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BOUNDARY BEHAVIOR AND QUASI-NORMALITY OF FINITELY VALENT HOLOMORPHIC FUNCTIONS

DAVID C. HADDAD

A function defined in a domain D is n-valent in D if $f(z) - w_0$ has at most n zeros in D for each complex number w_0. Let $\mathscr{V} = \mathscr{V}(r_0, n)$ denote the class of nonconstant, holomorphic functions f in the unit disc that are n-valent in each component of the set $\{z: |f(z)| > r_0\}$. MacLane's class $\mathscr{A}$ is the class of nonconstant, holomorphic functions in the unit disc that have asymptotic values at a dense subset of $|z| = 1$. (For a detailed discussion of $\mathscr{A}$ see MacLane [**4**].)

In [**2**, Theorem 3] we showed that $\mathscr{V} \subset \mathscr{A}$. Bagemihl and Seidel [**1**] and MacLane [**4**] independently showed that $\mathscr{N} \subset \mathscr{A}$, where $\mathscr{N}$ is the class of nonconstant holomorphic functions in the unit disc that are normal in the sense of Lehto and Virtanen [**3**]. Furthermore, Lehto and Virtanen showed [**3**, Theorem 2] that a normal function having asymptotic value c at $e^{i\theta}$ has angular limit c at $e^{i\theta}$.

Is there any relationship between the two classes $\mathscr{V}$ and $\mathscr{N}$? Clearly, $\mathscr{N} \not\subset \mathscr{V}$ since $e^{1/(1-z)}$ belongs to $\mathscr{N}$ but not to $\mathscr{V}$. In this paper we show that $\mathscr{V}$ is a quasi-normal family of order n and each function $f \in \mathscr{V}$ is a quasi-normal function of order at most $n - 1$ (the definitions are below). We show that this result is the best possible so that $\mathscr{V} \not\subset \mathscr{N}$. Furthermore, Letho and Virtanen's result on angular limits is true for functions in $\mathscr{V}$. Thus each function in $\mathscr{V}$ has angular limits at a dense subset of $|z| = 1$.

A general reference on quasi-normal families is Montel [**5**, Chapter 2]. However, it is necessary for our purposes to elaborate on some of his definitions.

A sequence of functions defined in a domain D converges *subuniformly* in D if the sequence converges uniformly on compact subsets of D. A set $E \subset D$ is *sparse* in D, if E is a finite set of points or if $E = \{z_n\}$ is a countable set and the distance (on the Riemann sphere) from z_n to ∂D tends to zero as n tends to infinity.

A family Q of holomorphic functions in D is a *quasi-normal family* in D if every sequence of functions in Q has a subsequence which converges subuniformly in $D - E$, where E is a sparse subset of D. (In general, E depends on the particular subsequence.)

If $\{f_k\}$ is a sequence of holomorphic functions in D converging to f subuniformly in $D - E$, then a point $z_0 \in E$ is an *irregular point* for the sequence

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$\{f_k\}$ if $\{f_k\}$ does not converge to f subuniformly in any neighborhood of z_0. Irregular points occur only in the case when $f \equiv \infty$. The *order of the sequence* $\{f_n\}$ is the number of irregular points for the sequence. A point z_0 is a *strongly irregular* point for $\{f_k\}$ if for k sufficiently large, $f_k(z)$ takes on all complex numbers in every neighborhood of z_0.

A sequence $\{f_k\}$ is *strong* if either it converges subuniformly in D to a holomorphic function or it converges subuniformly in $D - E$ to infinity and each point of the sparse set E is strongly irregular.

It is not hard to show that every sequence in a quasi-normal family has a subsequence that is strong.

If Q is a quasi-normal family of holomorphic functions in D, then the *order* of Q is the supremum of the orders of the strong sequences in Q. (This definition differs from Montel's [**5**, p. 66]; Montel takes the supremum over all sequences in Q, not just the strong sequences.)

A holomorphic function f in $|z| < 1$ is a *quasi-normal fnuction* of order n if the family $\{f \circ \psi\}$, where ψ runs through all the Möbius transformations of $|z| < 1$ onto itself, is a quasi-normal family of order n. (This is the obvious extension of Lehto and Virtanen's definition of a normal function.)

It is convenient to introduce the following notation. If $w = f(z)$ is a non-constant, holomorphic function in $|z| < 1$, we denote by F the Riemann surface of f^{-1} as a covering surface of the w-plane. Let p denote the projection from F onto the w-plane and let $\hat{f}$ be the one-to-one conformal map of $|z| < 1$ onto F so that $f = p \circ \hat{f}$. If T is a curve in $|z| < 1$, we let

$$m_F(T) = \int_T |f'(z)|\,|dz|.$$

A component of the set $\{z : |f(z)| = r > 0\}$ is called a *level curve* of f.

LEMMA 1. *Let Q be a family of holomorphic functions in a domain D. Let $\{D_k\}$ be a sequence of domains such that $\bar{D}_k \subset D$, $D_k \subset D_{k+1}$, and $\bigcup_{k=k}^{\infty} D_k = D$. The family Q is quasi-normal in D if there is a sequence $\{j_k\}$ of positive integers and two distinct complex numbers a and b such that $f(z) - a$ and $f(z) - b$ have at most j_k zeros in D_k for every $f \in Q$.*

Proof. That Q is a quasi-normal family in D_k of order at most j_k follows from a theorem [**5**, p. 67] of Montel. Thus each sequence $\{f_\alpha\} \subset Q$ has a subsequence $\{f_{1\alpha}\}$ that converges subuniformly in $D_1 - E_1$, where E_1 has at most j_1 points. The sequence $\{f_{1\alpha}\}$ has a subsequence $\{f_{2\alpha}\}$ that converges subuniformly in $D_2 - E_2$ where E_2 has at most j_2 points. Proceeding inductively, we obtain a sequence $\{f_{k\alpha}\} \subset \{f_\alpha\}$ for each k such that $\{f_{k\alpha}\}$ converges subuniformly in $D_k - E_k$, where E_k has at most j_k points. The diagonal sequence $\{f_{kk}\}$ is a subsequence of $\{f_\alpha\}$ that converges subuniformly in D minus the sparse set $\bigcup_{1=1}^{\infty} E_k$. Thus Q is a quasi-normal family in D.

LEMMA 2. *Every function $f \in \mathscr{V} = \mathscr{V}(r_0, n)$ takes on each value w_0 ($|w_0| > 2r_0$) at most $q(s)$ times in $|z| < s$, where q is a function of s and not of f or w_0.*

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Remark. Of course, q depends on r_0 and n in addition to s.

Proof. Let $f \in \mathscr{V}$ and let $\{C_j(r)\}$ denote the level curves of $\{z : |f(z)| = r\}$. Let $L_j(r)$ denote the length of $C_j(r)$, let $\hat{C}_j(r) = \hat{f}(C_j(r))$, and let $g = \hat{f}^{-1}$. Let $R = \{r : r_0 < r < 2r_0$, and F has no branch points lying over $|w| = r\}$. For each $r \in R$,

$$L_j(r)^2 = \left(\int_{\hat{C}_j(r)} |g'(w)|\,|dw| \right)^2$$
$$\leqq 4\pi r_0 n \int_{\hat{C}_j(r)} |g'(w)|^2 |dw|.$$

If we let α denote the area of $\{z : r_0 < |f(z)| < 2r_0\}$, then it readily follows that for each integer k,

$$\int_R \sum_{j=1}^{k} L_j(r)^2 dr \leqq 4\pi r_0 n \alpha.$$

Thus $\sum_{j=1}^{k} L_j(r_1)^2 \leqq 4\pi r_0 \alpha n \leqq 4\pi^2 r_0 n$ for some $r_1 \in R$ and for each positive integer k.

If a component $D(r_1)$ of the set $\{z : |f(z)| > r_1\}$ meets $|z| < s$ and if all the level curves of $\partial D(r_1)$ that meet $|z| < s$ are relatively compact in $|z| < 1$, then $D(r_1)$ must be the only component of $\{z : |f(z)| > r_1\}$ meeting $|z| < s$. Hence, $f(z)$ assumes each value w ($|w| > 2r_0$) at most n times in $|z| < s$ because $f \in \mathscr{V}$.

Hence, we may as well assume that every component $D_j(r_1)$ of $\{z : |f(z)| > r_1\}$ meeting $|z| < s$ has a noncompact level curve C_j on its boundary that meets $|z| < s$. Clearly, the length of each C_j is bounded below by $2(1 - s)$. If k is the number of components $D_j(r_1)$ that meet $|z| < s$, then

$$4k(1 - s)^2 \leqq \sum_{j=1}^{k} L_j(r_1)^2 \leqq 4\pi^2 r_0 n.$$

Thus, if $|z| < s$ then $f(z)$ assumes each value w ($|w| > 2r_0$) at most $nk \leqq \pi^2 r_0 n^2/(1 - s)^2$ times. This completes the proof of the lemma.

LEMMA 3. *Let f be a nonconstant, holomorphic function in $|z| < 1$ that is n-valent in a component $D(r_0)$ of $\{z : |f(z)| > r_0\}$. Let $D \subset D(r_0)$ be a component of $\{z : |f(z)| > r > r_0\}$, and let k be the number of zeros of f in D^*, the simply connected domain obtained by adding to D those components of $\{z : |f(z)| \leqq r\}$ that punch holes in D. Then $k \leqq n$, and the connectivity of D is bounded above by $k + 1$.*

Proof. Each component G_j of $D^* - D$ is bounded by a closed level curve $T_j \subset \{z : |f(z)| = r\}$. By the minimum principle f has at least one zero in G_j. Thus the connectivity of D is bounded above by $k + 1$. If q_j denotes the number of zeros of f in G_j, then by the argument principle $\Delta_{T_j} \arg f(z) = 2\pi q_j$. Since f is n-valent in $D(r_0)$, then

$$2\pi k = 2\pi \sum_j q_j = \sum_j (\Delta_{T_j} \arg f(z)) \leqq 2\pi n.$$

Thus, $k \leqq n$.

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THEOREM 1. *The family* $\mathscr{V}$ *is a quasi-normal family of order* n *in* $|z| < 1$.

Proof. That $\mathscr{V}$ is a quasi-normal family in $|z| < 1$ follows immediately from Lemma 1 and Lemma 2.

Let

$$g_k(z) = k \prod_{j=0}^{n-1} \left(z - \frac{j}{k}\right).$$

Clearly, $\{g_k\} \subset \mathscr{V}$ and $\{g_k\}$ is a strong sequence of order n. Hence, the order of $\mathscr{V}$ is at least n.

To obtain an upper bound on the order of $\mathscr{V}$, let $\{f_k\}$ be a strong sequence converging to infinity subuniformly in $|z| < 1$ minus a sparse set. Choose s $(0 < s < 1)$ such that $|z| = s$ contains no irregular points for $\{f_k\}$. For k sufficiently large, $|z| = s$ lies in a component G_k of $\{z : |f_k(z)| > 2r_0\}$. If q irregular points for $\{f_k\}$ lie inside $|z| < s$, then the connectivity of G_k is at least $q + 1$. By Lemma 3, $n + 1$ is an upper bound on the connectivity of G_k. Hence, $q \leqq n$, and, since s can be chosen arbitrarily near 1, the order of $\{f_k\}$ cannot exceed n. Thus, $\mathscr{V}$ is a quasi-normal family of order precisely n.

THEOREM 2. *Each function* $f \in \mathscr{V}$ *is a quasi-normal function of order at most* $n - 1$.

Proof. The family $\{f \circ \psi\}$ where ψ runs through all of the Möbius transformations of $|z| < 1$ onto itself is a subfamily of $\mathscr{V}$ and hence is quasi-normal of order at most n.

Suppose $\{f_k = f \circ \psi_k\}$ is a strong sequence with n irregular points. Choose s $(0 < s < 1)$ so that the n irregular points for $\{f_k\}$ lie inside $|z| < s$. Thus, we can choose k_0 so that the circle $|z| = s$ lies in a component G_k of $\{z : |f_k(z)| > 2r_0\}$ for each $k > k_0$. Let $k > k_0$. By Lemma 3, f_k has at most n zeros in $|z| < s$. Hence, it follows from the argument principle that $m_F[f_k(\partial G_k \cap \{|z| < s\})] \geqq 4\pi n r_0$. On the other hand, since $f \in \mathscr{V}$, $m_F[f_k(\partial G_k \cap \{|z| < 1\})] \leqq 4\pi n r_0$. Hence, all the level curves of ∂G_k lie inside $|z| < s$, and so $|f_k(z)| > 2r_0$ for z in the annulus $A = \{z : s < |z| < 1\}$. Thus, $\psi_k(A)$ contains an annulus $B = \{\zeta : t < |\zeta| < 1\}$ in which $|f(z)| > 2r_0$. If $M = \max_{|\zeta| \leqq t} |f(\zeta)|$, then for k sufficiently large, $\min_{|z|=s} |f(\psi_k(z))| > M$ since the irregular points for $\{f_k\}$ lie inside $|z| < s$. Thus for k sufficiently large $\psi_k(|z| = s) \subset B$, which implies $\psi_k(|z| \leqq s) \subset B$. Hence, $|f(\psi_k(z))| > 2r_0$ for $|z| \leqq s$ and k sufficiently large. This is inconsistent with the fact that the n irregular points for $\{f_k\}$ lie inside $|z| < s$. Therefore, the order of each sequence $\{f \circ \psi_k\}$ is bounded above by $n - 1$, and consequently f is a quasi-normal function of order at most $n - 1$.

THEOREM 3. *If a function* $f \in \mathscr{V}$ *has asymptotic value* c *at* $e^{i\theta}$ *then* f *has angular limit* c *at* $e^{i\theta}$.

Proof. If c is finite, Lehto and Virtanen's argument [**3**, pp. 52-53] shows that f has angular limit c at $e^{i\theta}$ whenever f has asymptotic value c at $e^{i\theta}$.

Suppose f has asymptotic value infinity at $e^{i\theta}$ but does not have angular limit infinity at $e^{i\theta}$. Then by [**3**, Theorem 1], there is an asymptotic path T ending at $e^{i\theta}$ on which $f(z)$ tends to infinity and a sequence of points z_k converging to $e^{i\theta}$ at which $f(z_k) = a \neq \infty$ $(k = 1, 2, \ldots)$ such that the hyperbolic distance $\sigma(T, z_k)$ from T to z_k is bounded by a constant b for all k. By Lemma 3 we can choose $r > \max(|a|, r_0)$ so that F has no branch points over $|w| = r$ and so that the component $D = D(r)$ of $\{z : |f(z)| > r\}$ containing a terminal subarc of T is simply connected.

Let $H(b) = \{z : \sigma(z, T) < b\}$, where $\sigma(z, T)$ denotes the hyperbolic distance from z to T. Every neighborhood of $e^{i\theta}$ contains a subsequence of $\{z_k\}$ in $H(b) - D$ and a subarc of T in $H(b) \cap D$. Thus we can find a sequence of distinct points $t_k \in \partial D \cap H(b)$ that converges to $e^{i\theta}$.

Since D is simply connected, ∂D contains no compact level curves. It follows from [**2**, Corollary 1 and Theorem 2] that each level curve of ∂D is a crosscut of $|z| < 1$ ending at points other than $e^{i\theta}$. Hence only finitely many t_k can belong to the same level curve. Thus we may assume that $t_k \in L_k$, a level curve of ∂D ending at points of $|z| = 1$ other than $e^{i\theta}$ and that $L_k \cap L_j = \emptyset$ for $j \neq k$. Since $t_k \in H(b)$ and since $H(2b)$ meets $|z| = 1$ only at the point $e^{i\theta}$, each curve L_k contains a subarc lying in $H(2b)$ with initial point t_k and terminal point on $\partial H(2b)$.

Let $\zeta = \psi_k(z)$ be a Möbius transformation of $|z| < 1$ onto $|\zeta| < 1$ such that $\psi_k(t_k) = 0$. Since f is a quasi-normal function (Theorem 2), the sequence $\{f \circ \psi_k^{-1}\}$ contains a subsequence $\{f \circ \psi_\alpha^{-1}\}$ which converges either to a holomorphic function g subuniformly in $|\zeta| < 1$ or to $g \equiv \infty$ subuniformly in $|\zeta| < 1$ minus at most $n - 1$ points.

Let $K(b)$ be the hyperbolic disc with center $\zeta = 0$ and radius b. Since $\inf_{z \in T} \sigma(z, t_\alpha) < b$ and $\sup_{z \in T} \sigma(z, t_\alpha) = \infty$ for each α and since the hyperbolic metric is invariant under one-to-one conformal mappings, each $\psi_\alpha(T)$ contains a subarc with one end point on $\partial K(b)$, the other on $\partial K(2b)$. Furthermore, each $\psi_\alpha(L_\alpha)$ contains a subarc lying in $K(b)$ with one end point at $\zeta = 0$, the other on $\partial K(b)$. Hence, each of the sequences $\{\psi_\alpha(L_\alpha)\}$ and $\{\psi_\alpha(T)\}$ has at least one accumulation continuum J and S lying in $|\zeta| < 1$. Since $|g(\zeta)| = r$ for $\zeta \in J$, then g must be holomorphic in all of $|\zeta| < 1$. On the other hand, $g(\zeta) = \infty$ for $\zeta \in S$ since f has asymptotic value ∞ along T. This contradiction completes the proof of the theorem.

The following corollary follows immediately from Theorem 3 and [**2**, Theorem 2].

COROLLARY 1. *If $f \in \mathscr{V}$ then f has angular limits at a dense subset of $|z| = 1$.*

Example 1. We shall construct a function $w = f(z)$ holomorphic in $|z| < 1$ such that f is quasi-normal of order n and $f \in \mathscr{V}$ $(r_0, n + 1)$ for each $r_0 > 1$. This example shows that Theorem 3 cannot be improved.

Our method is to construct a hyperbolic Riemann surface F lying over the

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w-plane and then to let $\hat{f}$ be the conformal map of $|z| < 1$ onto F; then $f = p \circ \hat{f}$ where p is the projection of F onto the w-plane.

Let $D_1^1, D_1^2, \ldots, D_1^n$ be n copies of the unit disc $|w| < 1$. Join D_1^k to D_1^{k+1} $(k = 1, 2, \ldots, n - 1)$ by a snake-like strip lying in $|w| > 1$ (see Figure 1). The resulting surface F_1 is a simply connected smooth covering of the w-plane.

For each positive integer j let F_j be the surface obtained by stretching F_1 by a factor of j, that is, $F_j = jF_1$. We join F_j to F_{j+1} as follows: joint D_j^n to D_{j+1}^1 by a snake-like strip S_j passing through $|w| < 1$ so that $S_j \cap S_{j+1} = \emptyset$ (see Figure 2). The resulting surface F consisting of $F_1, S_1, F_2, S_2, \ldots, F_j, S_j, \ldots$ is a smooth simply connected covering surface of the w-plane such that each component of F lying over $|w| > r_0 > 1$ has at most $n + 1$ points lying over any given point in the w-plane.

Let $\hat{f}$ be a one-to-one conformal map of $|z| < 1$ onto F. Clearly, $f \in \mathscr{V}(r_0, n + 1)$ for each $r_0 > 1$.

To show that f is a quasi-normal function of order n we need to produce a sequence $\{\psi_\beta\}$ of Möbius transformations of $|s| < 1$ onto $|z| < 1$ and a set of points $s_1, s_2, \ldots, s_n$ that are strongly irregular for the sequence $\{f \circ \psi_\beta\}$.

Let $\theta_j^1, \theta_j^2, \ldots, \theta_j^n$ be the n points of F_j lying over the point $w = 0$. Let h_1 be a one-to-one conformal map of F_1 onto $|\zeta| < 1$ such that $h_1(\theta_1^1) = 0$; denote by ζ_j the point $h_1(\theta_1^j)$ $(j = 1, 2, \ldots, n)$. Let h_j be the one-to-one

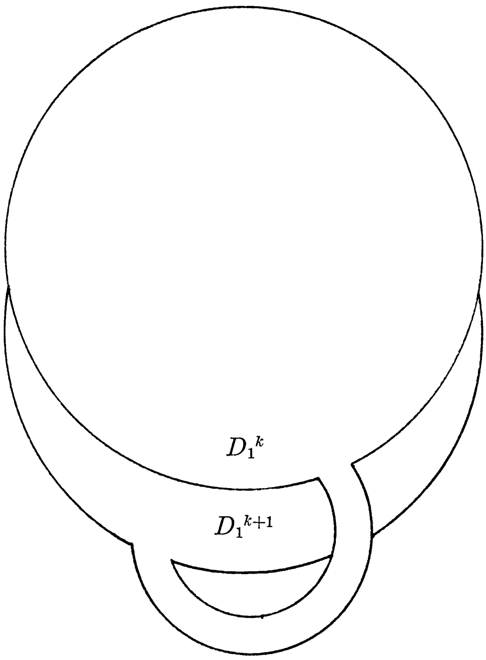

FIGURE 1

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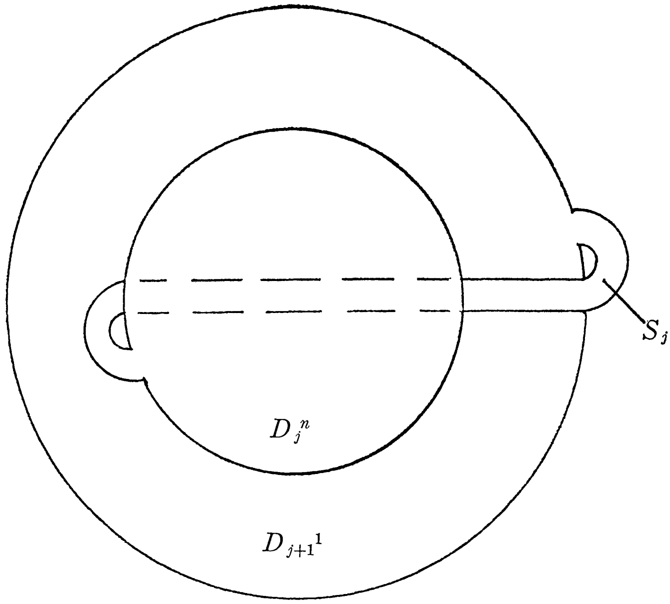

FIGURE 2

conformal map of F_j onto $|\zeta| < 1$ defined by $h_j(t) = h_1(t/j)$. Let ψ_j be a Möbius transformation of $|s| < 1$ onto $|z| < 1$ such that $\psi_j(0) = \hat{f}^{-1}(\theta_j^1)$. Denote by χ_j the map $\psi_j^{-1} \circ \hat{f}^{-1} \circ h_j^{-1}$ of $|\zeta| < 1$ into $|s| < 1$.

We shall show that the sequence $\{\chi_j\}$ has a subsequence $\{\chi_\alpha\}$ that converges to a one-to-one holomorphic function from $|\zeta| < 1$ into $|s| < 1$ such that $|\chi(\zeta)| \leqq |\zeta|$. Let us accept this for now and proceed to show that $f \circ \psi_\alpha$ has a strong subsequence with n irregular points.

Let $s_j = \chi(\zeta_j)$ ($j = 1, 2, \ldots, n$). Since χ is one-to-one and $|\chi(\zeta)| \leqq |\zeta|$, it follows that $s_j \neq s_k$ for $j \neq k$ and $|s_j| < 1$. Each s_j will be a strongly irregular point for any subsequence of $\{f \circ \psi_\alpha\}$ converging to infinity because $\chi_\alpha(\zeta_j)$ is a zero of $f \circ \psi_\alpha$ and $\chi_\alpha(\zeta_j) \rightarrow s_j$ as $\alpha \rightarrow \infty$.

Each surface F_j has a crosscut lying over $|w| = j$ that separates θ_j^1 and θ_j^2. Hence, for each j there is a crosscut of $|s| < 1$ by an arc of $\{s : |f(\psi_j(s))| = j\}$ that separates $s = 0$ from $\chi_j(\zeta_2)$. These crosscuts have at least one accumulation continuum G in $|s| < 1$ because $\chi_\alpha(\zeta_2) \rightarrow s_2$ as $\alpha \rightarrow \infty$ and $|s_2| \neq 1$. Since $f \in \mathscr{V}(r_0, n + 1)$, then by Theorem 3, f is a quasi-normal function of order at most n. Thus the sequence $\{f \circ \psi_\alpha\}$ has a subsequence which converges to a function g subuniformly in $|s| < 1$ minus at most n points. Since $g(s) = \infty$ on G, it follows that $g \equiv \infty$ and $s_1, \ldots, s_n$ are strongly irregular points for a subsequence of $\{f \circ \psi_\alpha\}$. Hence, we will have shown f is a quasi-normal function of order precisely n once we show that a subsequence of $\{\chi_j\}$ converges to a one-to-one holomorphic function χ on $|\zeta| < 1$ such that $|\chi(\zeta)| \leqq |\zeta|$.

Each χ_j is a one-to-one holomorphic map of $|\zeta| < 1$ into $|s| < 1$ satisfying

$\chi_j(0) = 0$. Thus the sequence $\{\chi_j\}$ is a normal family and hence has a subsequence $\{\chi_\alpha\}$ that converges subuniformly on $|\zeta| < 1$ to a holomorphic function χ with $\chi(0) = 0$. By Schwarz's lemma, $|\chi(\zeta) \leqq |\zeta|$. Either χ is one-to-one or $\chi \equiv 0$ by Hurwitz's theorem. We shall show $\chi \not\equiv 0$.

Since $|\chi_j(\zeta)| = 1$ on an arc $L \subset \{|\zeta| = 1\}$, we can extend χ_j to be holomorphic and one-to-one in $K = L \cup \{\zeta : |\zeta| \neq 1\}$. Let q be a conformal mapping of K onto $|s| < 1$ such that $q(0) = 0$ and $q'(0) > 0$. Then

$$\left[\left|\frac{d}{ds}\chi_j(q^{-1}(s))\right|\right]_{s=0} \leqq \frac{1}{q'(0)}$$

since $|\chi_j'(0)| \leqq 1$ by Schwarz's lemma. Thus

$$|\chi_j(q^{-1}(s))| \leqq \frac{|s|}{(1 - |s|)^2 q'(0)} \qquad (|s| < 1)$$

by a distortion theorem of Koebe. Thus $\{\chi_j(q^{-1}(s))\}$ is uniformly bounded on compact subsets of $|s| < 1$ and consequently $\{\chi_j(\zeta)\}$ is uniformly bounded on compact subsets of K. Hence χ_j is a normal family in K, and therefore a subsequence of the sequence $\{\chi_\alpha\}$ converges subuniformly in K to a function $\tilde{\chi}$ holomorphic in K. Thus, $\chi \not\equiv 0$ because $|\tilde{\chi}(\zeta)| \geqq 1$ for $|\zeta| > 1$ and $\tilde{\chi}(\zeta) = \chi(\zeta)$ for $|\zeta| < 1$.

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West Virginia College of Graduate Studies,
Institute, West Virginia