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POSITIVE CENTRE SETS OF CONVEX CURVES AND A BONNESEN TYPE INEQUALITY

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Abstract

We consider the positive centre sets of regular n-gons, rectangles and half discs, and conjecture a Bonnesen type inequality concerning positive centre sets which is stronger than the classical isoperimetric inequality.

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1. Introduction

The classical isoperimetric inequality asserts that for any simple closed curve,

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

where L and A denote, respectively, its length and its enclosed area, and equality in (1.1) holds if and only if the curve is a circle. Related topics, generalisations and applications can be found in the book [3] and the survey paper [10]. One sharpened form of (1.1) is the Bonnesen type inequality,

$$L^2 - 4\pi A \geq \pi^2(r_{\text{out}} - r_{\text{in}})^2, \tag{1.2}$$

where r_{in} and r_{out} are the inradius and circumradius of the convex curve γ, respectively. Inequality (1.2) can be deduced from the well-known Bonnesen inequality (see [1]),

$$Lr - A - \pi r^2 \geq 0 \quad \text{for } r_{\text{in}} \leq r \leq r_{\text{out}}. \tag{1.3}$$

There are many elegant proofs and applications of (1.3) (see [2, 4, 7, 11, 14, 15]). A more comprehensive account of various aspects of Bonnesen's inequality can be found in [13, pages 321–327]. Based on a method of symmetrisation and (1.3), Gage [5]

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proved an 'isoperimetric inequality' for a convex curve γ involving the total squared curvature of the curve:

$$\int_\gamma \kappa^2\, ds \geq \frac{\pi L}{A}, \tag{1.4}$$

where κ is the curvature of γ, L is its length and A is its enclosed area. To simplify the proof of inequality (1.4) and help the reader understand the curve shortening problem in the plane, Gage [6] introduced the concept of the positive centre of a closed convex curve γ and showed that the centre of the minimal annulus of γ must be its positive centre.

Let M, N be two compact convex domains in $\mathbb{R}^2$. The *Minkowski sum* of M and N is

$$M + N = \{x + y \mid x \in M, y \in N\}.$$

Denote by r_{in} the inradius of M and by B_2 the unit disc in $\mathbb{R}^2$. The set

$$M_{-\lambda} = \{x \in \mathbb{R}^2 \mid x + \lambda B_2 \subseteq M\}, \quad 0 \leq \lambda \leq r_{\text{in}},$$

is called the *inner parallel body of M at distance λ*. Denote by K the domain bounded by γ and by $\operatorname{int} K$ its interior. For a point $c \in K$, let

$$r_{\text{in}}(c) = \max\{r \geq 0 \mid c + rB_2 \subseteq K\}, \quad r_{\text{out}}(c) = \min\{r > 0 \mid c + rB_2 \supseteq K\}.$$

The positive centre of γ can be defined by means of its Bonnesen function,

$$B(r) = Lr - A - \pi r^2.$$

A point $c \in \operatorname{int} K$ is called a *positive centre* of γ if it satisfies

$$B(r_{\text{in}}(c)) \geq 0 \quad \text{and} \quad B(r_{\text{out}}(c)) \geq 0.$$

This definition is equivalent to that of Gage (see [6, 8]). Denote by $\mathfrak{P}(\gamma)$ the set of all positive centres of γ.

Huang *et al.* [8] showed that positive centre sets are convex for closed convex curves and concluded that the positive centre sets of curves of constant width and squares are all their inner parallel bodies K_{-r_1}, where r_1 is the left zero point of the Bonnesen function $B(r)$. Motivated by the work of [8], we ask the following two questions.

QUESTION 1.1. *If γ is a regular n-gon, is $\mathfrak{P}(\gamma)$ its inner parallel body K_{-r_1}?*

QUESTION 1.2. *Is there another convex curve γ such that $\mathfrak{P}(\gamma)$ is also its inner parallel body K_{-r_1} besides the curves of constant width and regular n-gons?*

The purpose of the present paper is to investigate these two questions. Further aspects of the positive centre set can be found in [9, 12].

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2. Main results

Firstly, we state two propositions from [8].

Proposition 2.1 [8]. *If γ is a convex and centrally symmetric planar curve, then $\mathfrak{P}(\gamma)$ is also convex and centrally symmetric with respect to the centre of the minimal annulus of γ.*

Proposition 2.2 [8]. *If γ is a convex and axially symmetric planar curve, then $\mathfrak{P}(\gamma)$ is also convex and axially symmetric and the symmetric axes pass through the centre of the minimal annulus of γ.*

The following theorem describes the positive centre set of a regular n-gon and gives a positive answer to Question 1.1.

Theorem 2.3. *If γ is a regular n-gon with enclosed area A, then*

(i) *$\mathfrak{P}(\gamma)$ is its inner parallel body K_{-r_1}, where r_1 is the left zero point of the Bonnesen function $B(r)$; and*

(ii) *the boundary of $\mathfrak{P}(\gamma)$ is also a regular n-gon which is a homothet of γ and*

$$A(\mathfrak{P}(\gamma)) = \left(1 - \frac{n\sin\frac{\pi}{n} - \sqrt{n^2\sin^2\frac{\pi}{n} - n\pi\sin\frac{\pi}{n}\cos\frac{\pi}{n}}}{\pi\cos\frac{\pi}{n}}\right)^2 A. \tag{2.1}$$

Proof. (i) Without loss of generality, let γ be a regular n-gon with circumradius 1. Hence $L = 2n\sin(\pi/n)$, $A = n\sin(\pi/n)\cos(\pi/n)$ and the two roots of $B(r) = 0$ are

$$r_1 = \frac{n\sin\frac{\pi}{n} - \sqrt{n^2\sin^2\frac{\pi}{n} - n\pi\sin\frac{\pi}{n}\cos\frac{\pi}{n}}}{\pi}$$

and

$$r_2 = \frac{n\sin\frac{\pi}{n} + \sqrt{n^2\sin^2\frac{\pi}{n} - n\pi\sin\frac{\pi}{n}\cos\frac{\pi}{n}}}{\pi}.$$

From [6, Theorem 1.8], the centre O of the minimal annulus of γ is a positive centre.

To determine the shape of $\mathfrak{P}(\gamma)$, we must find the furthest point of $\mathfrak{P}(\gamma)$ on the directions from the point O to the midpoint of each side and that to each vertex. In other words, we have to find $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$ for any point $c \in K$, where K is the domain enclosed by γ. From the definitions of $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$, we see that $r_{\text{in}}(c)$ is the nearest distance from c to each side and $r_{\text{out}}(c)$ is the furthest distance from c to each vertex.

Suppose n is odd. If $B \in K$ is a point on the line from the point O to the midpoint of one side and x is the distance from O to B, then

$$r_{\text{in}}(B) = \cos\frac{\pi}{n} - x \quad \text{and} \quad r_{\text{out}}(B) = x + 1$$

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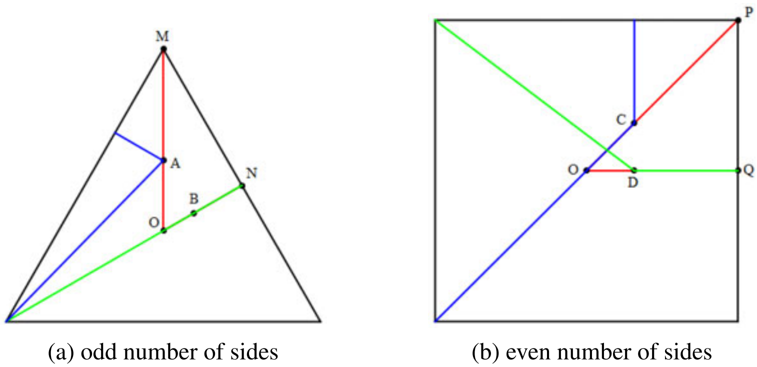

FIGURE 1. $r_{\text{in}}(\cdot)$ and $r_{\text{out}}(\cdot)$.

($r_{\text{in}}(B)$ and $r_{\text{out}}(B)$ are the green segments through B in Figure 1(a)). If $A \in K$ is a point on the line from the point O to one endpoint of the above side and y is the distance from O to A, then

$$r_{\text{in}}(A) = (1-y)\cos\frac{\pi}{n} \quad \text{and} \quad r_{\text{out}}(A) = \sqrt{\sin^2\frac{\pi}{n} + \left(y + \cos\frac{\pi}{n}\right)^2}$$

($r_{\text{in}}(A)$ and $r_{\text{out}}(A)$ are the blue segments in Figure 1(a)). Combining these results with the definition of positive centre set gives

$$\begin{cases} 0 \le x \le \cos\dfrac{\pi}{n}, \\ \cos\dfrac{\pi}{n} - x \ge r_1, \\ x + 1 \le r_2 \end{cases} \quad \text{and} \quad \begin{cases} 0 \le y \le 1, \\ (1-y)\cos\dfrac{\pi}{n} \ge r_1, \\ \sqrt{\sin^2\dfrac{\pi}{n} + \left(y + \cos\dfrac{\pi}{n}\right)^2} \le r_2. \end{cases}$$

From computations and comparisons in Mathematica 7.0,

$$0 \le x \le \cos\frac{\pi}{n} - r_1 \quad \text{and} \quad 0 \le y \le \frac{\cos\frac{\pi}{n} - r_1}{\cos\frac{\pi}{n}}. \tag{2.2}$$

Now suppose n is even. If $D \in K$ is a point on the line from the point O to the midpoint of one side and z denotes the distance from O to D, then

$$r_{\text{in}}(D) = \cos\frac{\pi}{n} - z \quad \text{and} \quad r_{\text{out}}(D) = \sqrt{\sin^2\frac{\pi}{n} + \left(z + \cos\frac{\pi}{n}\right)^2}$$

($r_{\text{in}}(D)$ and $r_{\text{out}}(D)$ are the green segments through D in Figure 1(b)). If $C \in K$ is a point on the line from the point O to one endpoint of the above side and w denotes the distance from O to C, then

$$r_{\text{in}}(C) = (1-w)\cos\frac{\pi}{n} \quad \text{and} \quad r_{\text{out}}(C) = w + 1$$

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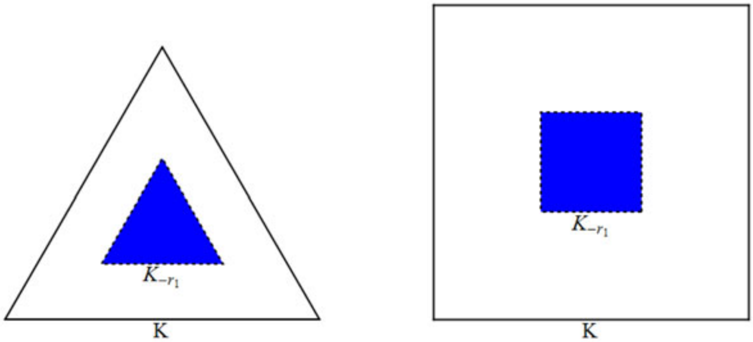

FIGURE 2. The positive center sets of regular n-gons.

($r_{\text{in}}(C)$ and $r_{\text{out}}(C)$ are the blue segments through C in Figure 1(b)). Hence,

$$\begin{cases} 0 \le z \le \cos \dfrac{\pi}{n}, \\ \cos \dfrac{\pi}{n} - z \ge r_1, \\ \sqrt{\sin^2 \dfrac{\pi}{n} + \left(z + \cos \dfrac{\pi}{n}\right)^2} \le r_2 \end{cases} \quad \text{and} \quad \begin{cases} 0 \le w \le 1, \\ (1 - w) \cos \dfrac{\pi}{n} \ge r_1, \\ w + 1 \le r_2. \end{cases}$$

By some computations and comparisons in Mathematica 7.0,

$$0 \le z \le \cos \frac{\pi}{n} - r_1 \quad \text{and} \quad 0 \le w \le \frac{\cos \frac{\pi}{n} - r_1}{\cos \frac{\pi}{n}}. \tag{2.3}$$

Since γ is a regular n-gon, γ is a cap-body. From [13, Lemma 3.1.10], (2.2) and (2.3), the boundary of $\mathfrak{P}(\gamma)$ is homothetic to γ. Again by Proposition 2.2 and the convexity of $\mathfrak{P}(\gamma)$ (see [8, Theorem 2.1]), $\mathfrak{P}(\gamma) = K_{-r_1}$ (see Figure 2).

(ii) Since $\mathfrak{P}(\gamma) = K_{-r_1}$ and the boundary of $\mathfrak{P}(\gamma)$ is homothetic to γ, it follows that equality (2.1) holds. □

To investigate Question 1.2, we examine the positive centre sets of rectangles.

THEOREM 2.4. *There are rectangles such that their positive centre sets are also their inner parallel bodies K_{-r_1}, where r_1 is the left zero point of the Bonnesen function $B(r)$.*

PROOF. Let γ be a rectangle of length $2m$ and width 2. Then its length is $L = 4(m + 1)$, its enclosed area is $A = 4m$ and the two roots of the Bonnesen function $B(r)$ are

$$r_1 = \frac{2(m + 1) - 2\sqrt{(m + 1)^2 - m\pi}}{\pi} \quad \text{and} \quad r_2 = \frac{2(m + 1) + 2\sqrt{(m + 1)^2 - m\pi}}{\pi}.$$

To determine the shape of $\mathfrak{P}(\gamma)$, from Propositions 2.1 and 2.2, we need only describe a quarter of the positive centre set of γ. Divide the quarter rectangle into

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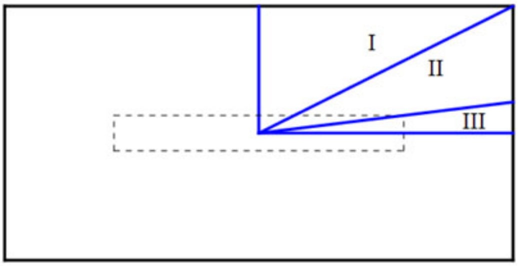

FIGURE 3. Dividing the quarter rectangle into three parts.

three parts as shown in Figure 3 by lines of slope k passing through the centre of the rectangle and a point of γ. For an arbitrary point P in these domains,

$$r_{\text{in}}(P) = \min\{1 - k(m - v), v\} \quad \text{and} \quad r_{\text{out}}(P) = \sqrt{(2m - v)^2 + (1 + k(m - v))^2},$$

where v is the distance from P to the right side of the rectangle.

A similar argument to the proof of Theorem 2.3 shows that $(km + r_1 - 1)/k \le v \le m$ for the points of domain (I). Set $k_1 = (1 - r_1)/(m - r_1)$ and $k_2 = 1/m$. Then, for domain (II) where $k_1 \le k \le k_2$ and domain (III) where $0 \le k < k_1$,

$$\begin{cases} 0 \le v \le m, \\ k > (1 - v)/(m - v), 1 - k(m - v) \ge r_1, \\ \sqrt{(2m - v)^2 + (1 + k(m - v))^2} \le r_2 \end{cases} \quad \text{and} \quad \begin{cases} 0 \le v \le m, \\ k \le (1 - v)/(m - v), v \ge r_1, \\ \sqrt{(2m - v)^2 + (1 + k(m - v))^2} \le r_2. \end{cases}$$

By some tedious comparisons in Mathematica 7.0, we find $r_1 \le v \le m$ for $0 \le k \le k_1$ and $(km + r_1 - 1)/k \le v \le m$ for $k_1 \le k \le k_2$ when $m \in (1, 1.04)$, which shows that the positive centre sets are all inner parallel bodies K_{-r_1}. □

REMARK 2.5. From the proof of Theorem 2.4, we find the positive centre set of a rectangle is extremely complicated for different choices of m. For example, if $m = 2$, then $q_1 \le x \le 2$ for $0 \le k \le k_3$ and $(2k + r_1 - 1)/k \le x \le 2$ for $k_3 \le k \le k_2$, where q_1 is the smaller root of $(2m - x)^2 + (1 + k(m - x))^2 = r_2^2$ and k_3 is the solution of the expression $(mk + r_1 - 1)/k = q_1$, where k_1 and k_2 are as in the proof of Theorem 2.4. By Mathematica 7.0, we find the quarter of the positive centre set for this case as shown in Figure 4.

3. Another example and a conjecture

If γ is the boundary of a half disc with radius 1, then its length is $L = \pi + 2$, its enclosed area is $A = \frac{1}{2}\pi$ and the two zeros of its Bonnesen function $B(r)$ are

$$r_1 = \frac{\pi + 2 - \sqrt{4 + 4\pi - \pi^2}}{2\pi} \quad \text{and} \quad r_2 = \frac{\pi + 2 + \sqrt{4 + 4\pi - \pi^2}}{2\pi}.$$

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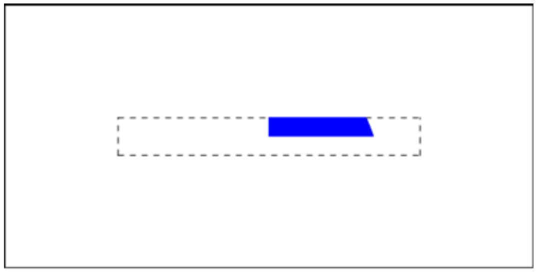
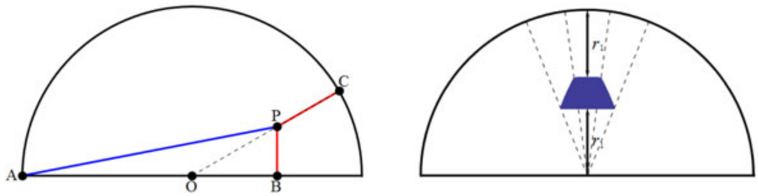


FIGURE 4. A quarter of the positive centre set of a rectangle for $m = 2$.



(a) (b)

FIGURE 5. A half disc and its positive centre set.

To describe the shape of $\mathfrak{P}(\gamma)$, from Proposition 2.2, we only need to consider the positive centre set of the right side of γ. For an arbitrary point P of the right side of a disc, set $|OP| = r$ and denote by θ the angle from OB to OP (see Figure 5(a)). Then

$$r_{\text{in}}(P) = \min\{1 - r, r\sin\theta\} = \begin{cases} 1 - r & \text{for } 1/(1 + \sin\theta) \le r \le 1, \\ r\sin\theta & \text{for } 0 < r < 1/(1 + \sin\theta), \end{cases}$$

and

$$r_{\text{out}}(P) = \sqrt{r^2 + 2r\cos\theta + 1}.$$

Hence,

$$\begin{cases} 0 \le r < 1/(1 + \sin\theta), & \theta \in \left[0, \dfrac{\pi}{2}\right], \\ r\sin\theta \ge r_1, \\ \sqrt{r^2 + 2r\cos\theta + 1} \le r_2 \end{cases} \quad \text{and} \quad \begin{cases} 1/(1 + \sin\theta) \le r \le 1, & \theta \in \left[0, \dfrac{\pi}{2}\right], \\ 1 - r \ge r_1, \\ \sqrt{r^2 + 2r\cos\theta + 1} \le r_2. \end{cases}$$

From computations in Mathematica 7.0, $r \in [r_1/\sin\theta, s_2]$ for $\theta_1 \le \theta < \theta_2$ and $r \in [r_1/\sin\theta, 1 - r_1]$ for $\theta_2 \le \theta \le \pi/2$, where $\theta_1 \doteq 1.19346$, $\theta_2 \doteq 1.43474$ and $s_2 = -\cos\theta + \sqrt{\cos^2\theta - 1 + r_2^2}$. Combining this with Proposition 2.2, we have the positive centre set of γ as shown in Figure 5(b).

Based on numerical results from Mathematica 7.0 for regular n-gons, half discs, rectangles and Reuleaux triangles (see Tables 1 and 2), we are led to the following conjecture.

TABLE 1. Some numerical results for regular n-gons with circumradius 1.

n	L	A	$A(K_{-r_1})$	Q
3	$3\sqrt{3}$	$\frac{3\sqrt{3}}{4}$	0.1936059	8.01479867
10	6.1803399	2.9389263	0.06966603	1.171656193
100	6.2821518	3.1395360	9.9645422×10^{-4}	0.01198560359
10^5	6.2831853	3.1415927	1.0335051×10^{-9}	$1.195437374 \times 10^{-8}$
10^{10}	6.28318531	3.14159265	$1.03354256 \times 10^{-19}$	$1.195433624889859 \times 10^{-18}$

TABLE 2. Some numerical results for a Reuleaux triangle, half disc and rectangles.

Examples	L	A	$A(K_{-r_1})$	r_{out}	r_{in}	Q
Reuleaux triangle with width 1	π	$\frac{\pi-\sqrt{3}}{2}$	0.032105	$\frac{\sqrt{3}}{3}$	$1-\frac{\sqrt{3}}{3}$	0.37355
Half disc of radius 1	$\pi+2$	$\frac{\pi}{2}$	0.111849	1	$\frac{1}{2}$	2.82383
Rectangle of length 4 and wide 2	12	8	0.635667	2	1	20.40160
Rectangle of length 20 and wide 2	44	40	0.81985	$\sqrt{101}$	1	614.7196

CONJECTURE 3.1. If γ is a convex curve with length L and enclosed area A, then

$$L^2 - 4\pi A \geq \pi^2 (r_{\text{out}} - r_{\text{in}})^2 + 4\pi A(\mathfrak{P}(\gamma)), \tag{3.1}$$

where $\mathfrak{P}(\gamma)$ is the positive centre set of γ, and the equality holds if and only if γ is a circle.

Next, we suggest a method to attack this conjecture. Let

$$Q = L^2 - 4\pi A - \pi^2 (r_{\text{out}} - r_{\text{in}})^2 - 4\pi A(K_{-r_1}).$$

From the definition of the positive centre, $A(K_{-r_1}) \geq A(\mathfrak{P}(\gamma))$. Hence, if we can show that $Q \geq 0$, then inequality (3.1) is correct.

The Mathematica models and programs used in this paper can be found via the link: https://pan.baidu.com/disk/home#/all?vmode=list&path=%2Fpaper%20mode.

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