeros of $T_\lambda f$.

THEOREM 5. *If* $1 \le \lambda \le \sqrt{1 + \pi^2/4n^2}$, *then* $T_\lambda f$ *has only real zeros whose separation satisfies*

$$\delta(T_\lambda f) \ge \frac{2d}{\pi}\tanh\left(\frac{\pi c}{d}\right)\cos^{-1}\left(\frac{\lambda}{\sqrt{1 + \pi^2/4n^2}}\right)$$

where $d = \delta(f) = \min_j(a_j - a_{j-1})$.

The reality of the zeros follows at once from Lemma 4, since the restriction on λ ensures that $\lambda|\phi_2| \leq 1$ on I. The proof of the estimate will follow after we have established Lemma 6 below.

The following example shows that there can be no corresponding result for infinite products, without further restrictions on the location of their zeros.

EXAMPLE. Let $f(x) = \prod_1^\infty \{(1 - x/n)e^{x/n}\}$. Then $T_\lambda f$ has some non-real roots for any $\lambda > 1$.

In this example we have $\theta_n = \cot^{-1}\{(n-x)/c\}$ and we write $n \pm ic = x + r_n e^{\pm i\theta_n}$ as in Lemma 1. Then

$$T_\lambda f(x) = \prod_1^\infty \left\{\left(1 - \frac{x+ic}{n}\right) e^{(x+ic)/n}\right\} + \prod_1^\infty \left\{\left(1 - \frac{x-ic}{n}\right) e^{(x-ic)/n}\right\}$$
$$+ 2\lambda \prod_1^\infty \left\{\left(1 - \frac{x}{n}\right) e^{x/n}\right\}$$
$$= \prod_1^\infty \left\{\frac{r_n}{n} e^{x/n+i(c/n-\theta_n)}\right\} + \prod_1^\infty \left\{\frac{r_n}{n} e^{x/n-i(c/n-\theta_n)}\right\} + 2\lambda \prod_1^\infty \left\{\frac{r_n}{n} e^{x/n} \cos\theta_n\right\}$$
$$= 2 \prod_1^\infty \left\{\frac{r_n}{n} e^{x/n}\right\} \left[\cos\left(\sum_1^\infty \left(\theta_n - \frac{c}{n}\right)\right) + \lambda \prod_1^\infty \cos\theta_n\right],$$

where it is easily checked that the sums and products are absolutely convergent as grouped. This gives an obvious extension of Lemma 1. We shall show that the sum $\sum_1^\infty(\theta_n - c/n) \to -\infty$ as $x \to -\infty$; this will establish the result since $\phi_1 = \cos\left(\sum_1^\infty(\theta_n - c/n)\right)$ oscillates infinitely often between 1 and -1, while $\phi_2 = \prod_1^\infty \cos\theta_n \to 1$ as $x \to -\infty$ since each term $\to 1$ and the product is absolutely convergent. Thus as soon as $\lambda > 1$, some real intersections of ϕ_1 and $-\lambda\phi_2$ will be replaced by corresponding non-real roots. It remains to show that $\sum_1^\infty(\theta_n - c/n) \to -\infty$ as $x \to -\infty$.

Given $K > 0$, choose $N \geq 3$ such that (i) $\sum_{N+1}^\infty(\theta_n - c/n) < 1$ and (ii) $\sum_1^N c/n > 2K$. Since each $\theta_n \to 0$ as $x \to -\infty$ we can choose x sufficiently large and negative such that $\theta_n < 1/N$ for $n = 1, 2, \ldots, N$. Then for such x we will have

$$\sum_1^\infty \left(\theta_n - \frac{c}{n}\right) = \sum_1^N \theta_n - \sum_1^N \frac{c}{n} + \sum_{N+1}^\infty \left(\theta_n - \frac{c}{n}\right) < 1 - 2K + 1$$

which is less than $-K$ when $K > 2$. This shows that $\sum_1^\infty(\theta_n - c/n) \to -\infty$ as $x \to -\infty$ as required.

The following estimate for the rate of increase of the sum of the angles θ_j is needed to complete the proof of Theorem 5.

LEMMA 6. *For all $x \in \mathbb{R}$,*

$$\frac{d}{dx}\left(\sum_{j=1}^{n} \theta_j\right) < \frac{\pi}{d} \coth\left(\frac{\pi c}{d}\right),$$

where $d = \delta(f) = \min(a_j - a_{j-1})$.

PROOF. We have $\theta_j = \cot^{-1}((a_j - x)/c)$ and hence $\theta_j' = c/(c^2 + (a_j - x)^2)$. Let $h = \min_j |a_j - x|$. Since the zeros are separated by at least $d = \delta(f)$ the sum $\sum_1^n \theta_j'$ is bounded above by

$$\sum_{j=-\infty}^{\infty} \frac{c}{c^2 + (h + jd)^2} = \frac{\pi}{d} \frac{\sinh(2\pi c/d)}{\cosh(2\pi c/d) - \cos(2\pi h/d)}$$

$$\leq \frac{\pi}{d} \frac{\sinh(2\pi c/d)}{\cosh(2\pi c/d) - 1} = \frac{\pi}{d} \coth\left(\frac{\pi c}{d}\right). \qquad \square$$

We can now complete the proof of Theorem 5.

PROOF. Put $\lambda_0 = \sqrt{1 + \pi^2/4n^2}$ and let σ denote any real number in $(1, \lambda_0)$. Then for $x \in I$ and $1 \leq \lambda \leq \sigma$ we have from (2) that

$$\lambda|\phi_2(x)| \leq \lambda/\lambda_0 \leq \sigma/\lambda_0 < 1.$$

In particular, each zero p_{j-1} of $T_\lambda f$ lies in that part of the interval (c_{j-1}, c_j) on which $|\phi_1(x)| \leq \sigma/\lambda_0$. It follows that between p_{j-1} and c_j the sum $\sum \theta_j$ must increase by at least $\cos^{-1}(\sigma/\lambda_0)$. Similarly between c_j and p_j the sum $\sum \theta_j$ must increase by at least $\cos^{-1}(\sigma/\lambda_0)$, and so on $[p_{j-1}, p_j]$ the sum $\sum \theta_j$ must increase by at least $2\cos^{-1}(\sigma/\lambda_0)$.

But $\phi_1(x) = \cos\left(\sum \theta_j\right)$ and so $|\phi_1'(x)| \leq \sum \theta_j' \leq (\pi/d)\coth(\pi c/d)$ from Lemma 6. Hence an increase in x of at least Δ is required to make $\sum \theta_j$ increase by $2\cos^{-1}(\sigma/\lambda_0)$ where

$$\Delta \frac{\pi}{d} \coth\left(\frac{\pi c}{d}\right) \geq 2\cos^{-1}\left(\frac{\sigma}{\lambda_0}\right).$$

Hence the separation $\delta(T_\lambda f)$ is at least $(2d/\pi) \tanh(\pi c/d) \cos^{-1}(\sigma/\lambda_0)$ and putting $\lambda = \sigma$ gives the required result. $\qquad \square$

We complement Theorem 5 with the following result which shows that for $\lambda > 1$, some restriction on f is necessary to ensure reality of the zeros of $T_\lambda f$.

THEOREM 7. *Given any $\lambda > 1$, there is a polynomial f with real distinct zeros such that not all zeros of $T_\lambda f$ are real.*

PROOF. Consider first $f(x) = x^n$, where all zeros are at the origin. With the notation of Lemma 1, we have all $\theta_j = \theta := \cot^{-1}(x/c)$, $0 < \theta < \pi$, and so the zeros of $T_\lambda f$ are where $\cos(n\theta) + \lambda(\cos\theta)^n = 0$.

We see by inspection of the graphs of $\cos(n\theta)$ and $-\lambda(\cos\theta)^n$ that when $\lambda = 1$ there are two real zeros on the interval $[\pi/2n, 3\pi/2n]$. However if λ is increased so that $-\lambda(\cos\theta)^n = -1$ at $\theta = 3\pi/2n$ then these real zeros are lost. Thus we have to show that for any $\lambda > 1$ we can choose n so that $\lambda(\cos(3\pi/2n))^n > 1$. This follows at once since $(\cos(3\pi/2n))^n \to 1$ as $n \to \infty$, and we can estimate the size of n from the elementary inequalities

$$\left(\cos\left(\frac{3\pi}{2n}\right)\right)^n > \left(1 - \frac{1}{2}\left(\frac{3\pi}{2n}\right)^2\right)^n > 1 - \frac{9\pi^2}{8n}.$$

Thus given $\lambda > 1$, we can say that not all zeros of $T_\lambda(x^n)$ are real if

$$\lambda > \frac{1}{1 - 9\pi^2/8n}, \quad \text{or equivalently} \quad n > \frac{9\pi^2\lambda}{8(\lambda - 1)}.$$

Finally to find an example with distinct roots, we can replace x^n by a polynomial n zeros equally spaced on $[-h, h]$ say, and take h sufficiently close to zero.   □

It would be interesting to find which of the estimates $1 + O(n^{-2})$ given by Theorem 5, or $1 + O(n^{-1})$ for the example of Theorem 7 is closer to the correct order of magnitude for the value of λ required to make the zeros of $T_\lambda f$ real.

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