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SOLUTION SPACE DECOMPOSITIONS FOR nth ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. Consider the nth order scalar ordinary differential equation

$$(1.1)\quad y^{(n)}(t) + \sum_{r=0}^{n-1} p_r(t)y^{(r)}(t) = 0$$

with $p_r \in C([0, \infty) \to \mathbf{R})$. The purpose of this paper is to establish the following:

DECOMPOSITION THEOREM. *The solution space X of* (1.1) *has a direct sum decomposition*

$$X = M_1 \oplus M_2$$

where M_1 and M_2 are subspaces of X such that

(1) *each solution in $M_1 \backslash \{0\}$ is nonzero for sufficiently large t* (*nonoscillatory*);
(2) *each solution in M_2 has infinitely many zeros* (*oscillatory*).

This result includes results of F. Neuman [**6**], J. M. Dolan and G. A. Klaasen [**2**] for $n = 3$, and establishes the generalizations conjectured in [**2**].

The importance of this result for the oscillation theory of (1.1) is that we now have the correct interpretation of "decomposing (1.1) into products of lower order operators," in the absence of the usual factorization hypothesis of disconjugacy.

It should be mentioned here that the dimensions of M_1 and M_2 are not at one's disposal for a given equation (1.1). Indeed, using the techniques in [**5**], it is relatively easy to construct examples of equations (1.1) for which $\dim M_1 = n - \dim M_2 = k$ for any integer k, $0 \leqq k \leqq n$. For $n = 3$, and $\dim M_2 = 2$, there exist examples of equations (1.1) such that M_2 is unique. As the dimension n of the solution space grows, this phenomena becomes less likely to occur. It would be interesting to pursue the relationship between uniqueness of M_2, satisfying $\dim M_2 = k$, and the oscillation properties of solutions of (1.1).

The techniques used here are classical, and depend only on standard separation properties of convex sets in linear spaces. In section 2 we add to the theory of cones in Banach spaces a geometric theorem on tangential supporting hyperplanes. This theorem, plus a simple induction argument, allows us to prove a

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decomposition theorem for $\mathbf{R}^n$ relative to a double convex cone in $\mathbf{R}^n$. This double cone decomposition theorem was conjectured to be true by Dolan and Klaasen [**2**], and following their methods, we deduce the subspace decomposition theorem in section 3 from the cone theory results of section 2.

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2. Cone theory. Let X be a Banach space with norm $||\cdot||$, over the real numbers $\mathbf{R}$. Define $A + B = \{a + b : a \in A, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$. A convex cone K in X is defined by (1) $K + K \subseteq K$, (2) $\lambda K \subseteq K$ for $\lambda > 0$, (3) 0 belongs to the boundary ∂K of K. This definition differs from that of Krasnoselskii [**7**] in that K need not be closed, and 0 need not be an extremal point of K.

We assume that the reader is familiar with standard separation theorems for convex sets in Banach spaces, especially the material and terminology in Dunford-Schwartz [**3**]. For convenience, we record for use in the sequel the following easily proved special results:

LEMMA 2.1. *Let K be a convex cone in X, $f \in X^*$, $f \neq 0$, $f(K) \geqq 0$ and let $x_0 \in \operatorname{int}(K)$. Then $f(x_0) > 0$.*

LEMMA 2.2. *Let K be a convex cone in X, with closure $\bar{K}$ not all of X. Then $\bar{K}$ is also a convex cone, and*

$$\bar{K} = \bigcap_{f \in K^*} \{x \in X : f(x) \geqq 0\}$$

where K^* is the dual cone of functionals $f \in X^*$ with $f(K) \geqq 0$.

Definition 2.3. Let K be a convex cone in the Banach space X, $\bar{K} \neq X$. We define the *tangential periphery* of K to be the subspace

$$H = \bigcap_{g \in K^*} \{x : g(x) = 0\}.$$

For example, if $X = \mathbf{R}^3$ and $K = \{(x, y, z) \in \mathbf{R}^3 : z > 0\}$, then H is the xy-plane.

The Theorem (2.5) on tangential supporting hyperplanes is a consequence of the following lemma. It is interesting to note that when K satisfies Krasnoselskii's conditions [**7**], the result can be strengthened to: $f(x) > 0$ for $x \neq 0$, $x \in K$.

Positive Functional Lemma

LEMMA 2.4. *Let K be a convex cone in a separable Banach space X, $\bar{K} \neq X$, $K \backslash H \neq \emptyset$, then there exists a nonzero functional $f \in X^*$ such that $f(H) = 0$ and $f(K \backslash H) > 0$.*

Proof. For each $x \in K \backslash H$, select $f_x \in K^*$ with $||f_x|| = 1$ such that $f_x(x) \neq 0$; this is possible, by the definition of H.

Select for each $x \in K\backslash H$, a corresponding sphere B_x, centered at x, such that $f_x(B_x) > 0$.

The space X, being separable, has a countable basis for its topology, hence is hereditarily Lindelöf. Therefore, the covering $\{B_x\}$ of $K\backslash H$ has a countable subcover $\{B_{x_i}\}_{i \geqq 1}$. Put

$$f = \sum_{i \geqq 1} 2^{-i} f_{x_i}.$$

Then $f \in X^*$, $0 < ||f|| \leqq 1$, and $f(K) \geqq 0$. To show $f(H) = 0$, we utilize $H \subseteq \{x : f(x) = 0\}$, due to $f \in K^*$. Finally, if $x \in K\backslash H$, then $x \in B_{x_i}$ for some index i, whereby $f_{x_i}(x) > 0$. The representation for f gives $f(x) > 0$; this completes the proof.

Tangential Supporting Hyperplanes

A closed right circular cone in $\mathbf{R}^3$ has a supporting hyperplane at 0 which meets the cone only at 0; for an arbitrary convex cone in $\mathbf{R}^3$ this need not be true, and in analogy with the geometry of this situation we define a *tangential supporting hyperplane* to be the kernel of a functional $f \in X^*$ satisfying $f(H) = 0$, $f(K\backslash H) > 0$. If we can insure somehow that $K\backslash H \neq \emptyset$, then the positive functional lemma supplies the existence of a tangential supporting hyperplane at 0. Thus we have:

THEOREM 2.5. *Let K be a convex cone in a separable Banach space X. The following are sufficient for K to have a tangential supporting hyperplane*:

(1) int $K \neq \emptyset$, *or*

(2) *span* $(K) = X$ *and* $\bar{K} \neq X$.

Double Cone Decomposition

We turn now to finite-dimensional Euclidean space $\mathbf{E}$. In this setting, a convex cone K has a supporting hyperplane at 0, hence $\bar{K} \neq \mathbf{E}$. We prove now the following decomposition theorem for double cones.

THEOREM 2.6. *Let K be a convex cone in a finite dimensional Euclidean space* $\mathbf{E}$. *Then there exists subspaces N_1 and N_2 in* $\mathbf{E}$ *such that*

(1) $E = N_1 \oplus N_2$,

(2) $N_1\backslash\{0\} \subseteq K \cup (-K)$, *and*

(3) $N_2\backslash\{0\} \subseteq E\backslash[K \cup (-K)]$.

Proof. The proof proceeds by induction on the dimension n of the space $\mathbf{E}$; the result is immediate for $n = 1$. Suppose $n \geqq 1$ and the theorem has been proved for all Euclidean spaces $\mathbf{E}_1$ of finite dimension $\leqq n$. Let us establish the theorem for an arbitrary Euclidean space $\mathbf{E}$ of finite dimension $n + 1$.

First of all, we may assume that K does not lie in a lower dimensional subspace $\mathbf{E}_1$ of $\mathbf{E}$. Indeed, in this case, the decomposition of $\mathbf{E}_1$ given by the

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induction hypothesis induces the required decomposition of $\mathbf{E}$ in a natural way. In particular, we may assume span$(K) = \mathbf{E}$.

By Theorem 2.5, K has a tangential supporting hyperplane $F = \{x : f(x) = 0\}$, where $f \in K^*$ satisfies $f(H) = 0$, $f(K \backslash H) > 0$. Let us decompose F into complementary subspaces H and L: $F = H \oplus L$.

The induction hypothesis is now invoked to decompose the Euclidean space H, applied to the convex cone $K_1 = K \cap H = K \cap F$. Let $H = P_1 \oplus P_2$ be the decomposition so obtained, satisfying (1), (2), (3).

Choose any $x_0 \in \operatorname{int}(K)$; this is possible because span$(K) = \mathbf{E}$ implies $\operatorname{int}(K) \neq \emptyset$.

Define $N_1 = \operatorname{span}\{x_0, P_1\}$, $N_2 = P_2 \oplus L$. Since $H = P_1 \oplus P_2$ and $F = H \oplus L$, then $\mathbf{E} = N_1 \oplus N_2$. The condition $N_2 \backslash \{0\} \subseteq \mathbf{E} \backslash K \cup (-K)$ follows immediately.

To prove $N_1 \backslash \{0\} \subseteq K \cup (-K)$, it suffices to show that the variety $x_0 + H \subseteq K$.

Suppose y is any element of $x_0 + H$; we show $y \in K$. Suppose $y \notin K$, then by the separation theorem, there is a $g \in E^*$ with $g(y) \leqq g(K)$, $g \neq 0$. Then $g(K) \geqq 0$, hence $g(y) \leqq 0$. However, $g(y) = g(x_0)$ because $g(H) = 0$. This contradicts $x_0 \in \operatorname{int}(K)$ by Lemma 2.1. Hence $y \in K$.

The proof is complete.

3. Proof of the solution space decomposition theorem. Following Dolan and Klaasen [**2**], let $y_1, \ldots, y_n$ be a basis for the solution space of (1.1), and define

$$K = \left\{(a_1, \ldots, a_n) \in \mathbf{R}^n : \sum_{i=1}^{n} a_i y_i(t) > 0 \text{ for large } t\right\}.$$

If $K = \emptyset$, then we are done. Assume $K \neq \emptyset$, then it is easy to verify that K is a convex cone in $\mathbf{R}^n$.

Let us apply Theorem 2.6, and obtain a decomposition $\mathbf{R}^n = N_1 \oplus N_2$, where $N_1 \backslash \{0\} \subseteq K \cup (-K)$, $N_2 \backslash \{0\} \subseteq \mathbf{R}^n \backslash [K \cup (-K)]$. Define

$$M_j = \left\{\sum_{i=1}^{n} a_i y_i : (a_1, \ldots, a_n) \in N_j\right\}, \quad j = 1, 2.$$

It is routine to verify that solutions in M_1 are eventually positive or negative, hence M_1 consists of nonoscillatory solutions of (1.1). On the other hand, no nontrivial solution in M_2 can be one-signed for large t, by definition, hence M_2 consists of oscillatory solutions.

The decomposition $X = M_1 \oplus M_2$ is verified in a routine manner, using the relation $\mathbf{R}^n = N_1 \oplus N_2$. The proof is complete.

4. Remarks. In [**4**], an example is claimed which violates the conclusion of the solution subspace decomposition theorem. The discrepancy was eventually found by the author, in an effort to resolve the differences in results obtained by Neuman [**6**] and Dolan-Klaasen [**2**]. The example constructed and the theorems presented in [**4**] are without error; the error is in the first step of the verification on page 251. It is conjectured by the author that the example in [**4**] has a unique subspace M_2, dim $M_2 = 2$, in any splitting $X = M_1 \oplus M_2$.

Extensions of the results herein to nth order linear differential equations in Banach spaces can be made without essential modifications of the foregoing techniques. In this case, "oscillatory" means that $x^*(x(t))$ has infinitely many zeros on $[0, \infty)$ for each x^* in the dual E^* of the Banach space E. The solution space $\mathscr{S}$ of the differential equation therefore admits an algebraic decomposition $\mathscr{S} = M_1 \oplus M_2$, where M_1 is nonoscillatory and all solutions in M_2 are oscillatory.

In a similar manner, one can write down decompositions of finite-dimensional solution spaces of functional differential equations. In this connection, it would be interesting to extend Theorem 2.6 to separable Banach Spaces. Whether or not this can be done remains undecided at present.

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