Canad. J. Math. Vol. 67 (5), 2015 pp. 1144–1160
http://dx.doi.org/10.4153/CJM-2014-043-8

Outer Partial Actions and Partial Skew Group Rings

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Abstract. We extend the classical notion of an outer action α of a group G on a unital ring A to the case when α is a partial action on ideals, all of which have local units. We show that if α is an outer partial action of an abelian group G, then its associated partial skew group ring $A \star_\alpha G$ is simple if and only if A is G-simple. This result is applied to partial skew group rings associated with two different types of partial dynamical systems.

1 Introduction

The notion of a partial action of a group on a C*-algebra and the construction of its associated crossed product C*-algebra, were introduced by R. Exel [9, 12] for partial actions of the integers and then extended by K. McClanahan [20] to partial actions of discrete groups. Since then, the theory of (twisted) partial actions on C*-algebras has developed into a rich theory that has become an important tool in the study of C*-algebras. It is now known that several important classes of C*-algebras can be realized as crossed product C*-algebras by (twisted) partial actions, *e.g.*, AF-algebras [11], Bunce–Deddens algebras [10], Cuntz-Krieger algebras [14], and Cuntz-Li algebras [4].

In a purely algebraic context, partial skew group rings were introduced by M. Dokuchaev and R. Exel [6] as a generalization of classical skew group rings and as an algebraic analogue of partial crossed product C*-algebras. Compared to the abundance of results in the context of skew group rings or partial crossed product C*-algebras, the theory of partial skew group rings is still underdeveloped. In particular, apart from the results in [2, 3, 16, 17], very little is known about the ideal structure and simplicity criteria for partial skew group rings.

The primary goal of this article is to establish a generalization (see Theorem 1.2) of a result due to K. Crow [5] (see Theorem 1.1) concerning a connection between outer actions and simplicity of unital skew group rings, to partial skew group rings that have local units. The secondary goal is to apply this result to show generalizations (see Theorems 1.3 and 1.4) of recent results by D. Gonçalves [16] concerning partial skew group rings associated with two different types of partial dynamical systems.

Received by the editors June 11, 2014; revised November 5, 2014.
Published electronically March 3, 2015.
The second author was partially supported by The Swedish Research Council (repatriation grant no. 2012-6113).
AMS subject classification: **16S35**, 16W50, 37B05, 37B99, 54H15, 54H20.
Keywords: outer action, partial action, minimality, topological dynamics, partial skew group ring, simplicity.

Before we describe these results, we first need to recall the following notions. Let G be a group with identity element e and let X be a set. A *partial action* α of G on X is a collection of subsets $\{X_g\}_{g\in G}$ of X and a collection of bijections $\alpha_g: X_{g^{-1}} \to X_g$, for $g \in G$, such that for all $g, h \in G$ and every $x \in X_{h^{-1}} \cap X_{(gh)^{-1}}$, the following three relations hold:

(a) $\alpha_e = \mathrm{id}_X$;
(b) $\alpha_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$;
(c) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$.

It often happens that the set X carries an additional structure. By requiring that the subsets $\{X_g\}_{g\in G}$ and the bijections $\{\alpha_g\}_{g\in G}$ be compatible with the given structure on X, we may define a partial action of a certain type. If X is a topological space, then we require that for each $g \in G$, X_g is an open set and α_g is a homeomorphism. If X is a semigroup (ring, algebra), then we require that, for each $g \in G$, the subset X_g is an ideal of X and the map α_g is a semigroup (ring, algebra) isomorphism. A subset I of X is called G-*invariant* if for each $g \in G$ the inclusion $\alpha_g(I \cap D_{g^{-1}}) \subseteq I$ holds. In case X is a semigroup (ring, algebra), we say that X is G-*simple* if there is no G-invariant ideal of X other than X itself and $\{0\}$ (which need not exist). The action α is called *global* if the equality $X_g = X$ holds for each $g \in G$.

As a preparation for Crow's result below, we shall now recall a couple of important notions from the classical setting, *i.e.*, when X is a unital ring (algebra) and $\alpha: G \ni g \mapsto \alpha_g \in \mathrm{Aut}(X)$ is a global action of G on X. If $g \in G$, then the map α_g is said to be *inner* if there is an invertible $a \in X$ such that the relation $\alpha_g(x) = a^{-1}xa$ holds for all $x \in X$. The action α is said to be *outer* if the identity element e is the only element of G that maps to an inner automorphism of X.

Theorem 1.1 (Crow [5]) *If $\alpha: G \to \mathrm{Aut}(A)$ is an outer action (in the classical sense) of an abelian group G on a unital ring A, then the associated skew group ring $A *_\alpha G$ is simple if and only if A is G-simple.*

To describe our generalization of Theorem 1.1 and its applications, we first need to answer the following question:

What should it mean for a partial action of a group on a ring to be outer?

As far as we know, this question has not previously been analysed in the literature in the C*-algebra context or in the purely algebraical setting. The starting point for our investigations is the observation that many of the concepts concerning partial actions on rings are formulated by using only the operation of multiplication, and thus forgetting the additive structure. In other words, we are working in the multiplicative semigroup of a ring.

In Section 2, we therefore begin our explorations in a general semigroup S. In addition, since we want to establish a non-unital version of Theorem 1.1, we also have to decide what it should mean for isomorphisms $\alpha: I \to J$ of ideals I and J in S to be outer, *locally at idempotents* $u \in I \cap J$. To motivate the approach taken later, let us briefly describe the train of reasoning that lead us to the formal definition. The restricted map $\alpha|_{uSu}: uSu \to \alpha(u)S\alpha(u)$ is also an isomorphism of semigroups. So by mimicking the global case, the map $\alpha|_{uSu}$ should be called *inner* if there are $a, b \in S$

https://doi.org/10.4153/CJM-2014-043-8 Published online by Cambridge University Press

such that $\alpha|_{uSu}(x) = bxa$ holds for all $x \in uSu$. However, for such a definition to make sense, we need to assume that $a \in uS\alpha(u)$ and $b \in \alpha(u)Su$. From the fact that $\alpha(u) = \alpha|_{uSu}(u) = bua = ba$, we get that $ba = \alpha(u)$. Also, the inverse of $\alpha|_{uSu}$ should be defined by the "reversed" map $a(\cdot)b$ from which we get that $ab = u$. Therefore, if such a and b exist, we say that α is *inner at u*; otherwise, α is called *outer at u* (see Definition 2.4 for more details).

In Section 3, we recall a result (see Theorem 3.1) from [21] by the authors of this article concerning simplicity of group graded rings that we will need in the subsequent section for application to partial skew group rings, which, in a natural way, are group graded rings.

In Section 4, we use the definition of outer actions in semigroups from Section 2 to define outer partial actions $\alpha_g: D_{g^{-1}} \to D_g$ of a group G on a ring A in the following way (see Definition 4.9 for more details). Consider A as a semigroup with respect to multiplication. If $g \in G$, then we say that α_g is inner (outer) at an idempotent $u \in A$ if it is inner (outer) at u in the sense defined above. Furthermore, we say that α is *outer* (or *outer at u*) if there is a non-zero idempotent $u \in A$ such that for each non-identity $g \in G$, the map α_g is outer at u. In the classical setting, *i.e.*, when A is unital and α is a global action of G on A, our definition of outerness coincides with the classical definition of outerness described above (see Remark 4.10). At the end of Section 4, we show, with the aid of the result in Section 3, the following generalization of Theorem 1.1.

Theorem 1.2 *If $\alpha_g: D_{g^{-1}} \to D_g$, for $g \in G$, is an outer partial action of an abelian group G on a ring A such that D_g, for each $g \in G$, has local units, then the associated partial skew group ring $A \star_\alpha G$ is simple if and only if A is G-simple.*

In Sections 5 and 6, we show that Theorem 1.2 can be effectively applied to set dynamics respectively topological dynamics. To be more precise, let us recall the following notions for a partial action α of a group G on a set (topological space) X. If for each non-identity $g \in G$, there is some $x \in X_{g^{-1}}$ such that $\alpha_g(x) \neq x$, then α is said to be *faithful*. If for each non-identity $g \in G$, the set of $x \in X_{g^{-1}}$ that satisfy $\alpha_g(x) = x$ is the empty set (has empty interior), then α is called (topologically) *free*. Clearly, freeness implies topological freeness. If X and $\varnothing$ are the only G-invariant (closed) subsets of X, then α is said to be (topologically) *minimal*.

In the set dynamical case, we are given a partial action α of a group G on a (non-empty) set X and consider the partial skew group ring $F_0(X,B) \star_\alpha G$. Here $F_0(X,B)$ denotes the algebra of finitely supported functions $X \to B$, where B is a simple associative ring that has local units.

Theorem 1.3 *If G is abelian, then the following three assertions are equivalent:*

(i) *$F_0(X,B) \star_\alpha G$ is simple;*
(ii) *θ is minimal and free;*
(iii) *θ is minimal and faithful.*

In the topological dynamical case, we are given a partial action α of a group G on a compact Hausdorff space X such that X_g is clopen for each $g \in G$. Note that if G is a countable discrete group, then these partial actions are exactly the ones for which the enveloping space is Hausdorff (see [13, Proposition 3.1]). We then consider the partial skew group ring $C_E(X,B) \star_\alpha G$. Here B denotes a simple associative topological real algebra that has a set E of local units. (Some additional assumptions are made on B; see Section 6.) The algebra $C_E(X,B)$ is the directed union of the "local" algebras $C(X,\epsilon B\epsilon) = \{\text{continuous } f\colon X \to \epsilon B\epsilon\}$, where ϵ runs over all elements in E.

Theorem 1.4 *If G is abelian, X is compact Hausdorff, and each X_g, for $g \in G$, is clopen, then the following three assertions are equivalent:*

(i) *$C_E(X,B) \star_\alpha G$ is simple;*
(ii) *θ is topologically minimal and topologically free;*
(iii) *θ is topologically minimal and faithful.*

Note that Theorems 1.3 and 1.4 generalize recent results by D. Gonçalves [16] to also include cases when the coefficients are taken from *non-commutative* rings which have local units.

2 Outer Actions of Ideals in Semigroups

In this section, we introduce the concepts of innerness and outerness of homomorphisms of ideals in semigroups at idempotents (see Definition 2.4). We also show that the innerness is preserved by the classical partial order on the idempotents in the semigroup (see Proposition 2.7). We begin by fixing some notation.

Throughout this section, S denotes a semigroup. By this we mean that S is a non-empty set equipped with an associative binary operation $S \times S \ni (x,y) \mapsto xy \in S$, which is referred to as the *multiplication* of the semigroup. For subsets I and J of S we let IJ denote the set of all products of the form xy for $x \in I$ and $y \in J$. A non-empty subset I of S is called a subsemigroup (left ideal, right ideal, ideal) of S if $II \subseteq I$ ($SI \subseteq I$, $IS \subseteq I$, $SI \cup IS \subseteq I$). If T is another semigroup, then a map $\alpha\colon S \to T$ is a homomorphism of semigroups if it respects the multiplication in S and T. Suppose that I and J are right ideals of S. Then a map $\alpha\colon I \to J$ is called a homomorphism of right ideals if $\alpha(xy) = \alpha(x)y$, for $x \in I$ and $y \in S$. We let $\mathrm{Hom}_S(I,J)$ denote the set of all homomorphisms $I \to J$ of right ideals. The concept of a homomorphism of (left) ideals is defined analogously.

The first two propositions below have already appeared in the context of ideals in rings (see *e.g.*, [19, Propositions (21.6) and (21.20)]), except for the last part of the first proposition. However, we were not able to find an appropriate reference for the case of semigroups. The proofs are a close adaptation to semigroups of the proofs given in loc. cit., and we include them for the convenience of the reader.

Proposition 2.1 *Let u, v, and w be idempotents in S and suppose that I is a right ideal of S. Then the map of sets $\lambda\colon \mathrm{Hom}_S(uS,I) \to Iu$ defined by $\lambda(\beta) = \beta(u)$ for $\beta \in \mathrm{Hom}_S(uS,I)$ is a bijection. In particular, if we put $I = vS$, then the corresponding map $\lambda_{v,u}\colon \mathrm{Hom}_S(uS,vS) \to vSu$ is a bijection. Moreover, if $\beta \in \mathrm{Hom}_S(uS,vS)$ and $\beta' \in \mathrm{Hom}_S(vS,wS)$, then $\lambda_{w,v}(\beta')\lambda_{v,u}(\beta) = (\lambda_{w,u})(\beta' \circ \beta)$.*

Proof First we show that λ is well defined. Suppose that $\beta\colon uS \to I$ is a right ideal homomorphism. Then $\lambda(\beta) = \beta(u) = \beta(u^2) = \beta(u)u \in Iu$. Next, we show that λ is injective. Suppose that β and β' are right ideal homomorphisms $uS \to I$ such that $\lambda(\beta) = \lambda(\beta')$. Take $s \in S$. Then $\beta(us) = \beta(u)s = \lambda(\beta)s = \lambda(\beta')s = \beta'(u)s = \beta'(us)$. Therefore, $\beta = \beta'$. Finally, we show that λ is surjective. Take $iu \in Iu$, where $i \in I$. Define $\beta_{iu} \in \operatorname{Hom}_S(uS, I)$ by $\beta_{iu}(us) = ius$, for $s \in S$. We claim that β_{iu} is well defined. If we assume that the claim holds, then $\lambda(\beta_{iu}) = \beta_{iu}(u) = \beta_{iu}(uu) = iuu = iu$, and thus λ is surjective. Now we show the claim. Suppose that $us = us'$ for some $s, s' \in S$. Then $\beta_{iu}(us) = ius = ius' = \beta_{iu}(us')$. The second part follows immediately from the first part. Now we show the last part of the proof. Take $\beta \in \operatorname{Hom}_S(uS, vS)$ and $\beta' \in \operatorname{Hom}_S(vS, wS)$. Then $\lambda_{w,v}(\beta')\lambda_{v,u}(\beta) = \beta'(v)\beta(u) = \beta'(v\beta(u)) = \beta'(\beta(u)) = \lambda_{w,u}(\beta' \circ \beta)$. ∎

Proposition 2.2 *If u and v are idempotents of S, then the following four assertions are equivalent:*

(i) *$uS \cong vS$ as right ideals of S;*
(ii) *$Su \cong Sv$ as left ideals of S;*
(iii) *there exist $a \in uSv$ and $b \in vSu$ such that $ab = u$ and $ba = v$;*
(iv) *there exist $a, b \in S$ such that $ab = u$ and $ba = v$.*

Proof By left-right symmetry it is enough to show (i)⇒(iii)⇒(iv)⇒(i).

(i)⇒(iii): Let $\beta\colon uS \to vS$ be an isomorphism of right ideals. Put $a = \lambda_{v,u}(\beta)$ and $b = \lambda_{u,v}(\beta^{-1})$. Then, by the last part of Proposition 2.1, we get

$$u = \lambda_{u,u}(\mathrm{id}_{uS}) = \lambda_{u,u}(\beta^{-1} \circ \beta) = \lambda_{u,v}(\beta^{-1})\lambda_{v,u}(\beta) = ba,$$
$$v = \lambda_{v,v}(\mathrm{id}_{vS}) = \lambda_{v,v}(\beta \circ \beta^{-1}) = \lambda_{v,u}(\beta)\lambda_{u,v}(\beta^{-1}) = ab.$$

(iii)⇒(iv): Trivial.

(iv)⇒(i): Suppose that there are $a, b \in S$ such that $ab = u$ and $ba = v$. Define $\beta\colon uS \to vS$ and $\gamma\colon vS \to uS$ by the relations $\beta(x) = bx$, for $x \in uS$, and $\gamma(y) = ay$, for $y \in vS$, respectively. Since $bx = bux = babx = vbx$, for $x \in uS$, and $ay = avy = abay = uay$, for $y \in vS$, it follows that β and γ are well-defined homomorphisms of right ideals. Now we show that $\gamma \circ \beta = \mathrm{id}_{uS}$ and $\beta \circ \gamma = \mathrm{id}_{vS}$. Take $x \in uS$ and $y \in vS$. Then

$$(\gamma \circ \beta)(x) = \gamma(bx) = abx = ux = x \text{ and } (\beta \circ \gamma)(y) = \beta(ay) = bay = vy = y. \quad ∎$$

Definition 2.3 Let u and v be idempotents of S. We say that u and v are *equivalent* and denote this by $u \sim v$, if u and v satisfy any (and hence all) of the equivalent conditions (i)–(iv) above.

Definition 2.4 Suppose that I and J are ideals of S and $\alpha\colon I \to J$ is a semigroup homomorphism. Let u be an idempotent of S. We say that α is *inner at u* if $u \in I$ and $u \sim \alpha(u)$ where this equivalence is defined by an isomorphism $\beta\colon uS \to \alpha(u)S$ of right ideals of S such that $\alpha(x) = \beta(u)x\beta^{-1}(\alpha(u))$ for all $x \in uSu$. We say that α is *outer at u* if α is not inner at u. We say that α is *strongly outer* if it is outer at *all* non-zero idempotents of S.

Remark 2.5 Suppose that I and J are ideals of S and that $\alpha: I \to J$ is a semigroup homomorphism which is inner at an idempotent u of I.

(a) Although in the above definition we only assume that $\alpha: I \to J$ is a semigroup homomorphism, the restricted map $\alpha|_{uSu}: uSu \to \alpha(u)S\alpha(u)$ is always an *isomorphism* of semigroups. In fact, if we put $a = \beta^{-1}(\alpha(u))$ and $b = \beta(u)$, then $ba = \alpha(u)$ and $ab = u$ and $\alpha(x) = bxa$ for all $x \in uSu$. It is now clear that $\alpha|_{uSu}^{-1}: \beta(u)S\beta(u) \to uSu$ is defined by $\alpha|_{uSu}^{-1}(x) = axb$ for all $x \in \beta(u)S\beta(u)$.
(b) It follows that $u \in I \cap J$, since $u = ab = a\alpha(u)b \in aJb \subseteq J$.
(c) If S is a monoid and we let u be the identity element of S, then $\alpha: S \to S$ is inner at u precisely when it is inner in the classical case, *i.e.*, if there is an invertible $y \in S$ such that $\alpha(x) = yxy^{-1}$ for all $x \in S$. In particular, by (a), this forces α to be a semigroup automorphism of S.

Definition 2.6 Recall that the idempotents of S can be partially ordered by saying that $v \leq u$ if $uv = vu = v$. An idempotent is called *minimal* if it is minimal with respect to $\leq$.

Proposition 2.7 *Suppose that I and J are ideals of S and that $\alpha: I \to J$ is a semigroup homomorphism that is inner at an idempotent u of I. If v is another idempotent of I with $v \leq u$, then α is inner at v.*

Proof Suppose that there is an isomorphism $\beta: uS \to \alpha(u)S$ of right ideals of S such that $\alpha(x) = \beta(u)x\beta^{-1}(\alpha(u))$ for all $x \in uSu$. Put $b = \beta(u)$ and $a = \beta^{-1}(\alpha(u))$. Then $ab = u$ and $ba = \alpha(u)$, and there are some $d, d' \in S$ such that $a = ud\alpha(u)$ and $b = \alpha(u)d'u$.

Consider the elements $a' = vd\alpha(v)$ and $b' = \alpha(v)d'v$. Then $a\alpha(x)b = a(bxa)b = uxu = x$ holds for any $x \in uSu$. In particular, for $x = v$, this yields $a\alpha(v)b = v$, and hence

$$\begin{aligned} a'b' &= (vd\alpha(v))(\alpha(v)d'v) = vd\alpha(v)d'v = vud\alpha(u)\alpha(v)\alpha(u)d'uv \\ &= v(ud\alpha(u))\alpha(v)(\alpha(u)d'u)v = vvv = v. \end{aligned}$$

Moreover, $bva = \alpha(v)$, and hence

$$\begin{aligned} b'a' &= (\alpha(v)d'v)(vd\alpha(v)) = \alpha(v)d'vd\alpha(v) = \alpha(v)\alpha(u)d'uvud\alpha(u)\alpha(v) \\ &= \alpha(v)(\alpha(u)d'u)v(ud\alpha(u))\alpha(v) = \alpha(v)\alpha(v)\alpha(v) = \alpha(v). \end{aligned}$$

Take $x \in vSv \subseteq uSu$. There is some $z \in S$ such that $x = vzv$. Hence, $\alpha(x) = \alpha(vzv) = \alpha(v)\alpha(zv) = \alpha(vz)\alpha(v)$. This shows that $\alpha(x) = \alpha(v)\alpha(x)\alpha(v)$. Then

$$\begin{aligned} \alpha(x) &= (\alpha(u)d'u)x(ud\alpha(u)) = (\alpha(u)d'u)vxv(ud\alpha(u)) = \alpha(u)d'vxvd\alpha(u) \\ &= \alpha(v)(\alpha(u)d'vxvd\alpha(u))\alpha(v) = (\alpha(v)d'v)x(vd\alpha(v)) = b'xa'. \end{aligned}$$

This shows that α is inner at v. ∎

Remark 2.8 The conclusion of Proposition 2.7 does not hold, in general, if $v \le u$ is replaced by $u \le v$. In particular, local innerness cannot always be lifted to global innerness. To be more precise, suppose that I and J are ideals of S and that $\alpha\colon I \to J$ is a semigroup homomorphism. If $u, v \in S$ are idempotents such that $v \le u$ and α is inner at v, then this does not in general imply that α is inner at u. In fact, let $S = I = J$ denote the multiplicative semigroup of functions from $\{1, 2, 3\}$ to a field K. Let $u, v \in S$ be defined by $u(1) = u(2) = u(3) = 1_K$, resp. $v(1) = 1_K$ and $v(2) = v(3) = 0$. Then $v \le u$. If we define $\alpha\colon S \to S$ by $\alpha(f)(1) = f(1)$, $\alpha(f)(2) = f(3)$ and $\alpha(f)(3) = f(2)$, for all $f \in S$, then it is easy to see that $\alpha|_{vSv} = \mathrm{id}_{vSv}$. Clearly, α is inner at v, but outer at u.

Definition 2.9 We say that a set E of minimal non-zero idempotents of S is a *complete set of minimal idempotents* if for each non-zero idempotent $u \in S$, there is $v \in E$ such that $v \le u$.

Corollary 2.10 *Suppose that there is a complete set E of minimal idempotents of S. Let I and J be ideals of S and suppose that $\alpha\colon I \to J$ is a semigroup homomorphism. Then α is strongly outer if and only if it is outer at each $u \in E$.*

Proof This follows immediately from Proposition 2.7 and Definition 2.9. ■

Remark 2.11 Innerness of ring automorphisms at idempotents (however not in the generality of semigroup homomorphisms of ideals) was considered by J. Haefner and A. del Rio in [18, Definition 1.2, p. 38].

3 Simple Group Graded Rings

In this section, we recall a result (see Theorem 3.1) from [21] by the authors of this article concerning simple group graded rings, which we will need in the sequel. We begin by fixing some notation.

Let R denote a ring that is associative but not necessarily unital. If R is unital, then we let 1_R denote its multiplicative identity element. By an *ideal* of R we always mean a two-sided ideal of R. The *center* of R, denoted by $Z(R)$, is the set of elements $x \in R$ with the property that $xy = yx$ holds for each $y \in R$. Recall from [1] that R is said to have *local units* if there exists a set E of idempotents of R such that for every finite subset X of R, there exists an $f \in E$ such that $X \subseteq fRf$. It then follows that $x = fx = xf$ holds for each $x \in X$.

Let G denote a group with identity element e. Recall that R is said to be *graded* (by G), if there for each $g \in G$ is an additive subgroup R_g of R such that $R = \oplus_{g\in G} R_g$ and the inclusion $R_g R_h \subseteq R_{gh}$ holds for all $g, h \in G$. Take $r \in R$. There are unique $r_g \in R_g$, for $g \in G$, such that all but finitely many of them are zero and $r = \sum_{g\in G} r_g$. We let the *support* of r, denoted by $\mathrm{Supp}(r)$, be the set of $g \in G$ such that $r_g \ne 0$. The element r is called *homogeneous* if $|\mathrm{Supp}(r)| \le 1$. If $r \in R_g \setminus \{0\}$, for some $g \in G$, then we write $\deg(r) = g$. An additive subgroup A of R is called *graded* if $A = \oplus_{g\in G}(A \cap R_g)$ holds. The ring R is said to be *graded simple* if R and $\{0\}$ are its only graded ideals. Clearly, graded simplicity is a necessary condition for simplicity.

Theorem 3.1 *If R is a ring graded by an abelian group G and R_e contains a non-zero idempotent u, then R is simple if and only if it is graded simple and $Z(uRu)$ is a field.*

Proof This follows from a more general result by the authors of this article, concerning simplicity of semigroup graded rings (see [21, Theorem 2.1]). For the convenience of the reader, we now give a direct proof. The "only if" statement is straightforward. Now we show the "if" statement. Let I be a non-zero ideal of R. Take $r \in I \setminus \{0\}$ such that $|\operatorname{Supp}(r)|$ is minimal. Choose some $g \in G$ such that r_g is non-zero.

Since R is graded simple, there are homogeneous $s_i, t_i \in R$, for $i = \{1, \ldots, n\}$, such that $\sum_{i=1}^{n} s_i r_g t_i = u$. In particular, there is $j \in \{1, \ldots, n\}$ such that $s_j r_g t_i \in R_e \setminus \{0\}$. By replacing r with $s_j r t_j$, we can from now on assume that r_e is non-zero.

Next we show that we can suppose that $r_e = u$. Put

$$J = \{s_e \mid s \in RrR,\ \operatorname{Supp}(s) \subseteq \operatorname{Supp}(r)\}.$$

Then J is a non-zero ideal of R_e, and hence RJR is a non-zero graded ideal of R. By graded simplicity of R we get that there are $s^{(i)} \in RrR$ and $v_i, w_i \in R$, for $i \in \{1, \ldots, n\}$, such that $\operatorname{Supp}(s^{(i)}) \subseteq \operatorname{Supp}(r)$ and $u = \sum_{i=1}^{n} v_i s_e^{(i)} w_i$. From the last equality it follows that we can suppose that $\deg(v_i)\deg(w_i) = e$ for all i such that $v_i s_e^{(i)} w_i \neq 0$. Put $s = \sum_{i=1}^{n} v_i s^{(i)} w_i$. Then $s \in I$, and since $\operatorname{Supp}(s^{(i)}) \subseteq \operatorname{Supp}(r)$ for all i and G is abelian, we get that $\operatorname{Supp}(s) \subseteq \operatorname{Supp}(r)$. Therefore, $u = \sum_{i=1}^{n} v_i s_e^{(i)} w_i = s_e \in J$.

Finally, we show that $I = R$. Take $h \in G$ and $t \in uR_h u$. Since $r_e = u$ and G is abelian, we get that $|\operatorname{Supp}(rt - tr)| < |\operatorname{Supp}(r)|$. By the assumption that $|\operatorname{Supp}(r)|$ is minimal and the fact that $rt - tr \in I$, we get that $\operatorname{Supp}(rt - tr) = \varnothing$ and hence that $rt - tr = 0$. Since $h \in G$ was arbitrarily chosen, we get that $r \in Z(uRu) \cap I$. Using that $Z(uRu)$ is a field, we get that $u \in I$. Therefore, since R is graded simple, we get that $R = RuR \subseteq I$. ∎

4 Partial Actions and Partial Skew Group Rings

In this section, we introduce outer partial actions of groups on rings (see Definition 4.9) and we prove the main result of this article concerning simplicity of partial skew group rings (see Theorem 1.2).

Assumption *Throughout this section, α will denote a partial action of a group G on a ring A, and the corresponding ideals of A are denoted by D_g, for $g \in G$.*

Definition 4.1 The *partial skew group ring* $A \star_\alpha G$ is defined as the set of all finite formal sums $\sum_{g \in G} a_g \delta_g$, where for each $g \in G$, $a_g \in D_g$ and δ_g is a symbol. Addition is defined in the obvious way, and multiplication is defined as the linear extension of the rule $(a_g \delta_g)(b_h \delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g) b_h)\delta_{gh}$ for $g, h \in G$, $a_g \in D_g$ and $b_h \in D_h$. Clearly, each classical skew group ring (see *e.g.*, [5,15,22]) is a partial skew group ring where $D_g = A$ for all $g \in G$.

Remark 4.2 A partial skew group ring $A \star_\alpha G$ need not in general be associative (see [6, Example 3.5]). However, if each D_g, for $g \in G$, has local units, then, in particular, each D_g, for $g \in G$, is an idempotent ring, *i.e.*, $D_g^2 = D_g$, which by [6, Corollary 3.2], ensures that $A \star_\alpha G$ is associative. In that case, the set $E\delta_e = \{f\delta_e \mid f \in E\}$ is a set of local units for $A \star_\alpha G$ if E is a set of local units for A.

Definition 4.3 If there does not exist any non-identity $g \in G$ such that $D_g \cap D_{g^{-1}}$ is non-zero and $\alpha_g|_{D_g \cap D_{g^{-1}}} = \mathrm{id}_{D_g \cap D_{g^{-1}}}$, then α is said to be *injective*.

The next result extends a well-known result for group actions on rings (see *e.g.*, [22]) to the case of partial actions.

Proposition 4.4 *If the partial skew group ring $A \star_\alpha G$ is simple, then α is injective.*

Proof Suppose that α is not injective. Then there is a non-identity $g \in G$ such that $D_g \cap D_{g^{-1}} \neq \{0\}$ and $\alpha_g|_{D_g \cap D_{g^{-1}}} = \mathrm{id}_{D_g \cap D_{g^{-1}}}$. Take a non-zero element $i \in D_g \cap D_{g^{-1}}$. Let J be the ideal of $A \star_\alpha G$ generated by the element $i\delta_e - i\delta_g$. It is clear that J is non-zero and strictly contained in $A \star_\alpha G$. Therefore, $A \star_\alpha G$ is not simple. ∎

Remark 4.5 Note that $A \star_\alpha G$ need not be associative for Proposition 4.4 to hold.

Remark 4.6 It is easy to check that if we put $(A \star_\alpha G)_g = D_g\delta_g$, for $g \in G$, then this defines a gradation on the ring $A \star_\alpha G$. In the sequel, whenever we speak of *graded* or *graded simple* it will be with respect to this gradation.

Proposition 4.7 *If each D_g, for $g \in G$, has local units, then $A \star_\alpha G$ is graded simple if and only if A is G-simple.*

Proof We begin by showing the "only if" statement. Suppose that $A \star_\alpha G$ is graded simple. Let I be a non-zero G-invariant ideal of A. Define $I \star_\alpha G$ to be the set of all finite sums of the form $\sum_{g \in G} a_g\delta_g$, where $a_g \in I \cap D_g$, for $g \in G$. Note that $I \star_\alpha G$ is a non-zero two-sided graded ideal of $A \star_\alpha G$. Hence, $I \star_\alpha G = A \star_\alpha G$. In particular, $A\delta_e \subseteq I \star_\alpha G$, which shows that $I \subseteq A \subseteq I$. We conclude that $I = A$. Thus, A is G-simple.

Now we show the "if" statement. Suppose that A is G-simple. Let J be a non-zero graded ideal of $A \star_\alpha G$. We claim that $J_e = J \cap A$ is a non-zero G-invariant ideal of A. If we assume that the claim holds, then $A = J_e = A \cap J \subseteq J$ from which it follows that $J = A \star_\alpha G$. Now we show the claim. First we show that J_e is non-zero. Since J is non-zero, there is $g \in G$ and a non-zero $a_g \in D_g$ with $a_g\delta_g \in J$. Let $b_{g^{-1}} \in D_{g^{-1}}$ be a local unit for $\alpha_{g^{-1}}(a_g)$. Then

$$J \ni a_g\delta_g b_{g^{-1}}\delta_{g^{-1}} = \alpha_g\big(\alpha_{g^{-1}}(a_g)b_{g^{-1}}\big)\delta_e = \alpha_g\big(\alpha_{g^{-1}}(a_g)\big)\delta_e = a_g\delta_e$$

which is non-zero. Now we show that J_e is G-invariant. Take $g \in G$ and $a \in J_e \cap D_{g^{-1}}$. Let $c_g \in D_g$ be such that $\alpha_{g^{-1}}(c_g)$ is a local unit for a. Then $\alpha_g(a)\delta_e = \alpha_g(\alpha_{g^{-1}}(c_g)a)\delta_e = c_g\delta_g a\delta_{g^{-1}} \in J$. ∎

Remark 4.8 Note that even if there is some $g \in G$ such that D_g does not have local units, the first half of the above proposition still holds as long as $A \star_\alpha G$ is associative. That is, graded simplicity of $A \star_\alpha G$ implies G-simplicity of A. In particular, simplicity of $A \star_\alpha G$ implies G-simplicity of A.

https://doi.org/10.4153/CJM-2014-043-8 Published online by Cambridge University Press

Definition 4.9 Consider A as a semigroup with respect to multiplication. If $g \in G$, then we say that α_g is inner at an idempotent $u \in A$ if it is *inner at* u in the sense of Definition 2.4. Moreover, we say that α is *outer* (or *outer at* u) if there is a non-zero idempotent $u \in A$ such that for each non-identity $g \in G$, the map α_g is outer at u in the sense of Definition 2.4. We say that α is *strongly outer* if for every non-identity $g \in G$, the map α_g is strongly outer in the sense of Definition 2.4.

Remark 4.10 Suppose that A is unital and that $\alpha: G \to \operatorname{Aut}(A)$ is a global action. Then α is outer in the classical sense if and only if it is outer in our sense, *i.e.*, in the sense of Definition 4.9. This follows from Proposition 2.7 and the fact that $u \le 1$ holds for any idempotent u of A.

Suppose that β is a global action of a group G on a ring B and that A is an ideal of B. If, for each $g \in G$, we define $D_g = A \cap \beta_g(A)$ and $\alpha_g(x) = \beta_g(x)$ for $x \in D_{g^{-1}}$, then it is easily verified that α is a partial action of G on A. In this situation, α is referred to as a *restriction* of β, and β is referred to as a *globalization* of α. (See *e.g.*, [6, 8].)

Proposition 4.11 *Let α be a partial action of a group G on a ring A and suppose that α has a globalization β (on a ring B). The following two assertions hold.*

(i) *If u is a non-zero idempotent of A, then, for $g \in G$, the map α_g is inner at u if and only if β_g is inner at u.*
(ii) *If α is outer, then β is outer. Moreover, if B is unital, then β is outer in the classical sense.*

Proof (i) We first show the "if" statement. Suppose that β_g is inner at u. There are elements $a \in uB\beta_g(u)$ and $b \in \beta_g(u)Bu$, satisfying $ab = u$ and $ba = \beta_g(u)$ such that $\beta_g(x) = bxa$ holds for each $x \in uBu$. Note that $bua \in A$, since A is an ideal of B, and that $u = \beta_{g^{-1}}(\beta_g(u)) = \beta_{g^{-1}}(bua)$. This shows that $u \in D_{g^{-1}}$. For any $x \in D_{g^{-1}} \cap uBu$ we have that $\alpha_g(x) = \beta_g(x) = bxa$. In particular, $\alpha_g(u) = \beta_g(u)$. Now, define $a' = ua \in uA\alpha_g(u)$ and $b' = bu \in \alpha_g(u)Au$. It is easy to see that $a'b' = u$ and $b'a' = \alpha_g(u)$. From the fact that $\beta_{g^{-1}}(A) \ni u$ is an ideal of B we get that $uAu \subseteq uBu \subseteq D_{g^{-1}}$. We conclude that $\alpha_g(x) = b'xa'$ holds for any $x \in uAu$. This shows that α_g is inner at u.

We now show the "only if" statement. Suppose that α_g is inner at u. There are elements $a \in uA\alpha_g(u)$ and $b \in \alpha_g(u)Au$, satisfying $ab = u$ and $ba = \alpha_g(u)$, such that $\alpha_g(x) = bxa$ holds for each $x \in uAu$. Using that α is a restriction of β, we know that $\alpha_g(x) = \beta_g(x)$ holds for each $x \in D_{g^{-1}}$. Note that $uAu = uBu$, since u is an idempotent of A that is an ideal of B. Hence, $uBu \subseteq D_{g^{-1}}$ and we conclude that $\beta_g(x) = \alpha_g(x) = bxa$ holds for each $x \in uBu$. In particular, $\beta_g(u) = \alpha_g(u)$, which makes it easy to see that a and b have the desired properties. This shows that β_g is outer at u.

(ii) Suppose that α is outer. There is a non-zero idempotent $u \in A$ such that for each non-identity $g \in G$, the map α_g is outer at u. It now follows immediately from (i) that, for each non-identity $g \in G$, the map β_g is outer at u. This shows that β is outer. For the proof of the last part, we assume that B is unital. Seeking a contradiction, suppose that β is not outer (in the classical sense). Then there is a non-identity $g \in G$

such that the automorphism $\beta_g\colon B \to B$ is inner at 1. Since $u \le 1$, Proposition 2.7 yields that β_g is inner at u, which is a contradiction. ∎

Remark 4.12 Note that Proposition 4.11 does not make use of the assumption, on the existence of local units, that is made in the beginning of Section 4.

Remark 4.13 Note that the converse of Proposition 4.11(ii) does not hold in general. In light of Remark 2.8, we want to underline that even if α_g, for some $g \in G$, is inner at an idempotent of A, it is fully possible for the globalization β to be outer (in the classical sense). In fact, β could potentially be outer at any idempotent, as long as the idempotent lies outside of A.

Proof of Theorem 1.2 The "only if" statement follows from Proposition 4.7 and the fact that graded simplicity is a necessary condition for simplicity. Now we show the "if" statement. Suppose that A is a G-simple ring. Let u be a non-zero idempotent of A such that for each non-identity $g \in G$, the map α_g is outer at u. Put $S = (u\delta_e)(A \star_\alpha G)(u\delta_e)$. By Theorem 3.1, we are done if we can show that $Z(S)$ is a field. Let $(u\delta_e)(\sum_{g\in G} a_g\delta_g)(u\delta_e)$ be a non-zero element of $Z(S)$, where $a_g \in D_g$ is zero for all but finitely many $g \in G$. Fix $g \in G$ so that $(u\delta_e)(a_g\delta_g)(u\delta_e) \neq 0$. Since G is abelian, we get that $(u\delta_e)(a_g\delta_g)(u\delta_e) \in Z(S)$. Since $A \star_\alpha G$ is graded simple, it is easy to see that S is also graded simple. Therefore, the graded ideal of S generated by $(u\delta_e)(a_g\delta_g)(u\delta_e)$ equals S. So, in particular, there is $k \in D_{g^{-1}}$ such that

$$(u\delta_e)(a_g\delta_g)(u\delta_e)(k\delta_{g^{-1}})(u\delta_e) = u\delta_e, \tag{4.1}$$

which is equivalent to the following four equivalent equations

$$\begin{aligned}(ua_g\delta_g)(uk\delta_{g^{-1}})(u\delta_e) = u\delta_e &\iff (\alpha_g(\alpha_{g^{-1}}(ua_g)uk)\delta_e)(u\delta_e) = u\delta_e\\ &\iff (ua_g\alpha_g(uk)\delta_e)(u\delta_e) = u\delta_e\\ &\iff ua_g\alpha_g(uk)u\delta_e = u\delta_e,\end{aligned}$$

which finally gives us that

$$ua_g\alpha_g(uk)u = u. \tag{4.2}$$

Note that equation (4.2) implies that $u \in D_g$. Since $(u\delta_e)(a_g\delta_g)(u\delta_e) \in Z(S)$, we can change the order of the factors on the left-hand side of equation (4.1) and obtain the following three equivalent equations

$$\begin{aligned}(u\delta_e)(k\delta_{g^{-1}})(u\delta_e)(a_g\delta_g)(u\delta_e) = u\delta_e &\iff (uk\delta_{g^{-1}})(ua_g\delta_g)(u\delta_e) = u\delta_e\\ &\iff \alpha_{g^{-1}}(\alpha_g(uk)ua_g)\delta_e(u\delta_e) = u\delta_e,\end{aligned}$$

which are equivalent to

$$\alpha_{g^{-1}}(\alpha_g(uk)ua_g)u = u. \tag{4.3}$$

Note that equation (4.3) implies that $u \in D_{g^{-1}}$, and therefore

$$\alpha_g(uk)ua_g\alpha_g(u) = \alpha_g(u). \tag{4.4}$$

Using again that $u \in D_{g^{-1}}$, we can rewrite equations (4.2) and (4.4) as

$$ua_g\alpha_g(u)\alpha_g(u)\alpha_g(k)u = u$$

and

$$\alpha_g(u)\alpha_g(k)uua_g\alpha_g(u) = \alpha_g(u) \tag{4.5}$$

respectively. Furthermore, for every $b \in A$, the following three equivalent equations hold

$$\begin{aligned}(u\delta_e)(a_g\delta_g)(u\delta_e)(b\delta_e)(u\delta_e) &= (u\delta_e)(b\delta_e)(u\delta_e)(a_g\delta_g)(u\delta_e)\\ \Longleftrightarrow (ua_g\delta_g)(ubu\delta_e) &= (ubu\delta_e)(\alpha_g(\alpha_{g^{-1}}(ua_g)u)\delta_g)\\ \Longleftrightarrow \alpha_g(\alpha_{g^{-1}}(ua_g)ubu)\delta_g &= ubua_g\alpha_g(u)\delta_g.\end{aligned}$$

The last equation yields

$$ua_g\alpha_g(u)\alpha_g(ubu) = ubua_g\alpha_g(u).$$

By equation (4.5), the last equation implies that

$$\alpha_g(ubu) = \alpha_g(u)\alpha_g(k)uubuua_g\alpha_g(u)$$

which shows that α_g is inner at u. But since α_g is outer, at u, for non-identity $g \in G$, we conclude that $g = e$. Hence, finally, by equation (4.1), we get that $Z(S)$ is a field. ∎

Remark 4.14 We will now make a couple of important observations.

(a) Outerness is not a necessary condition for simplicity of a partial skew group ring $A \star_\alpha G$. Indeed, consider the simple skew group ring $M_2(\mathbb{R}) \rtimes_\sigma \mathbb{Z}/2\mathbb{Z}$ in [22, Example 4.1].

(b) Theorem 1.2 does not hold for arbitrary (non-abelian) groups. Indeed, consider [22, Example 5.1] where $X = S^1$ is the circle, $G = \mathrm{Homeo}(S^1)$ is the group of all homeomorphisms of S^1. One can define $\sigma: G \to \mathrm{Aut}(C(X))$ in the usual way. It then turns out that $C(X)$ is G-simple and that the action is outer. However, the skew group ring $C(X) \rtimes_\sigma G$ is not simple.

5 An Application to Set Dynamics

At the end of this section, we use Theorem 1.2 to prove Theorem 1.3.

Assumption *Throughout this section, B denotes a simple associative ring that has local units, θ denotes a partial action of a group G on a non-empty set X, and the corresponding subsets of X are denoted by X_g, for $g \in G$.*

Definition 5.1 We let $F_0(X,B)$ denote the set of functions $X \to B$ with finite support. For each $g \in G$, let D_g denote the set of $f \in F_0(X,B)$ such that $f(x) = 0$ for all $x \in X \setminus X_g$. It is clear that D_g is an ideal of $F_0(X,B)$ and that the map

$$G \ni g \mapsto (\alpha_g: D_{g^{-1}} \to D_g),$$

defined by $\alpha_g(f) = f \circ \theta_{g^{-1}}$, for $f \in D_{g^{-1}}$, defines a partial action of G on $F_0(X,B)$.

Remark 5.2 For each subset S of X and each $b \in B$, let b_S denote the function $X \to B$ defined by $b_S(x) = b$, if $x \in S$, and $b_S(x) = 0$, otherwise. If $S = \{y\}$ for some $y \in X$, and $b \in B$, then we let b_S be denoted by b_y. It is clear that for each $g \in G$, the set of ϵ_S, where S is a finite subset of X_g and ϵ is a local unit in B is a set of local units for D_g. In particular,

$$E = \{\epsilon_S \mid S \text{ is a finite subset of } X \text{ and } \epsilon \text{ is a local unit in } B\}$$

is a set of local units for $F_0(X, B)$.

For future reference we record the following result.

Proposition 5.3 *If θ is a partial action of an abelian group G on a set (Hausdorff topological space) X such that θ is faithful and (topologically) minimal, then θ is free.*

Proof Take a non-identity $g \in G$ and consider the set

$$F_g = \{x \in X_{g^{-1}} \mid \theta_g(x) = x\}.$$

We need to show that F_g is empty. Take $h \in G$ and $x \in F_g \cap X_{h^{-1}}$. By the relations (b)–(c) in the definition of a partial action, and the fact that G is abelian, we get that $\theta_h(x) = \theta_h(\theta_g(x)) = \theta_{hg}(x) = \theta_{gh}(x) = \theta_g(\theta_h(x))$. Thus, F_g is G-invariant (and closed since X is Hausdorff). Since θ is faithful, we get that $F_g \neq X$. Hence, we get that $F_g = \varnothing$. Thus, θ is free. ∎

Proposition 5.4 *θ is minimal if and only if $F_0(X, B)$ is G-simple.*

Proof Suppose that $F_0(X, B)$ is not G-simple. Then there is a non-trivial G-invariant ideal I of $F_0(X, B)$. Let $N_I = \bigcap_{f \in I} f^{-1}(\{0\})$. Since I is G-invariant, the same is true for N_I. Since I is non-zero, it follows that N_I is a proper subset of X. Seeking a contradiction, suppose that N_I is empty. Take $x \in X$ and $b \in B$. We claim that $b_x \in I$. If we assume that the claim holds, then since the set of b_x, for $x \in X$ and $b \in B$, generates $F_0(X, B)$, we will get the contradiction $I = F_0(X, B)$. Now we show the claim. From $N_I = \varnothing$, it follows that there is a non-zero $c \in B$ such that $c_x \in I$. By simplicity of B, there is a natural number n and $d^{(1)}, \dots, d^{(n)}, d'^{(1)}, \dots, d'^{(n)} \in B$ such that $b = \sum_{i=1}^n d^{(i)} c d'^{(i)}$. But then $b_x = \sum_{i=1}^n d_x^{(i)} c_x d_x'^{(i)} \in I$, which proves the claim. Therefore, N_I is a non-empty G-invariant subset of X, and hence θ is not minimal.

Now suppose that θ is not minimal. Let Y be a non-trivial G-invariant subset of X. Let I_Y denote the ideal of $F_0(X, B)$ consisting of all $f \in F_0(X, B)$ that vanish on Y. Since Y is G-invariant, it follows that I_Y is G-invariant. Using that $\varnothing \neq Y \neq X$, we conclude that I_Y is a non-zero proper ideal of $F_0(X, B)$. Thus, $F_0(X, B)$ is not G-simple. ∎

Proposition 5.5 *If α is injective, then θ is faithful.*

Proof Suppose that θ is not faithful. Then there is a non-identity $g \in G$ such that $\theta_g(x) = x$ for $x \in X_{g^{-1}}$. This implies that $X_g = X_{g^{-1}}$ and thus that $D_g = D_{g^{-1}}$ and $\alpha_g(f) = f$, for $f \in D_{g^{-1}}$. Thus, α is not injective. ∎

Proposition 5.6 *If θ is free, then α is strongly outer.*

Proof Suppose that α is not strongly outer. We show that θ is not free. Choose a non-zero idempotent $u \in F_0(X,B)$ and a non-identity $g \in G$ such that α_g is inner at u. Pick $x \in X$ such that $b = u(x) \neq 0$. Then $b_x \leq u$ in the sense of Definition 2.6. By Proposition 2.7, we get that α_g is inner at b_x. In particular, there are $f, f' \in F_0(X,B)$ such that $b_x f \alpha_g(b_x) f' b_x = b_x$, or equivalently, $b_x f b_{\theta_g(x)} f' b_x = b_x$. Therefore, we get that

$$b_x(x) f(x) b_{\theta_g(x)}(x) f'(x) b_x(x) = b_x(x) = b \neq 0$$

from which it follows that $\theta_g(x) = x$. This shows that θ is not free. ∎

Proof of Theorem 1.3 (i)⇒(iii): Suppose that $F_0(X,B) \star_\alpha G$ is simple. Clearly, $F_0(X,B) \star_\alpha G$ is graded simple and hence, by Proposition 4.7, we get that $F_0(X,B)$ is G-simple. By Proposition 5.4, we get that θ is minimal. By Proposition 4.4, we conclude that α is injective, and hence, by Proposition 5.5, θ is faithful.

(iii)⇒(ii): This follows immediately from Proposition 5.3.

(ii)⇒(i): Suppose that θ is minimal and free. By Propositions 5.4 and 5.6, we get, respectively, that $F_0(X,B)$ is G-simple and that α is strongly outer. Theorem 1.2 implies that $F_0(X,B) \star_\alpha G$ is simple. ∎

6 An Application to Topological Dynamics

At the end of this section, we use Theorem 1.2 to prove Theorem 1.4.

Assumption *Throughout this section, θ denotes a partial action of a group G on a topological space X, and the corresponding subsets of X are denoted by X_g, for $g \in G$. Let B denote a simple associative topological real algebra which has a set E of local units. Let $C_E(X,B) = \cup_{\epsilon \in E} C(X, \epsilon B \epsilon)$ where*

$$C(X, \epsilon B \epsilon) = \{\text{continuous } f : X \to \epsilon B \epsilon\}.$$

We postulate that B satisfies the following property:

(P) *There is a continuous map $q: B \to \mathbb{R}_{\geq 0}$ satisfying $q(b) > 0$, for non-zero $b \in B$, and $(q \circ f)\epsilon_X \in I$ for every ideal I of $C_E(X,B)$ and every $f \in I \cap C(X, \epsilon B \epsilon)$.*

Remark 6.1 If E and E' are sets of local units for B, then $C_E(X,B) = C_{E'}(X,B)$. In particular, if B is unital, then $C_E(X,B) = C(X,B)$, and the postulate (P) simplifies to

(P1) *There is a continuous map $q: B \to \mathbb{R}_{\geq 0}$ satisfying $q(b) > 0$, for non-zero $b \in B$, and $q \circ f \in I$ for every ideal I of $C(X,B)$ and every $f \in I$.*

Now we show that there are lots of rings B that satisfy the postulate (P).

Example 6.2 Suppose that K denotes any of the unital rings of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$ equipped with their respective conjugation $\bar{\cdot}$, norm $|\cdot|$ and topology. Define $q: K \to \mathbb{R}_{\geq 0}$ by $q(k) = k\bar{k} = |k|^2$. Then, of course, $q(k) > 0$, for non-zero $k \in K$. If I is an ideal of $C(X,K)$, then $q \circ I \subseteq I\bar{I} \subseteq I$, so (P1) is satisfied.

Example 6.3 Let K be defined as in Example 6.2. Let n denote a positive integer and let B denote the unital ring $M_n(K)$ of $n \times n$ matrices over K. Extend $\bar{\cdot}$ to B by elementwise conjugation. For $1 \le i, j \le n$, let e_{ij} denote the matrix with 1 in the ij-th position and 0 elsewhere. For a matrix $b = (a_{ij})$ in B, let $q(b) = \sum_{1 \le i,j \le n} |a_{ij}|^2$. It is clear that q is continuous as a map $B \to \mathbb{R}$ and that $q(b) > 0$ for non-zero $b \in B$. Let I be an ideal of $C(X, B)$ and suppose that $f \in I$. Then for every choice of $i, j \in \{1, \dots, n\}$, there is a continuous map $f_{ij}: X \to B$ such that $f = \sum_{1 \le i,j \le n} f_{ij} e_{ij}$. Therefore, we get that

$$\begin{aligned} q \circ f &= \sum_{1 \le i,j \le n} |f_{ij}|^2 = \sum_{1 \le i,j \le n} f_{ij} \overline{f}_{ij} \\ &= \sum_{1 \le i,j,k \le n} e_{ki} f e_{jk} \overline{f}_{ij} \in \sum_{1 \le i,j,k \le n} e_{ki} I e_{jk} \overline{f}_{ij} \subseteq I \end{aligned}$$

and hence (P1) holds.

Example 6.4 Let K be defined as in Example 6.2. Let $B = \cup_{n \in \mathbb{N}} M_n(K)$. Note that if $m, n \in \mathbb{N}$ satisfy $m \le n$, then we can consider $M_m(K) \subseteq M_n(K)$ in the classical way. Namely, with each $(a_{ij}) \in M_m(K)$, we associate $(a'_{ij}) \in M_n(K)$, where $a'_{ij} = a_{ij}$, if $1 \le i, j \le m$, and $a'_{ij} = 0$, otherwise. Then B is a ring that has a set of local units E consisting of the matrices $\epsilon^{(n)} = \sum_{i=1}^{n} e_{ii}$, for $n \in \mathbb{N}$. Take $b \in B$. Then $b \in M_n(K)$ for some $n \in \mathbb{N}$. Define $q(b)$ as in Example 6.3. It is clear that $q(b) > 0$ if b is non-zero. Take an ideal I of $C_E(X, B)$ and $f \in I \cap C(X, \epsilon^{(n)} B \epsilon^{(n)})$, for some $n \in \mathbb{N}$. Then f belongs to $\epsilon_X^{(n)} I \epsilon_X^{(n)}$, which is an ideal in the unital ring $C(X, \epsilon^{(n)} B \epsilon^{(n)})$. Hence, by Example 6.3, we get that $(q \circ f)\epsilon_X^{(n)} \in \epsilon_X^{(n)} I \epsilon_X^{(n)} \subseteq I$. Therefore, postulate (P) holds.

Definition 6.5 For each $g \in G$, let D_g denote the set of $f \in C_E(X, B)$ such that $f(x) = 0$ for all $x \in X \smallsetminus X_g$. It is clear that D_g is an ideal of $C_E(X, B)$.

Remark 6.6 The set of all ϵ_X, for $\epsilon \in E$, is a set of local units for $C_E(X, B)$.

Proposition 6.7 *If each X_g, for $g \in G$, is clopen, then the map*

$$G \ni g \mapsto (\alpha_g: D_{g^{-1}} \to D_g),$$

defined by $\alpha_g(f) = f \circ \theta_{g^{-1}}$, for $f \in D_{g^{-1}}$, defines a partial action of G on $C(X, B)$.

Proof All we need to show is that α_g is well defined. Take $f \in D_{g^{-1}}$. We need to show that the map $h: X \to B$ defined by $h(x) = f(\theta_{g^{-1}}(x))$, for $x \in X_g$, and $h(x) = 0$, for $x \in X \smallsetminus X_g$, is continuous. Suppose that U is an open ball in B. We now consider two cases.

Case 1: $0 \notin U$. Then $h^{-1}(U) = (f \circ \theta_{g^{-1}})^{-1}(U)$, which is open in X_g and hence is open in X.

Case 2: $0 \in U$. Then $h^{-1}(U) = (f \circ \theta_{g^{-1}})^{-1}(U) \cup (X \smallsetminus X_g)$, which, by Case 1 and the fact that X_g is clopen, is open in X. ∎

Proposition 6.8 *If X is compact Hausdorff and each X_g, for $g \in G$, is clopen, then θ is topologically minimal if and only if $C(X, B)$ is a G-simple ring.*

Proof Suppose that $C(X,B)$ is not G-simple. There is a non-trivial G-invariant ideal I of $C(X,B)$. For a subset J of I, let N_J be the set $\bigcap_{f\in J} f^{-1}(\{0\})$. We claim that N_I is a closed, non-empty, proper, G-invariant subset of X. If we assume that the claim holds, then θ is not minimal. Now we prove the claim. Since I is G-invariant, the same is true for N_I. Since I is non-zero, it follows that N_I is a proper subset of X. Since each set $f^{-1}(\{0\})$, for $f \in I$, is closed, the same is true for N_I. Seeking a contradiction, suppose that N_I is empty. Since X is compact, there is a finite subset J of I such that $N_J = N_I = \varnothing$. Take an arbitrary non-zero local unit ϵ in B. Take another non-zero local unit ϵ' in B such that $\epsilon\epsilon' = \epsilon'\epsilon = \epsilon$ and $f \in C(X, \epsilon' B \epsilon')$, for all $f \in J$. Now define $g \in I$ by $g = \sum_{f\in J}(q\circ f)\epsilon'_X$. Since N_J is empty, we get that $\sum_{f\in J}(q\circ f)(x) > 0$ for all $x \in X$. Therefore, we get that g is invertible in the ring $\epsilon'_X C(X,B)\epsilon'_X$, which in turn implies that $\epsilon'_X \in I$. Hence $\epsilon_X = \epsilon_X \epsilon'_X \in I$. Since ϵ was arbitrarily chosen, we get that $I = C(X,B)$, which is a contradiction, and therefore N_I is non-empty.

Now suppose that θ is not minimal. We show that $C(X,B)$ is not G-simple. Let Y be a non-trivial, closed, G-invariant subset of X. Let I_Y denote the ideal of $C(X,B)$ consisting of all $f \in C(X,B)$ that vanish on Y. Since Y is G-invariant, it follows that I_Y is G-invariant. Now we show that I_Y is non-zero. Suppose that ϵ is a non-zero local unit in B. Since X is compact Hausdorff it is completely regular. Hence there is a non-zero continuous $f\colon X \to \mathbb{R}$ such that $f|_Y = 0$. Pick a non-zero $\epsilon \in B$ and define a continuous $\widetilde{f}\colon X \to B$ by $\widetilde{f}(x) = f(x)\epsilon$, for $x \in X$. Then $\widetilde{f} \in I_Y$ and therefore $I_Y \neq \{0\}$. Also, $I_Y \neq C(X,B)$. In fact, for every non-zero $b \in B$, the constant function $b_X \in C(X,B) \smallsetminus I_Y$. Thus, $C(X,B)$ is not G-simple. ∎

Proposition 6.9 *Suppose that X is compact Hausdorff and each X_g, for $g \in G$, is clopen. If θ is topologically free, and $\epsilon \in E \smallsetminus \{0\}$, then α is outer at ϵ_X.*

Proof Suppose that α is not outer at ϵ_X. We show that θ is not topologically free. Choose a non-identity $g \in G$ such that α_g is inner at ϵ_X. This implies in particular that $\epsilon_X \in D_g \cap D_{g^{-1}}$ and thus $X_g = X_{g^{-1}} = X$. Therefore, there are $f, f' \in C(X,B)$ such that $\epsilon_X f \alpha_g(\epsilon_X) f' \epsilon_X = \epsilon_X$ and $\alpha_g(\epsilon_X) f' \epsilon_X f \alpha_g(\epsilon_X) = \alpha_g(\epsilon_X)$ and $\alpha_g(\epsilon_X h \epsilon_X) = f' h f$ for all $h \in C(X,B)$. In particular, if we insert $h = r\epsilon_X$, where $r \in C(X,\mathbb{R})$, into the last equation, then we get that $r \circ \theta_{g^{-1}} = r$ which, in turn, by Urysohn's lemma, implies that $\theta_g = \mathrm{id}_X$. Thus, θ is not topologically free. ∎

Proof of Theorem 1.4 (i)⇒(iii): Suppose that $C_E(X,B) \star_\alpha G$ is simple. Clearly, $C_E(X,B) \star_\alpha G$ is graded simple, and hence, by Proposition 4.7, we get that $C_E(X,B)$ is G-simple. By Proposition 6.8, we get that θ is topologically minimal. By Proposition 4.4 we conclude that α is injective, and hence, by Proposition 5.5, θ is faithful.

(iii)⇒(ii): This follows immediately from Proposition 5.3.

(ii)⇒(i): Suppose that θ is topologically minimal and topologically free. Take any non-zero $\epsilon \in E$. By Propositions 6.8 and 6.9, we get that $C_E(X,B)$ is G-simple and that α is outer at ϵ_X, respectively. Theorem 1.2 implies that $C_E(X,B) \star_\alpha G$ is simple. ∎

Acknowledgment The authors are grateful to an anonymous referee for having provided them with valuable feedback and useful comments.

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