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A NON-STANDARD CHARACTERIZATION OF PERFECT MAPPINGS

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The central theorem of this note was proposed to me by I. Juhász on a napkin in Canter's restaurant in Pittsburgh, Pennsylvania. We agreed that if the proposition could be proven it would have some elegant applications and although it required a slight change before it could be proven, hopefully those expectations will be realized in this note.

The reader is assumed to be familiar with the fundamentals of non-standard analysis. Throughout this paper all spaces are assumed to be Hausdorff and to belong to some model for which we take an arbitrary but fixed enlargement. For any topological space $(X, \mathscr{T})$, the near-standard points of the enlargement $*X$ of X will be denoted by $\mathrm{ns}(*X)$, that is, $\mathrm{ns}(*X) = \bigcup\{\mu(x): x \in X\}$, and the topology on $*X$ for which $*\mathscr{T}$ is a base is called the Q-topology. Most of the results concerning the Q-topology in this note can be found in [**1**] or [**4**]. We begin with two necessary technical results.

THEOREM 1. *If $(X, \mathscr{T})$ any regular topological space, then any closed subset A of X is compact if and only if for any enlargement $*X$ of X, $\mu(A) = \bigcup\{\mu(x): x \in A\}$.*

Proof. Suppose that $\mu(A) = \bigcup\{\mu(x): x \in A\}$. Then, since $*A \subseteq \mu(A)$, each point of $*A$ is near standard, which is equivalent to compactness for closed sets in regular spaces [**3**].

For any set $A \subseteq X$, $\mu(A) \supseteq \bigcup\{\mu(x): x \in A\}$. Let $A \subseteq X$ be compact and let $z \in \mu(A)$, so that $z \in *U$ for each neighborhood U of A. If

$$z \notin \bigcup\{\mu(x): x \in A\},$$

then $\mathscr{U} = \{U \in \mathscr{T}: z \notin *U\}$ is an open cover of A, since for each $x \in A$ there is an open neighborhood U of x with $z \notin *U$. Let $\{U_1, \ldots, U_n\}$ be a finite subcover of $\mathscr{U}$. Then $z \notin *U_1 \cup \ldots \cup *U_n = *(U_1 \cup \ldots \cup U_n)$, but $U_1 \cup \ldots \cup U_n$ is an open neighborhood of A, contradicting $z \in \mu(A)$. Hence, $z \in \bigcup\{\mu(x): x \in A\}$, or $\mu(A) \subseteq \bigcup\{\mu(x): x \in A\}$.

THEOREM 2. *Let X and Y be arbitrary topological spaces and let $f: X \rightarrow Y$ be continuous and closed. Then for each subset S of Y, $f^{-1}(\mu(S)) \subseteq \mu(f^{-1}(S))$.*

Proof. Let $S \subseteq Y$ be given and let $z \in *X \backslash \mu(f^{-1}(S))$. Then there is an open neighborhood U of f^{-1} with $z \notin *U$. Because f is a closed mapping there is an open neighborhood V of S with $f^{-1}(V) \subseteq U$. Because f is continuous, $f^{-1}(V) \supseteq f^{-1}(S)$ is open, so $\mu(f^{-1}(S)) \subseteq *(f^{-1}(V))$. Now, $z \notin *(f^{-1}(V))$ so $z \notin \mu(f^{-1}(S))$.

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Recall that a function $f: X \to Y$ is said to be *perfect* if it is a continuous closed surjection such that for each $y \in Y, f^{-1}(y)$ is compact in X.

THEOREM 3. *Let X be a regular space and let $f: X \to Y$ be a closed surjection. Then f is perfect if and only if:*

(i) *For each $x \in X, f(\mu(x)) \subseteq \mu(f(x))$.*

(ii) *For each remote $z \in {}^*X, f(z)$ is remote in ${}^*Y$.*

Proof. It is well known that condition (*i*) is equivalent to the continuity of f.

Suppose that f satisfies conditions (i) and (ii). We need to show that for each $y \in Y, f^{-1}(y)$ is compact in X. For each $y \in Y$, each point of ${}^*(f^{-1}(y))$ is near-standard by (ii), and $f^{-1}(y)$ is closed by the continuity of f, so $f^{-1}(y)$ is compact.

If f is perfect, then it is continuous and closed. By Theorem 2, for each $y \in Y$, $f^{-1}(\mu(y)) \subseteq \mu(f^{-1}(y))$. Also, $f^{-1}(y)$ is compact, so by Theorem 1, $\mu(f^{-1}(y)) = \bigcup\{\mu(x): x \in f^{-1}(y)\}$, so that each point of $f^{-1}(\mu(y))$ is near-standard. Hence, f maps the remote points of ${}^*X$ onto the remote points of ${}^*Y$.

THEOREM 4. *Let X and Y be any topological spaces and let $f: X \to Y$ be perfect. Then for each $x \in {}^*X$, x is remote if and only if $f(x)$ is remote in ${}^*Y$.*

Proof. In the second part of the proof of Theorem 1 the regularity of X was not needed.

Note that for any spaces X and Y, and perfect $f: X \to Y$, $f(\mathrm{ns}({}^*X)) = \mathrm{ns}({}^*Y)$, and $\mathrm{ns}({}^*X) = f^{-1}(\mathrm{ns}({}^*Y))$.

Let us now reprove some standard results using Theorem 4. The proofs which follow were selected because they are shorter and more elegant than their standard counterparts.

THEOREM 5. *Let $f: X \to Y$ be perfect. Then X is regular if and only if Y is regular.*

Proof. Any space X is regular if and only if $\mathrm{ns}({}^*X)$ is Q-closed. Any function $f: X \to Y$ is closed if and only if ${}^*f: {}^*X \to {}^*Y$ is Q-closed, and is continuous if and only if ${}^*f: {}^*X \to {}^*Y$ is Q-continuous [**1**]. By the above remark, $\mathrm{ns}({}^*X)$ is Q-closed if and only if $\mathrm{ns}({}^*Y)$ is Q-closed.

THEOREM 6. *If Y is compact and if $f: X \to Y$ is perfect, then X is compact.*

Proof. If X is not compact, then there is a remote point $x \in {}^*X$ and $f(x)$ is remote in ${}^*Y$, so Y is not compact.

In [**1**] it was shown that a topological space X is locally compact if and only if for each point $x \in X$ there is a *compact *neighborhood U of x contained in $\mu(x)$.

THEOREM 7. *Let $f: X \to Y$ be perfect. Then X is locally compact if and only if Y is locally compact.*

Proof. Let X be locally compact, $y \in Y$. Pick any $x \in f^{-1}(y)$ and let U be a *compact *neighborhood of x contained in $\mu(x)$ so $f(U) \subseteq \mu(y)$ is *compact.

If $V \subseteq U$ is an *open *neighborhood of x, then $^*Y \backslash f(^*X \backslash V) \subseteq f(U)$ is a *compact *neighborhood of y.

Now let Y be locally compact and let $x \in X$. There is a *compact *neighborhood $U \subseteq \mu(f(x))$ of $f(x)$ so $f^{-1}(U) \subseteq \mu(f^{-1}(f(x))$ is a *compact *neighborhood of $f^{-1}(f(x))$. $\mu(x)$ contains a *closed *neighborhood V of x by regularity, so $f^{-1}(U) \cap V$ is the required *compact *neighborhood of x.

References

1. R. W. Button, *Monads for regular and normal spaces*, Notre Dame Journal of Formal Logic *3* (1976) 449–456.
2. J. Dugundji, *Topology* (Allyn and Bacon, Inc., Boston, 1966).
3. M. Machover and J. Hirschfeld, *Lectures on non-standard analysis*, Lecture Notes in Mathematics *94* (Springer-Verlag, Berlin, 1969).
4. A. Robinson, *Non-standard analysis, studies in logic and the foundations of mathematics* (North-Holland Publishing Co., Amsterdam, 1966).

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