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EXTREMA OF A CLASS OF FUNCTIONS ON A FINITE SET

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In this paper, we are concerned with the following problem: Let S be a finite set and $S_m^* \subset 2^S$ a collection of subsets of S each of whose members has m elements (m a positive integer). Let f be a real-valued function on S and, for $p \in S_m^*$, define $f(P)$ as $\sum_{s \in P} f(s)$. We seek the minimum (or maximum) of the function f on the set S_m^*.

The Traveling Salesman Problem is to find the cheapest polygonal path through a given set of vertices, given the cost of getting from any vertex to any other. It is easily seen that the Traveling Salesman Problem is a special case of this system, where S is the set of all edges joining pairs of points in the vertex set, S_m^* is the set of polygons, each polygon has m elements (m = no. of points in the vertex set = no. of edges per polygon), and f is the cost function.

Definition. Let S be a set, $S_m^* \subset 2^S$ as above. A set $P \in S_m^*$ can be considered as a 0-1 vector in $|S|$-dimensional space (where $|\ |$ = cardinality).

1. Theory of rearrangeability. If $P, Q \in S_m^*$, we say that P, Q are rearrangeable with respect to S_m^* if a positive integral linear combination of P and Q (thought of as vectors) is equal to a positive integral linear combination $\sum a_i A_i$ of members of $S_m^* - \{P, Q\}$. In this event we call $\sum a_i A_i$ a rearrangement of (P, Q). This rearrangement can also be interpreted as a list $\{P_i\}$ of members of $S_m^* - \{P, Q\}$, i.e., if $B = \sum a_i A_i \quad (A_i \in S_m^* - \{P, Q\})$, then $P_1, \ldots P_{a_1} = A_1, P_{a_1+1}, \ldots P_{a_1+a_2} = A_2$, etc.

It is clear that P and Q are symmetric in this definition, so if B is a rearrangement of (P, Q), it is also a rearrangement of (Q, P). An example of 2 polygons being rearrangeable is: $P = (123456)$, $Q = (136245)$, where the notation has the obvious (cyclic) meaning, the points having been arbitrarily labeled 1–6. Letting $P_1 = (134562)$, $P_2 = (154236)$, we have a rearrangement. It will be noted that the list of edges formed from P and Q is identical to that from P_1 and P_2, so the traversing of P, then Q is just a reordering of the operations involved in traversing P_1, then P_2. Note also that we are using our assumption that each edge is one variable, independent of direction. Of course, (P, Q) is also a rearrangement of (P_1, P_2). We have conjectured that for polygons, if (P, Q) are rearrangeable, there exists a rearrangement of only 2 elements

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(for $B = \sum a_i A_i$, $\sum a_i = 2$), but have been unable to settle the question. This conjecture is definitely not true in general.

An alternate definition, given in [**2**], is that two members of S_m^* are not rearrangeable if they are identical or are polytopal neighbors in the polytype defined by the set of vectors S_m^* as vertices.

LEMMA 1. *$|P_i \cap P| > |P \cap Q|$, where $\{P_i\}$ is a rearrangement of (P, Q).*

Proof. $\bigcap P_i = P \cap Q$, so for each i $P \cap Q \subset P_i$, so $P \cap P \cap Q \subset P \cap P_i$. Suppose $P \cap Q = P \cap P_i$. Then $s \in P \cap P_i \Rightarrow s \in Q$. Also, $s \in P_i - P \Rightarrow s \in Q$, since $s \in P \cup Q$ by the definition of rearrangement. Thus, $s \in P_i \Rightarrow s \in Q$. Since $|P_i| = |Q| = m$ finite,and $P_i \subset Q$, $P_i = Q$, contradicting the definition of rearrangement. Thus it is impossible that $P \cap Q = P \cap P_i$, so $P \cap Q \subset P \cap P_i$, but $P \cap Q \neq P \cap P_i$, whence, $P \cap Q$ and $P \cap P_i$ both having finite order, $|P \cap P_i| > |P \cap Q|$.

LEMMA 2. *Let $\{P_i\}$ be a rearrangement of (P, Q). Then $a_P f(P) + a_Q f(Q) = \sum_i f(P_i)$.*

Proof. Defining $f(L)$ for any list of elements of S as $f(L) = \sum_{s \in L} f(s)$, we get

$$\sum_i f(P_i) = \sum_i \sum_{s \in P_i} f(s) = f\left(\sum_i P_i\right) = f(a_P P + a_Q Q) = a_P f(P) + a_Q f(Q).$$

Having established these basic properties of rearrangeability, we will now begin to study the very close relationship this concept bears to the problem of minimizing the function f.

THEOREM 1. *Let S, S_m^*, f be as before, and let $P \in S_m^*$ be such that $f(P) \leqq f(Q)$ for all $Q \in S_m^*$ such that (P, Q) are not rearrangeable* (with respect to S_m^*). *Then*

$$f(P) = \min_{Q \in S_m^*} f(Q).$$

Also, if $f(P) \geqq f(Q)$ for all such Q,

$$f(P) = \max_{Q \in S_m^*} f(Q).$$

In particular, if polygon P is minimal with respect to all polygons with its vertex set which are not rearrangeable with P, then P is a solution to the traveling salesman problem for that cost function.

We shall present a proof for the minimum problem; the proof for the maximum follows exactly the same scheme.

Proof. Let $Q \in S_m^*$. If (P, Q) are not rearrangeable, $f(P) \leqq f(Q)$ by hypothesis. Suppose then that (P, Q) are rearrangeable. Then there exists a rearrangement $\{P_i\}$ of (P, Q). For each i, either (P_i, P) are rearrangeable or not. If they are, let $\{P_{ij}\}$ be a rearrangement of (P_i, P). If not, let $P_{i1} = P_i$, $P_{i2} = P$. Continue in this way for each P_{ij}, forming $\{P_{ijk}\}$, etc. If (P_t, P) are not rearrangeable, where t is a sequence (of integers), then clearly (P_{ti}, P) are not, since $P_{t1} = P_t$ is not rearrangeable with P, and $P_{t2} = P$ is not rearrangeable with P. Now suppose P_t is rearrangeable with P. Then P_r, where r is an "initial segment" of t, is rearrangeable with P. Now P, Q have at least 0 sides in common, whence (P_i, P) have at least one by Lemma 1, (P_{ij}, P) have at least two, and (P_t, P) have at least the number of digits in the subscript t. Since $|P| = m$, it follows that if (P_t, P) are rearrangeable, t has no more than m digits. Thus, for t having $m + 1$ digits, (P_t, P) are not rearrangeable. Furthermore, $\sum_i f(P_{ti}) = a_{Pt}f(P_t) + a_{P^t}f(P)$ for each P_t, for either $P_{t_1} = P_t$, $P_{t_2} = P$ whence $\sum_i f(P_{ti}) = f(P_t) + f(P)$, or $\{P_{ti}\}$ is a rearrangement of (P_t, P) and the contention holds by Lemma 2. Now consider any t having m digits. Then $\{P_{ti}\}$ is such that $\sum_i f(P_{ti}) = a_{P_t}f(P_t) + a_{P^t}f(P)$. Also, (P, P_{ti}) are not rearrangeable, since ti has $m + 1$ digits. Therefore $f(P) \leqq f(P_{ti})$ by hypothesis. Clearly there are precisely $a_{P_t} + a_{P^t}$ of the P_{ti}'s. Thus

$$a_{P_t}f(P_t) + a_{P^t}f(P) = \sum_{i=1}^{a_{P_t}+a_{P^t}} f(P_{ti}),$$

and $f(P) \leqq f(P_{ti})$. Thus,

$$a_{P_t}f(P_t) = \sum_{i=1}^{a_{P_t}+a_{P^t}} f(P_{ti}) + a_{P_t}f(P) - (a_{P_t} + a_{P^t})f(P) =$$

$$\left(\sum_{i=1}^{a_{P_t}+a_{P^t}} f(P_{ti}) - (a_{P_t} + a_{P^t})f(P)\right) + a_{P_t}f(P) \geqslant a_{P_t}f(P).$$

Thus $f(P_t) \geqq f(P)$, where t has m digits. Repeating the same process, we get $f(P) \leqq f(P_t)$ for t with $m - 1$ digits. Iterating further, since m is finite, we eventually obtain $f(P) \leqq f(P_i)$, where $\{P_i\}$ is the rearrangement of (P, Q), and finally $f(P) \leqq f(Q)$. Thus $f(P) \leqq f(Q)_{Q \in S_m^*}$, i.e. $f(P) = \min_{Q \in S_m^*} f(Q)$.

Though this theorem illustrates the connection between rearrangeability and the salesman problem, it is not entirely obvious how the result could be applied in practice, since we still have to find the set (polygon) P. Our next theorem, which includes Theorem 1 as a special case, will allow us to make more systematic use of our information, and in particular develop Theorem 3, which is a primitive algorithm for the traveling salesman problem.

THEOREM 2. *Let S, S_m^*, f be as usual. Let $P \in S_m^*$, $s \in P$, and suppose $f(P) \leqq (\geqq) f(Q)$ for all Q such that $s \in Q$, $Q \in S_m^*$, and (P, Q) are not rearrangeable* (with respect to S_m^*). *Then there exists $R \in S_m^*$ such that*

$$f(R) = \min_{Q \in S_m^*} (\max) f(Q),$$

and (P, R) are not rearrangeable.

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Proof. Let S_{m-1}^* be the set of all elements of S_m^* containing s, each minus $\{s\}$. Then $f(P - \{s\}) \leqq f(Q)$ for all $Q \in S_{m-1}^*$, whence $f(P) \leqq f(Q)$ for all $Q \in S_m^*$ with $s \in Q$ (this is by Theorem 1 applied to S_{m-1}^*). Now consider any $Q \in S_m^*$ so that (P, Q) are rearrangeable. Letting $\{P_i\}$ be a rearrangement, we obtain $a_P f(P) + a_Q f(Q) = \sum f(P_i)$, $s \in P_i$ for exactly a_P of the P_i (if $s \in Q$, then $f(P) \leqq f(Q)$ by the previous argument), and $f(P) \leqq f(T)_{s \in T \in S_m^*}$. Therefore, there are a_P i's such that $f(P) \leqq f(P_i)$ for those i's. Thus $a_P f(P) + a_Q f(Q) = \sum f(P_i)$, the P_i's ordered so the last a_P are the ones containing s. Therefore

$$a_Q f(Q) = \sum_{i=1}^{a_Q + a_P} f(P_i) - a_P f(P) =$$

$$\sum_{i=1}^{a_Q} f(P_i) + \sum_{i=a_Q+1}^{a_Q+a_P} f(P_i) - a_P f(P) \geqslant \sum_{i=1}^{a_Q} f(P_i),$$

whence $f(Q) \geqq f(P_i)$ for some i, and Q is not a unique minimum. By Lemma 1, $|P_i \cap P| > |Q \cap P|$, and if (P_i, P) are rearrangeable, we will obtain a P_{ij} so that $|P_{ij} \cap P| > |P_i \cap P| > |Q \cap P|$, and $f(P_{ij}) \leqq f(P_i) \leqq f(Q)$, etc. Since $|P|$ is finite, an end is reached, i.e. there exists R so that $f(R) \leqq f(Q)$ and (P, R) are not rearrangeable. Thus, since, every Q rearrangeable with P satisfies $f(Q) \geqq f(R)$ for some $R \in S_m^*$ which is not rearrangeable with P, and S_m^* has a minimum (since it is finite), it has a minimum which is not rearrangeable with P.

We can see that Theorem 1 follows directly from Theorem 2, for if $f(P) \leqq f(Q)$ for all Q not rearrangeable with P, the conditions of Theorem 2 are satisfied, and there is a minimum R which is not rearrangeable with P. But, by hypothesis, P is itself a minimum among those elements of S_m^* which are not rearrangeable with P, so it must be a minimum in S_m^*.

We will now attempt to use the theory constructively.

THEOREM 3. *Let S, S_m^*, f, be as usual. Let $P \in S_m^*$. Let $P^* \subset P$ and let $P_1^* = \{Q \in S_m^* | Q \cap P \supset P^*\}$. Let P_1 be an element of P_1^* with minimal f. Then let $s_1 \in P^*$, and let $P_2^* = \{Q \in S_m^* | Q \cap P \supset P^* - \{s_1\}\}$. Let P_2 be an element of P_2^* with minimal f which is not rearrangeable with P_1. We can iterate this procedure, defining P_n^* as*

$$\left\{Q \in S_m^* | Q \cap P \supset P^* - \bigcup_{i=1}^{n-1} \{s_i\}\right\},$$

P_n as an element of P_n^* not rearrangeable with P_{n-1} and minimal with respect to f in P_n^*. Of course, s_{n-1} is always chosen to be different from all the previous s's, *and the iteration ends when P is exhausted of edges, i.e. when $\bigcup \{s_i\} = P^*$. Clearly $P_{1+|P^*|}^* = S_m^*$. Then $P_{1+|P^*|}$ has minimum f in S_m^*.*

Proof. At each stage, P_n can be constructed to be both minimal in P_n^* and not rearrangeable with P_{n-1}, by Theorem 2.

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The essential point to this theorem is that we can restrict ourselves to finding the minimal element of a small set and systematically work our way up to the desired set, making good use of the favorable rearrangeability properties.

In particular, a way to implement this algorithm for polygons would be to start with any set of vertices contained in the vertex set, label two as endpoints, and find the minimal open connection (one edge short of a polygon) through these points and with these endpoints (if these were joined, a polygon would be formed) (this is P_1). Add one point, which will be a new endpoint. Renumber the initial points so that $12 \ldots = P_1$. Measure all connections not rearrangeable with $12 \ldots$ (with the new endpoint tacked on) and the minimum of these is P_2. Add a new point, etc. At the last stage, instead of adding a new point, we will close a polygon. $P_{1+|P^*|}$ is the minimum polygon not rearrangeable with the one so obtained (minimum, of course, always refers to the function f in this context). In terms of Theorem 3, we are actually allowing one new edge (the old endpoint connected to the new one) to be abandoned at each stage, which is what the theorem requires. We will see in the next section that we can potentially improve our algorithm by selective choice of P, P^*, and the sequence of s_i.

As usual, the analogous result for a maximum is established by the same argument.

Having developed properties of rearrangeability useful for studying the traveling salesman problem, we wish to show now that the concept is really central in some sense; no other enjoys its main property.

THEOREM 4. *Let S, S_m^* be as usual, and let $P \in S_m^*$. Then let $R^* = \{Q \in S_m^* | (P, Q)$ are not rearrangeable with respect to $S_m^*\}$.*
Then
(1) *For any real f on S extended as usual to S_m^*,*

$$f(P) = \min_{Q \in R^*} f(Q) \Rightarrow f(P) = \min_{Q \in S_m^*} f(Q).$$

(2) *If $R^{**} \subset S_m^*$ is such that (1) holds with R^{**} in place of R^*, $R^{**} \supset R^*$. In other words, R^* is the best set satisfying Theorem 1.*

Proof. Property (1) is simply a restatement of Theorem 1. To prove property (2), let $Q \neq P \in S_m^* - R^{**}$. Then the hypothesis of (2) implies that if f satisfies

$$f(P) = \min_{T \in R^{**}} f(T), \text{ then } f(P) \leqslant f(Q).$$

This can be rewritten as

$$\sum_{s \in P} f(s) - \sum_{s \in Q_i} f(s) \leqslant 0 \Rightarrow \sum_{s \in P} f(s) - \sum_{s \in Q} f(s)$$

where the Q_i range over R^{**}. A famous lemma of Farkas [**1**] states that every simultaneous solution to the k linear homogeneous inequalities $\sum_{i=1}^{n} A_{ji}u_i \leqq 0$ is also a solution of a $k+1$st, $\sum_{i=1}^{n} A_i u_i \leqq 0$ if and only if the last inequality is a nonnegative linear combination of the first k. This statement holds whether the A's and u_i's are interpreted as real or rational coefficients and variables. In our case, $f(s)$ $(s \in S)$ are the u_i's, and the coefficients are all 0, 1, -1. We interpret the A's and u's as rational, but there will clearly be no problem later in letting the u's range over the reals.

Using Farkas' Lemma, then, we can find rational (and therefore integral) nonnegative coefficients a_i so that

$$\sum_i \left(a_i \left(\sum_{s \in P} f(s) - \sum_{s \in Q_i} f(s) \right) \right) = a_Q \left(\sum_{s \in P} f(s) - \sum_{s \in Q} f(s) \right).$$

Here the $f(s)$'s function as dummy real variables, so we may as well drop the f's and write:

$$\sum_i \left(a_i \left(\sum_{s \in P} s - \sum_{s \in Q_i} s \right) \right) = a_Q \left(\sum_{s \in P} s - \sum_{s \in Q} s \right).$$

If $\bar{s} \in P - Q$ (such elements exist), then $\bar{s}$ appears a_Q times in the right hand side of the above, so it must appear with a positive coefficient at least as many times on the left hand side. In addition, any element of $P \cap Q$ appears on the r.h.s. with coefficient 0, so each of these elements must be in all the Q_i, and any element of $S - P \cup Q$ appears with coefficients 0 on the r.h.s. and is therefore in no Q_i. Since $Q_i \neq Q$, some member of $P - Q$ appears in a Q_i, and so appears with coefficient $+1$ on the l.h.s. more than a_Q times. Thus $\sum_i a_i > a_Q$.

Rewriting again and using our earlier notation, we get:

$$\left(\sum_i a_i \right) P - a_Q P = \sum a_i Q_i - a_Q Q.$$

This yields $a_P P + a_Q Q = \sum a_i Q_i$, where $a_P = \sum a_i - a_Q > 0$, and $Q_i \in R^{**} \subset S_m{}^*$. Since $Q \notin R^{**}$ and $Q_i = P$ could clearly just be eliminated, this is just the condition for P and Q to be rearrangeable. This establishes the result. (This Theorem, together with Theorem 1, constitute Theorem 1 of [**2**].)

One thing to note is that P itself is the one unclear case. In one sense, $f(P) \leqq f(P)$ always, but on the other hand, $f(P)$ is always one of the values that needs to be checked to compare it to the others, so we consider $P \in R^{**}$, which makes the theorem exactly right.

Another observation is that this theorem establishes the best procedure of the type which starts at some P, looks in some subset of $S_m{}^*$ for a set with smaller f, takes the smallest and continues with that as a base, the operation terminating when the scanned sets all have f values at least as big as the base set. In order that the final base set always be a set with minimal f in $S_m{}^*$, the scanned set at each point must be at least the class of all members of $S_m{}^*$ not rearrangeable with the base set, and need be no bigger. Of course, if we were satisfied to miss the absolute minimum sometimes (or if we take note of more

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than merely which scanned set has minimum f at every step), this statement does not apply.

We can give a further picture of how strong a property rearrangeability is with the following, which ties together some of the earlier results.

THEOREM 5. *Let S, S_m^*, P be as usual. Then (P, Q) are not rearrangeable with respect to $S_m^* \Leftrightarrow$ there exists f as usual such that P is the unique minimal (maximal) element with respect to S_m^* and f and there exists $s \in Q$ such that Q is minimal with respect to all $R \in S_m^*$ with $s \in R$ (and f), i.e. Q has a minimal open connection.*

First, assume such an f. Then by Theorem 2, there exists $R \in S_m^*$ such that $f(R) = \min_{T \in S_m^*}$ (max) $f(T)$ with (Q, R) not rearrangeable. However, P is the only element of S_m^* such that this holds, whence (P, Q) are not rearrangeable with respect to S_m^*.

Going the other way, since (P, Q) are not rearrangeable,

$$f(Q) \underset{R \in S_m^* - \{P\}}{\leqslant} f(R) \not\Rightarrow f(Q) \leqslant f(P),$$

i.e. there exists f such that $f(Q) \leqq f(R)_{R \in S_m^* - \{P\}}$ and $f(P) < f(Q)$. But this means P is a unique minimum relative to f and S_m^* (we are assuming $P \neq Q$; if $P = Q$ the theorem is obvious). Let $s \in Q - P$ (since $Q \neq P$, s exists). Then $s \in R \in S_m^* \Rightarrow f(Q) \leqq f(R)$, i.e., Q has a minimal open connection. We see, in fact, that not only has Q a minimal open connection, but it is second to minimum in S_m^* [**2**, Lemma 6].

We will demonstrate the use of the algorithm (Theorem 3) in a trivial case, to try and make its workings more clear. Let S_m^* be the class of all subsets of S having m elements. Let $P^* \subset P$, so P_1^* is the set of all m-element subsets of S containing P^*. P_1, the minimal element of P_1^*, is that element containing P^* with all its other elements the smallest ones in S outside of P^*. We choose $s_1 \in P^*$. The minimal element of P_2^* which is not rearrangeable with P_1 is clearly that element which is P_1 with s_1 replaced by the smallest element of S which is not in P_1 (unless s_1 is itself smaller than this element, in which case $P_1 = P_2$). Two elements of S_m^* in this case are rearrangeable unless there is only one element in each which is not in the other, simply because otherwise an element in one could be traded for an element in the other to obtain a rearrangement. It is also intuitively true in this case (as we know it is true in all cases) that P_2 is the minimal element of P_2^* (not just the minimal non-rearrangeable with P_1). It is clear that if we continue this procedure, we will at each stage exchange an element of P^* for the smallest element which is not already a member of the previous P_i (with the exception that the element could stay in if it is as small as all alternatives), until P^* is exhausted. The resulting element of S_m^* will clearly consist of the m smallest elements of S, and this clearly is the minimal element (there could be more than one if several elements of S have the same f value). It might be profitable to look at this example with regard to the other theorems as well.

A possible further generalization of the theory would be allowing non-real functions. Many of the properties of real numbers are not at all needed in the development. If suitable addition and less than relations exist in a system, it may be possible to apply this theory to them. This idea has undergone little investigation. Another idea is to let S be infinite. This affects the theory only in that S_m^* must be assumed to have a minimal element for Theorem 2.

Certain key questions remain unanswered, among them the discovery of a useful necessary and sufficient (purely combinatorial) condition for rearrangeability (particularly for polygons). Also helpful would be an effective way of computing the number of elements rearrangeable with a given one, especially for polygons. There seems to be a strong correlation between number of elements in common and rearrangeability, namely, the fewer in common, the more likely to be rearrangeable, which is certainly what we would expect. For polygons, one or no sides in common seems to be a sufficient condition for rearrangeability, but this is unproved.

2. Rearrangeability with further information. The concept of rearrangeability as hitherto developed is entirely a combinatorial one. Unfortunately, investigatory computations leave considerable doubt that the number of rearrangeable polygons (with a given one) is satisfactorily large for large n (see [**4**]) (though its ratio with the total does seem to go to 1). It is hoped, therefore, that we can utilize other information to make our procedure fruitful. The likeliest idea is to check out some properties of the particular function f at hand before applying the theory.

Let us first examine the essentials of our rearrangeability algorithm, to determine what liberties we can take with it. We have two basic properties:

(1) If A, B are rearrangeable then $k_1A + k_2B$ can be expressed as a sum of elements of S_m^*.

(2) Every rearrangeability sequence terminates, that is, if $B, P_1, P_{11}, \ldots$ is a sequence of members of S_m^*, each a member of a rearrangement of A and the preceding element in the sequence, the sequence is finite. This idea was used in Theorem 1.

We now note that we can weaken both these essential conditions without destroying the theory; namely, supposing that we seek a minimal element, it is clearly sufficient for Theorems 1-3 that we can get $k_1f(A) + k_2f(B)$ to be greater than or equal to a sum of elements of S_m^*.

As for terminating sequences, we can guarantee the same effect using sides in common as we did in section 1. Namely, if A, B are rearrangeable to form $k_1 + k_2$ other elements of S_m^*, and if we know $f(A)$ is less than (or equal to) f (at least k_1 of these) (where $k_1A + k_2B$ is the relevant linear combination of A and B), then we know $k_2f(B) \geqq \sum_{i=1}^{k_2} P_i$, whence $f(B) \geqq f(P_i)$ (some i), so B is not the unique minimum (there is a minor problem in that if the minimum is not unique, this procedure could eliminate all the minima; a little care will prevent this happening).

One way we could attempt to get additional information would be to use the constraints of geometry (assuming f is a distance function). This idea has not been ignored, though it has not been related to rearrangeability. Another possibility is to actually measure some non-rearrangeable configurations (i.e. take two non-rearrangeable elements and compare them by measuring the edges of each not on the other; this could have the effect of comparing other non-rearrangeable pairs with different sets of common elements). Certain special cases of the problem supply special information. We shall now consider another class of information.

Generally, to actually solve a specific problem, we have to discover all the values of f on S, since this information is essential in determining the minimum. Once we do this, it would seem easy enough to list them in increasing order (making an arbitrary choice any time two are equal). We shall assume the mechanics of listing S, its elements numbered in accordance with the order induced by f.

Suppose now $A = \{a_1, a_2, \ldots a_n\}$ and $B = \{b_1, b_2, \ldots b_n\}$, with $a_1 < a_2 \ldots < a_n$ and likewise for the b's. Then if $a_i \leqq b_i$ for each i, $f(a_i) \leqq f(b_i)$ for all i (since the elements of S were numbered consistent with f), and clearly $f(A) \leqq f(B)$, and we say B dominates A. Also, the comparability of A and B can not be known unless one dominates the other (given only the ordering of the elements of S via f).

Consider a positive integer linear combination of (A, B) (let us say $k_1A + k_2B$). Write $C = k_1A + k_2B$ as a sum of terms with coefficient 1 (so, for example, every member of $A - B$ will appear as k_1 summands). Now, choose some (maybe none) of the terms of C and replace each by a member of S with a smaller label than itself. For any Y formed in such a manner, we say $k_1A + k_2B \geqq Y$, which is a reasonable statement, since certainly $k_1f(A) + k_2f(B) \geqq f(Y)$ (for the meaning of all the notation, see the introductory section). Let us call any such Y an f-rearrangement of A, B if Y is a positive integer linear combination of members of S_m^*, with not all its summands A or B. We define (A, B) f-rearrangeable with respect to S_m^* if such a rearrangement exists. Note that any rearrangement is an f-rearrangement.

For the rest of this section, we will be discussing f-rearrangeability, but calling it rearrangeability for convenience.

If the list of edges kB dominates the list $\sum a_iP_i (P_i \in S_m^*, \sum a_i = k)$, we say B is self-rearrangeable. Domination is a special case, where $k = 1$. It is easy to verify that (1) A self-rearrangeable polygon cannot be a minimal element of S_m^*, (2) that any two rearrangeable polygons will still be rearrangeable upon removal of a self-rearrangeable other than the polygons themselves (free use will be made of the terms "edge" for an element of S and "polygon" for an element of S_m^*). From here on, we will assume S_m^* contains no self-rearrangeable elements.

To develop this idea of rearrangeability satisfactorily, we will have to worry about terminating sequences; it can be shown without any great difficulty that rearrangeability sequences will not in general terminate in this formulation. We will see that, using the remark made earlier, it is not too difficult to analogize Theorems 2 and 3 of Section 1. Theorem 4 (which includes Theorem 1), however, takes much more doing.

THEOREM 6. *Let S, S_m^* be as usual, with S ordered. Let $P \in S_m^*$. Then let $R^* = \{Q \in S_m^* | (P, Q)$ are not f-rearrangeable with respect to $S_m^*\}$. Then*

(1) *For any real f on S consistent with the order on S extended as usual to S_m^*,*

$$f(P) = \min_{Q \in R^*} f(Q) \Rightarrow f(P) = \min_{Q \in S_m^*} f(Q).$$

(2) *If $R^{**} \subset S_m^*$ is such that* (1) *holds with R^{**} in place of R^*, $R^{**} \supset R^*$.*

Proof of (1). Suppose the hypotheses of (1) and let $Q \in S_m^* - R^*$. We must show that $f(P) \leqq f(Q)$. Since (P, Q) are rearrangeable with respect to S_m^*, there exists a rearrangement $\{P_i\}$. Define $a\{P_i\}$ to be the number of distinct P_i which are rearrangeable with P (from here on, we will assume that none of the elements in a rearrangement of (P, Q) are either P or Q; if they were, they could be subtracted off, still leaving a rearrangement, since neither polygon is self-rearrangeable). Let $a_1 = \min a\{P_i\}$. This is clearly defined, since $0 \leqq a\{P_i\} \leqq |S_m^*|$. Choose a rearrangement of (P, Q) which realizes this minimum. Rearrange each of the rearrangeable P_i's to form a collection of P_{ij}'s. Let $a\{P_{ij}\}$ be the number of distinct new P_{ij} rearrangeable with P ("new" means not equal to any P with a shorter subscript, or Q). Let $a_2 = \min a\{P_{ij}\}$ and pick a collection of rearrangements of P_i which realizes this minimum. Continue to define a_k inductively.

Now suppose $a_k \neq 0$, $a_{k+1} = 0$. Then the collection of polygons with subscripts k digits long, new, and rearrangeable with P, can be rearranged to provide no "new" rearrangeable polygons. Therefore each individual polygon of this type can be rearranged to form no new rearrangeable polygons. Let A be one of these, and let B be a polygon whose rearrangement with P had A as an element. Then $k_1B + k_2P \geqq \sum P_t + k_3A$. But $k_4A + k_5P \geqq \sum Q_i$, where all the Q_i are either B or are a P_t with the number of digits in $t \leqq$ the number of digits in t' (where $A = P_{t'}$). Multiplying the first inequality by k_4 and the second by k_3, we get:

$$k_1k_4B + k_2k_4P \geqq k_4 \sum P_t + k_3k_4A \geqq k_3 \sum Q_i - k_3k_5P,$$

which yields

$$k_1k_4B + (k_2k_4 + k_3k_5)P \geqq k_3 \sum Q_i,$$

whence P and B have been rearranged without the benefit of A or any additional "new" polygons (it is quite clear that an inequality like the bottom one can only be generally valid if a rearrangement is represented). This contradicts the definition of the a_k (note: if A appeared as an element in a rearrangement of more than one B, the foregoing can be done in each case). So either $a_1 = 0$, or $a_k \neq 0$, for all k. But in the latter case, we can form a sequence of polygons $P_0 = Q, P_1, P_2, \ldots$, so that each polygon is "new" in its turn, that is, all the P_i's are different. This is clearly impossible since S_m^* is finite, so $a_1 = 0$. Therefore there exists a rearrangement of P, Q with only non-rearrangeable elements, whence $f(P) \leqq f(P_i)$, where $\{P_i\}$ is the rearrangement. If "$k_1 f(P) + k_2 f(Q) \geqq \sum f(P_i)$", we can subtract k_1 equations of the type $f(P) \leqq f(P_i)$, leaving $k_2 f(Q) \geqq \sum_i^{k_2} f(P_i)$, whence $f(Q) \geqq f(P_i) \geqq f(P)$ for some P_i, so $f(Q) \geqq f(P)$.

Proof of (2). This parallels Part (2) of Theorem 4; by Farkas' Lemma, we can write:

$$(\Gamma)\ \sum_i a_i\Big(\sum_{s\in P} s - \sum_{s\in Q_i} s\Big) + \sum_j (s_j - s_j') = a_Q\Big(\sum_{s\in P} s - \sum_{s\in Q} s\Big),$$

since all the known information about the system can be summarized by inequalities $f(P) \leqq f(Q_i)$ and $s_j \leqq s_j'$. If any s_j is an s_k', we can replace $s_j \leqq s_j'$ and $s_k \leqq s_k'$ by $s_k \leqq s_j'$. If any s_j is a member of a Q_i, we can replace s_j on Q_i by the larger element s_j', doing away with one of the second type inequality. This leads us to a system where no s_j is an s_k' or a member of a Q_i. Therefore all s_j's have a positive net coefficient on the left hand side of (Γ), so they are members of P. In this case, no member of P that is an s_j ever appears in a negative term on the l.h.s. of (Γ). Thus the coefficient of any such s_j on the left is at least $\sum a_i + 1$, so $\sum a_i < a_Q$; furthermore, each member of P must appear equally often as an s_j; if this is k times, add the inequality $f(P) \leqq f(P)$ k times; now s_j's appear on Q_i's and we proceed as before to eliminate all s_j's, giving:

$$\Big(\sum_i a_i\Big)P - a_Q P = \sum_i a_i Q_i' - a_Q Q,$$

where the Q_i' are formed from the Q_i by changing some edges to larger ones. Thus we get $a_Q Q \geqq (a_Q - \sum_i a_i)P + \sum_i a_i Q_i$ and Q is self-rearrangeable. Thus, after reducing to a system where no s_j was an s_k' or an element of a Q_i, we must have had no s_j's remaining. In this case we get $(\sum_i a_i - a_Q)P + a_Q Q \geqq \sum_i a_i Q_i$, and since Q still cannot be self-rearrangeable, the coefficient of P is positive and (P, Q) are rearrangeable. This completes the proof.

Note. The following more general result can easily be proved by the methods of Theorem 6: If S, S_m^*, f are as usual, and if any system of linear inequalities among the elements of S is given, define $P, Q \in S_m^*$ rearrangeable if

$$a_1 f(P) + a_2 f(Q) \geqslant \sum_{i=1}^{a_1+a_2} f(P_i)$$

(a_i a positive integer, $P, Q \neq P_i \in S_m^*$) is a linear consequence of the given inequalities. Defining R^* in the obvious way, the first part of Theorem 6 goes

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through, and if the defining inequalities for rearrangeability summarize the entire system of inequalities, the second part goes through.

In particular, if $m = 1$, we get a result which deals with linear inequalities among a system of real variables (as opposed to sets of real variables).

We now attempt to prove something similar to Theorem 3. We arrive at a first approximation, A^*, to the minimum polygon as follows. Start with the smallest edge. In the most general problem, there may be no polygon containing this edge. In general, choose that edge which is the smallest of any which appears on a polygon. Next, choose the smallest remaining edge which appears on a polygon in conjunction with the previous edge. A polygon can be built up by choosing each time the smallest edge which appears on a polygon in conjunction with all the previously chosen edges. This is A^*, and it satisfies the following:

THEOREM 7. *The minimal polygon in* S_m^* *can be built up from* A^*, *much the same as in Theorem* 3, *as follows: take the smallest polygon having the smallest* i *edges of* A^*, *and consider all the polygons not rearrangeable with it which contain the smallest* $i - 1$ *elements of* A^*; *use the smallest of these as a new starting point to iterate the procedure. The final polygon,* B^*, *which is not constrained to have any edges of* A^*, *is the* (*a*) *minimal member of* S_m^*. *In particular, we can let* $i = m - 1$ *and our starting point is* A^* (this is obvious from the construction of A^*).

Proof. We shall merely prove that a typical step of the process leads from a minimal polygon having the smallest i edges of A^* to a minimal polygon having the smallest $i - 1$ edges of A^*.

So let A be the smallest member of S_m^* which contains all the i smallest elements of A^*, and let B be a member of S_m^* which is rearrangeable with A and contains the $i - 1$ smallest elements of A^*. Then suppose $\{P_i\}$ is a rearrangement of A, B, with $k_1A + k_2B$ the appropriate linear combination of A and B. In particular, there must be k_1 elements among the P_i which are $\leqq$ the ith edge of A^*. It is not hard to see that the first $i - 1$ are members of each P_i (for example, the smallest, being on both A and B unless this is the last step, must appear $k_1 + k_2$ times among the P_i, since no smaller edge ever appears on a polygon. Thus, it is a member of P_i. Due to the construction of A^*, each successive edge can in turn be shown to be a member of each P_i, until the $i - 1$ edges run out) and by similar reasoning that the ith edge is a member of at least k_1 of the P_i. Since A is minimal with respect to these i edges, $f(A) \leqq f(P_i)$ for those k i's. Since $k_1f(A) + k_2f(B) \geqq \sum f(P_i)$, we certainly get $f(B) \geqq f(P_i)$, some i, and B is not a unique minimum. If $f(B) = f(P_i)$ at best, then a slightly different f exists with $f(B) > f(P_i)$ for at least one i, provided we always choose the same one of two equals as the smaller. So B is not a minimum, and we are done. (Note: given a consistent set of choices, it is clear that an f can be found indistinguishable from the given one for these purposes so that no two polygons have the identical f value.)

A remark is in order at this point. We see that rearrangeability at each stage in the above process is equivalent to rearrangeability in the class of only those polygons containing the relevant initial edges of A^*, which in turn is equivalent to rearrangeability in the class of sets of edges appearing as the rest of a polygon in conjunction with these initial edges of A^*. It can be seen that this construction parallels the open-line buildup described for polygons as an adjunct to Theorem 3. However, this procedure incorporates the additional information of the edge order by making fewer arbitrary choices than were made there and being able to use the stronger definition of rearrangeability. In the actual case of polygons, this construction may in no way resemble an open-line buildup geometrically, since the sequence of edges cannot be chosen to be tacked on an end at each stage (i.e. the $(m-1)$th and mth edges, which are the first two in the buildup, might be 12 and 34, which are two disjoint edges, and do not represent an open line, which is a polygon minus an edge); it is interesting to note that from a geometric point of view, it would not be easy to think of such a construction; so generalizing the problem could be the key to solving the original problem in this case.

Another point to note is that the theorem can be used secondarily to find the successive approximations; once we have determined the class of polygons which are not rearrangeable with the previous approximation, we can determine the minimum in this class using secondary approximations as initial polygons, and then having to directly compare only a small number. Of course, in the first steps, the number of polygons is too small for it to pay doing this.

Though this procedure seems to be to offer a substantial chance of being useful for the Traveling Salesman Problem and related problems, there are several stumbling blocks. First, it seems to be difficult to convert S_m^* (and, in particular, polygons) to the appropriate form of lists of edge-numbers (in the case of actual polygons, for each m there is a fixed number of different orderings of S, some of which are isomorphic, but this number increases rapidly with m).

Second, in the case it is most reasonable to examine (small m), most polygons seem to be eliminated by dominance; this in itself is a good thing for the particular problem, and further study of dominance might be useful in itself, but it makes it hard to discover any patterns with regard to rearrangements.

Difficulties in computing the number of rearrangeable polygons (with a given one), considerable enough when it is a function of m alone (which it is in our original definition, for real polygons), certainly hinder the study further.

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