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The Fitting length of a finite soluble group and the number of conjugacy classes of its maximal nilpotent subgroups

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It is shown that there exists a logarithmic upper bound on the Fitting length $h(G)$ of a finite soluble group G in terms of the number $\nu(G)$ of the conjugacy classes of its maximal nilpotent subgroups. For $\nu(G) = 3$, the best possible bound on $h(G)$ is shown to be 4.

1. Introduction

All groups considered in this paper are finite and soluble, and, for a group G, $\nu(G)$ denotes the number of conjugacy classes of its maximal nilpotent subgroups and $h(G)$ denotes its Fitting length. In [5], it was shown that the Fitting length of a finite soluble group of odd order is bounded in terms of the number of conjugacy classes of its maximal nilpotent subgroups. Our main purpose of this paper is to show that the result is true for any finite soluble group, not necessarily of odd order. More precisely, we show here that

THEOREM 1.1. *For any finite soluble group* G ,

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$$h(G) \le \begin{cases} 1 & , \textit{ if } \nu(G) = 1 ; \\ 2\left\{1+\log_3\left(\frac{18\nu(G)-19}{2}\right)\right\} & , \textit{ if } \nu(G) > 1 . \end{cases}$$

This result improves considerably the bound obtained in [5] for a finite group of odd order. The precise form of the bound obtained (though not its order of magnitude) relies, however, on an unpublished result of Newman (see Theorem 3.5).

Theorem 1.1 is proved in Section 3. In Section 4, we obtain the best possible bound on $h(G)$ in the special case when $\nu(G) = 3$. The sharp bound on $h(G)$ in the case when $\nu(G) = 2$ or in the case when $\nu(G) = 3$ but $|G|$ is odd has been discussed in [5].

2. Notation

We will use the following notation in the rest of the paper:

$\|G\|$	= the order of the group G
$Z(G)$	= the centre of G
$N_G(H)$	= the normalizer in G of the subgroup H of G
$C_G(H)$	= the centralizer of H in G
$\Phi(G)$	= the Frattini subgroup of G
$F(G)$	= the Fitting subgroup of G
$O_{p'p}(G)$	= the largest normal p-nilpotent subgroup of G
$O_{p'}(G)$	= the largest normal p'-subgroup of G
$d(G)$	= the minimal number of generators of G
$H \le G$	reads H is a subgroup of G
$H < G$	reads H is a proper subgroup of G
$H \trianglelefteq G$	reads H is a normal subgroup of G
$H \vartriangleleft G$	reads H is a proper normal subgroup of G
$G \times H$	= the direct product of the groups G and H
p, q, r	will always denote primes
$\mathrm{GF}(p)$	= the field with p elements.

3. The proof of the theorem

First, we will prove some lemmas. We begin with the following elementary lemma.

LEMMA 3.1. *Let* G *be a group whose Fitting subgroup* F *is a* p*-group and let* Q *be a* q*-subgroup of* G , $q \neq p$. *Then there is a maximal nilpotent subgroup* W *of* G *such that* $C_F(Q) = W \cap F$. *Moreover, if* $Q \neq \{1\}$, *then* $W \cap F < F$.

Proof. Let W be a maximal nilpotent subgroup of G which contains $C_F(Q) \times Q$. Clearly $W \cap F = C_F(Q)$. Also, since $C_G(F) \leq F$ (see Theorem 6.1.3 in Gorenstein [3]), $F \nleq W$ unless $Q = \{1\}$. Thus, if $Q \neq \{1\}$, $W \cap F < F$, as required.

Next, we show

LEMMA 3.2. *Let* G *be a group whose Fitting subgroup* F *is an elementary abelian* p*-group, let* Q *be a non-trivial* q*-subgroup of* G , $q \neq p$, *and let* S *be a maximal element in the set*

$$\underline{\underline{X}} = \{Q^* \mid Q^* \triangleleft Q,\ C_F(Q) < C_F(Q^*)\} .$$

Then $C_Q(C_F(S)) = S$ *and, moreover,* $Z(Q/S)$ *is cyclic.*

Proof. Since $S \trianglelefteq Q$, $C_F(S)$ is Q-invariant. Thus $H = C_Q(C_F(S)) \triangleleft Q$. However, $S \leq H$ and also $C_F(Q) < C_F(S) \leq C_F(H)$. Hence, since S is a maximal element in $\underline{\underline{X}}$, it follows that $S = H$, as required.

In order to show that $Z(Q/S)$ is cyclic, we proceed as follows. Since $C_F(S)$ is Q-invariant, we observe first that, by Theorem 3.3.2 in Gorenstein [3], $C_F(S) = C_F(Q) \times L$, where L is Q-invariant. Clearly, $L \neq \{1\}$ since $C_F(Q) < C_F(S)$. Now, let L^* be a non-trivial Q-invariant subgroup of L of minimal order and let $K = C_Q(L^*)$. Since L^* is Q-invariant, $K \triangleleft Q$. Moreover, $K \leq C_Q(L^*C_F(Q))$. In particular, $C_F(Q) < C_F(K)$, and so, $K \in \underline{\underline{X}}$. However, $S \leq K$. Thus, since S is a maximal element of $\underline{\underline{X}}$, $K = S$, and hence Q/S is represented faithfully as well as irreducibly on L^* , regarded as a vector space over the field $GF(p)$. By Theorem 3.2.2 in Gorenstein [3], it follows then that $Z(Q/S)$ is cyclic, and the proof is complete.

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The following lemma extends Lemma 3 of [5].

LEMMA 3.3. *Let* G, F *and* Q *be as in Lemma* 3.2 *and let* l *be the largest integer for which there exists a chain of subgroups*

$$(3.4) \qquad C_F(Q) = V_l \cap F < V_{l-1} \cap F < \ldots < V_1 \cap F < F ,$$

where V_i *is a maximal nilpotent subgroup of* G *for* $i = 1, 2, \ldots, l$ *and* $C_F(Q) \times Q \le V$. *If* $Q/Z(Q)$ *is elementary abelian, then* $d(Q)$ *is at most* $2l$.

Proof. We proceed by induction on $|Q|$. Let $\underline{\underline{X}}$ be defined as in Lemma 3.2, and let R be a maximal element in $\underline{\underline{X}}$. Then, by the same lemma, $\bar{Z}/R = Z(Q/R)$ is cyclic. Also, since $C_F(Q) < C_F(R)$ and, by Lemma 3.1, there is a maximal nilpotent subgroup W of G such that $W \cap F = C_F(R)$, it follows, in view of our hypothesis, that a chain of subgroups of the type (3.4) which joins $C_F(R)$ to F has length at most $l - 1$.

Suppose first that $Q = \bar{Z}$, so that Q/R is cyclic. Then, since, by induction, $d(R) \le 2(l-1)$, $d(Q) \le 2l - 1$ and we are done. Hence, assume next that $Q \ne \bar{Z}$. Let A/R be a maximal abelian normal subgroup of Q/R . Since $Q/Z(Q)$, and therefore $Q/\bar{Z}$ is elementary abelian, it follows from Satz III, 13.7 in Huppert [4] that there is a maximal abelian normal subgroup B/R of Q/R such that $AB = Q$, $A \cap B = \bar{Z}$ and $d(A/\bar{Z}) = d(B/\bar{Z})$; consequently $d(Q/R) \le 2d(A/R)$. It remains now to show that $d(A/R) \le l$ and that a chain of subgroups of the type (3.4) which joins $C_F(R)$ to F has length at most $l - d(A/R)$, for then, by the inductive hypothesis, $d(R) \le 2l - 2d(A/R) \le 2l - d(Q/R)$, and hence $d(Q) \le d(R) + d(Q/R) \le 2l$. We show this as follows.

Let $A = A_0$ and for $i = 1, 2, \ldots$, define A_i to be a maximal element in the set

$$\{Q^* \mid R \le Q^* \le A_{i-1} \text{ and } C_F(A_{i-1}) < C_F(Q^*)\} .$$

For some integer $n \ge 1$, $A_n = R$. Now, let $\bar{G}$ be the semidirect product of $X = C_F(R)$ by $Y = A/R$. Since, by Lemma 3.2, $C_Y(X) = \{1\}$, X is clearly the Fitting subgroup of $\bar{G}$. Thus, by the same lemma and the fact that A/R is abelian, it follows now that A_{i-1}/A_i is cyclic for $i = 1, 2, \ldots, n$. In particular, $d(A/R) \le n$. On the other hand, by Lemma 3.1, there exist maximal nilpotent subgroups $W_0, W_1, \ldots, W_n$ of G such that

$$C_F(A_0) = W_0 \cap F < C_F(A_1) = W_1 \cap F < \ldots < C_F(A_n) = C_F(R) = W_n \cap F .$$

Hence, by our hypothesis, $n \le l$, whence $d(A/R) \le n \le l$. Also, for the same reason, the chain of subgroups of the type (3.4) joining $C_F(R)$ to F has length certainly at most $l - n \le l - d(A/R)$. This completes the proof.

REMARK. In Lemma 3.3 we always have $l \le \nu(G) - 1$, since each member of at least one conjugacy class of maximal nilpotent subgroups of G contains F, and trivially, if V and W are conjugate maximal nilpotent subgroups of G, neither $V \cap F < W \cap F$ nor $W \cap F < V \cap F$.

It has been well-known for some time that the Fitting length of a soluble linear group is bounded in terms of its degree. The best possible bound has been obtained in recent unpublished work of Newman, and determines the precise form of the bound in Theorem 1.1. We now state this unpublished result of Newman.

THEOREM 3.5 (Newman). *Let G be a soluble linear group of degree $n \ge 1$. Then*

$$h(G) \le \begin{cases} 1 & , \text{ if } n = 1 ; \\ 3 & , \text{ if } n = 2 ; \\ 2s + 4 , & \text{ if } 2.3^s < n \le 4.3^s ; \\ 2s + 5 , & \text{ if } 4.3^s < n \le 2.3^{s+1} . \end{cases}$$

In particular, $h(G) \le 2 \log_3 \left(\frac{9n-1}{4}\right)$.

We will deduce the main result of this section from the following lemma.

LEMMA 3.6. *Let G be a group whose Fitting subgroup F is an elementary abelian p-group. If H/K is a q-chief factor of G, where $q \ne p$, then*

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$$h\left(G/C_G(H/K)\right) \leq 2 \log_3 \left(\frac{18\nu(G)-19}{2}\right) .$$

Proof. Let Q be a Sylow q-subgroup of H and $N = N_G(Q)$. Then, by the Frattini argument, $G = NH$, and hence $G/C_G(H/K) \cong N/C_N(H/K) = N/C_N(Q/Q\cap K)$. Thus, clearly it will be sufficient to show that $h\left(N/C_N(Q/Q\cap K)\right) \leq 2 \log_3 \left(\frac{18\nu(G)-19}{2}\right)$.

Let E be a characteristic subgroup of Q given by Lemma 8.2 of Feit and Thompson [1]. Then, by the same lemma, E has, among others, the following two properties of interest to us:

(i) $E/Z(E)$ is elementary abelian;

(ii) every non-trivial q'-automorphism of Q induces a non-trivial automorphism of E.

In particular, by (ii), $C_N(E)/C_N(Q)$ is a q-group. Also, since E is a characteristic subgroup of $Q \trianglelefteq N$, it follows that E, and hence $C_N(E)$, is a normal subgroup of N. Now, by Theorem 3.1.3 in Gorenstein [3], $G/C_Q(H/K)$, and hence $N/C_N(Q/Q\cap K)$, has no non-trivial normal q-subgroups. Thus, since the normal subgroup $C_N(E)C_N(Q/Q\cap K)/C_N(Q/Q\cap K)$ of $N/C_N(Q/Q\cap K)$ is isomorphic to $C_N(E)/C_N(E) \cap C_N(Q/Q\cap K)$ which is a factor of the q-group $C_N(E)/C_N(Q)$, we have $C_N(E) \leq C_N(Q/Q\cap K)$. For similar reasons one can say even more, namely that $C_N\left(E/\Phi(E)\right) \leq C_N(Q/Q\cap K)$; for, by a result of Burnside (see Theorem 5.1.4 in Gorenstein [3], for example), $C_N\left(E/\Phi(E)\right)/C_N(E)$ is a q-group, and, moreover, $C_N\left(E/\Phi(E)\right) \trianglelefteq N$ since $\Phi(E) \triangleleft N$, being a characteristic subgroup of $E \triangleleft N$.

Thus, it suffices to show that $h\left\{N/C_N\left(E/\Phi(E)\right)\right\}$ is below the upper bound claimed. However, $N/C_N\left(E/\Phi(E)\right)$ is a soluble linear group of degree at most the dimension of $E/\Phi(E)$ regarded as a vector space over $\mathrm{GF}(q)$, and the latter is, in view of (i), Lemma 3.3 and the remark following the latter, at most $2\left(\nu(G)-1\right)$. Thus, by the preceding result of Newman, namely Theorem 3.5,

$$h\left(N/C_N\left(E/\Phi(E)\right)\right) \le 2\log_3\left(\frac{18\nu(G)-19}{2}\right),$$

and so the proof is complete.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Since each Sylow subgroup of G is contained in some maximal nilpotent subgroup of G, it is immediate that $h(G) \le 1$ when $\nu(G) = 1$. For a proof by contradiction, let G be a counter-example of minimal order. Then $\nu(G) > 1$, and, in view of Lemma 1 in [5] and the fact that $\underline{\underline{N}}^n$, the class of groups of nilpotent length at most n, where n is any positive integer, is a saturated formation (see Gaschütz [2], for example), G is a monolithic group with its Fitting subgroup F as its monolith. In particular, F is an elementary abelian p-group for some p. Let $k = \left[2\left\{1+\log_3\left(\frac{18\nu(G)-19}{2}\right)\right\}\right]$, the largest integer is less than or equal to $2\left\{1+\log_3\left(\frac{18\nu(G)-19}{2}\right)\right\}$. Since, for each each q dividing $|G|$, $O_{q'q}(G)$ is the intersection of the centralizers of the q-chief factors of G (see Satz VI, 5.4 (b) in Huppert [4]), Lemma 3.6 and the fact that the class $\underline{\underline{N}}^{k-2}$ is a formation (see Gaschütz [2]) give

$$h\left(G\Big/\bigcap_{q\ne p} O_{q'q}(G)\right) \le \left[2\log_3\left(\frac{18\nu(G)-19}{2}\right)\right] = k-2.$$

But $\bigcap_{q\ne p} O_{q'q}(G)$ is q-nilpotent for every q other than p, and hence it is an extension of a p-group by a nilpotent group. Thus $h(G) \le k$, and so G is not a counterexample after all. This contradiction completes the proof.

We conclude this section with the remark that the bound of Theorem 1.1 is, at least for certain values of $\nu(G)$, not the best possible, as the cases $\nu(G) = 2$ (see [5]) and $\nu(G) = 3$ (see the following section) show.

4. The case $\nu(G) = 3$

Here, in this section, we will obtain a sharp bound on the Fitting length of a group G for $\nu(G) = 3$. In particular, we show

PROPOSITION 4.1. *For* $\nu(G) = 3$, $h(G) \le 4$.

For the proof of Proposition 4.1, we will need the following result which is a slight extension of Lemma 6 in [5].

LEMMA 4.2. *Let* G *be a monolithic group with its Fitting subgroup* $F(G)$ *as its monolith. If* $h(G) > 1$ *, there is a normal subgroup* S *of* G *such that* $h(G/S) = h(G) - 1$ *and the Fitting subgroup* R/S *of* G/S *is the monolith of* G/S *. Moreover, if* V *is a maximal nilpotent subgroup of* G *such that* $VF(G)/F(G) \geq F(G/F(G))$ *, then* $VS/S \geq R/S$ *.*

Proof. For the proof of the first part of the lemma, we refer the readers to Lemma 6 in [5]. The second part of the lemma is immediate from the proof of the first part.

We now proceed to prove Proposition 4.1.

Proof of Proposition 4.1. Suppose the result is false and let G be a minimal counter-example. Then, in view of Lemma 1 of [5], the corollary following the proof of the main theorem in [5] and the fact that the class $\underline{\underline{N}}^4$ of finite groups of the Fitting length at most 4 is a saturated formation (see [2]), it follows that

(4.3) G is monolithic with $F(G)$ as its monolith, $\nu(G/F(G)) = 3 = \nu(G)$ and $h(G) = 5$.

Let $|F| = p^{\alpha}$, $\alpha > 0$. Then, since $C_G(F(G)) = F(G)$ by Theorem 6.1.3 in Gorenstein [3], a Sylow p-subgroup P of G is clearly a maximal nilpotent subgroup of G . Let V and W be representatives of the remaining two conjugacy classes of maximal nilpotent subgroups of G , respectively, and assume, without loss of generality, that $VF(G) \geq F_2$, where $F_2/F(G) = F(G/F(G))$. Since $F(G)$ is the largest normal p-subgroup of G , $F_2/F(G)$ is a p'-group. Hence, since $C_G(F_2/F(G)) \leq F_2/F(G)$ (see Theorem 6.1.3 in Gorenstein [3]), it follows that

(4.4) $$VF(G)/F(G) \text{ is a } p'\text{-group.}$$

Consequently,

(4.5) $$V \cap F(G) = 1 .$$

For, assume to the contrary that $V \cap F(G) > \{1\}$. Since $F(G)$ is an

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abelian p-group (see (4.3)) and $V/V\cap F(G)$ is a p'-group (see (4.4)), $V \cap F(G) \leq Z(F_2)$, so that, in view of our assumption, $Z(F_2) > \{1\}$. But $Z(F_2) \trianglelefteq G$, being a characteristic subgroup of a normal subgroup, namely F_2, of G. Therefore, since $F(G)$ is the monolith of G (see (4.3)), $Z(F_2) \geq F(G)$. However, since $C_G(F(G)) = F(G)$, this can only happen if $F(G) = G$. Thus, since $h(G) = 5$ (see (4.3)), we must have $V \cap F(G) = \{1\}$.

Now, since G is monolithic with its Fitting subgroup as its monolith and $h(G) > 1$ (see (4.3)), G has, in view of Lemma 4.2, a normal subgroup S such that $h(G/S) = h(G) - 1$ and the Fitting subgroup R/S of G/S is the monolith of G/S. Clearly $h(G/S) > 3$, since otherwise $h(G) \leq h(G/S) + 1 \leq 4$, contrary to G being a minimal counter-example. Thus, it follows from (4.3) that

(4.6) $$h(G/R) = 3 \text{ and } h(G/S) = 4 .$$

In particular,

(4.7) $$\nu(G/S) = \nu(G) = 3 .$$

For, if $\nu(G/S) \neq \nu(G)$, then by Lemma 1 of [5], $\nu(G/S) \leq 2$, and hence, by the corollary following the proof of the main theorem in [5], $h(G/S) \leq 3$, contrary to (4.6).

Let $|R/S| = q^{\beta}$, $\beta > 0$. Since, by Lemma 4.2, $VS/S \geq R/S$, it follows from (4.4) that

(4.8) $$q \neq p .$$

Also,

(4.9) $$q \neq 2 .$$

For, suppose to the contrary that $q = 2$. Then, the proof of Lemma 3.6 shows that $G/C_G(R/S) = G/R$ is a factor of $\mathrm{GL}(4, 2)$. Since $|\mathrm{GL}(4, 2)| = 2^6.3^2.5.7$, it follows, therefore, that G/R is a soluble group of order dividing $2^6.3^2.5.7$. Thus, in view of Theorem 1.3.10 (ii) of Gorenstein [3], the group of automorphisms induced by G on a 5-chief factor or a 7-chief factor of G/R is cyclic, and clearly that induced on a 3-chief factor of G/R is a 2-group. In particular, the group of automorphisms induced by G on each odd-ordered chief factor of G/R is nilpotent. But then, since $\underline{\underline{N}}$, the class of all finite nilpotent groups, is a formation, and, by Theorem 3.1.3 of Gorenstein [3], G/R has no non-trivial normal 2-subgroups so that $F(G/R)$ is the intersection of the centralizers of the odd-ordered chief factors of G/R (see Satz III, 4.3 in Huppert [4] and the proof of Theorem 1.1), we have $h(G/R) \leq 2$. However, this is impossible because of (4.6), and so we conclude that q cannot be 2.

But, in view of (4.3), Lemma 3 in [5] and the remark following the proof of Lemma 3.3,

(4.10) G has no elementary abelian r-subgroups of order r^3, for each $r \neq p$,

so that, by a result of Thompson, namely Lemma 5.24 in [6], every odd-ordered r-chief factor of G, for each $r \neq p$, is of rank at most 2. Thus, since G/R is represented faithfully and irreducibly on R/S and $h(G/R) = 3$ $\{$see (4.6)$\}$, we have, in view of Theorem 3.2.5 of Gorenstein [3], that

(4.11) $|R/S| = q^2$; consequently, G/R is isomorphic to a subgroup of $\mathrm{GL}(2, q)$.

It follows then that

(4.12) $$q \neq 3 .$$

For otherwise, $G/R \cong \mathrm{GL}(2, 3)$ since every proper subgroup of $\mathrm{GL}(2, 3)$ has the Fitting length at most 2, while $h(G/R) = 3$ $\{$see (4.6)$\}$. But then, contrary to (4.7), $\nu(G/S) \geq 4$, the Sylow 2-subgroups, the Sylow 3-subgroups and the two distinct conjugacy classes of 6-cycles of G/S constituting four distinct conjugacy classes of maximal nilpotent subgroups of G/S. Hence q cannot be 3.

Now, let Z/R be the centre of G/R. Since G/R is isomorphic to a subgroup of $\mathrm{GL}(2, q)$ of the Fitting length 3 $\{$see (4.6) and (4.11)$\}$, it follows that

(4.13) $G/Z \cong S_4$, the symmetric group on four letters, and $Z \neq R$.

Moreover, if $K \triangleleft G$ such that $K \leq R$ and G/K is isomorphic to a subgroup of $\mathrm{GL}(2, q)$, then Z/K is the centre of G/K.

For, let $H \leq GL(2, q)$ such that $H \cong G/R$ and let $\tilde{Z} = Z(GL(2, q))$. Then $H/H\cap\tilde{Z} \cong H\tilde{Z}/\tilde{Z} \leq PGL(2, q)$, and so, by Satz II, 8.20 (b) in Huppert [4], $H/H\cap\tilde{Z}$ is isomorphic to a subgroup of $PSL(2, q^2)$. But now, by a well-known result of Dickson (see Hauptsatz II, 8.27 in Huppert [4], for example) and Theorem 2.8.3 (iii) in Gorenstein [3], $H/H\cap\tilde{Z} \cong S_4$. In particular, $Z(H) = H \cap \tilde{Z}$, and so, $G/Z \cong S_4$. Hence, since $q > 3$ (see (4.9) and (4.12)), $Z \neq R$ as S_4 has no faithful, irreducible 2-dimensional representation over a field of characteristic greater than 3. The rest of (4.13) now follows.

As a consequence of (4.13), (4.9), (4.12) and the fact that G/R has no non-trivial normal q-subgroups, we thus have

(4.14) $q \nmid |G/R|$, and so R/S is a maximal nilpotent subgroup of G/S.

But,

(4.15) V contains a Sylow q-subgroup of G.

For, otherwise, W contains one, since a Sylow q-subgroup of G is contained in some maximal nilpotent subgroup of G. Since, by Lemma 4.2, $VS/S \geq R/S$, it follows then that $R/S \leq WS/S \cap VS/S$. Thus, since $C_G(R/S) = R/S$, both WS/S and VS/S are q-groups, and so $\nu(G/S) \leq 2$, contrary to (4.7).

Next, let $L = O_{q'}(G)$, N/L a minimal normal q-subgroup of G/L and $C = C_G(N/L)$. Since $C_{G/S}(R/S) = R/S$ and L avoids R/S, we have $L \leq S$. On the other hand, since V contains a Sylow q-subgroup of G (see (4.15)), and since the Fitting subgroup of G/L is a q-group and contains its own centralizer in G/L, it is clear that

(4.16) VL/L is a Sylow q-subgroup of G/L.

Thus, as G/Z is a q'-group (see (4.14)), it follows that

(4.17) $$VL \leq Z.$$

Suppose first that $WL \geq N$. Then clearly $(WL/L)_{q'} \geq C/L$, and so WC/C is a q-group. Hence, since, in view of (4.16), VC/C is a Sylow q-subgroup of G/C, $WC/C \leq V^gC/C$ for some g in G. But now, since $V \leq Z$ (see (4.17)), we may conclude that $W \leq V^gC \leq ZC$, whence, by Lemma

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1 of [5], G/ZC has only one class of maximal nilpotent subgroups, namely, that of PCZ/CZ . Thus, G/ZC is a p-group. Since $G/Z \cong S_4$ (see (4.13)), ZC/Z must, therefore, contain a subgroup isomorphic to the alternating group A_4 on 4 letters. In particular, the Hall q'-subgroups of ZC/Z , and hence those of $C/Z\cap C$ and C/L , cannot be nilpotent. But then the nilpotent q'-subgroup $(WL/L)_{q'}$ of C/L cannot be a Hall q'-subgroup; that is, for some $r(\neq q)$, $(WL/L)_{q'}$ cannot contain any Sylow r-subgroup $(C/L)_r$ of C/L . However, $(C/L)_r \times N/L$ is a non-primary nilpotent subgroup of G/L . Since VL/L and PL/L are primary, $(C/L)_r \times N/L$ must, therefore, be contained in some conjugate of WL/L . But now, some conjugate of $(C/L)_r$ is in $(WL/L)_{q'}$, and we have a contradiction.

Hence, $WL \ngeq N$. We claim that C/L is then a q-group. Suppose this is not so, and let T/L be a non-trivial Sylow r-subgroups of C/L for some $r \neq q$. Then $(N/L) \times (T/L)$ is a non-primary nilpotent subgroup of G/L , which, as before, must be contained in some conjugate of WL/L . But then $N/L \leq WL/L$, contrary to $WL \ngeq N$.

Thus, it follows that $C \leq R$ (for G/R is a q'-group (see (4.14)) and $L \geq R$), and so $|N/L| \geq q^2$, since the alternative $|N/L| = q$ implies that G/C , and hence G/R is cyclic, contrary to (4.6).

Next, we show that $|N/L| = q^2$ and $WL \cap N > L$. Let Q be a Sylow q-subgroup of N/L ; Q is then elementary abelian, and so, from what has been just shown and (4.10), $|Q| = |N/L| = q^2$. Moreover, by Lemma 3.2, Q has a non-trivial subgroup Q^* such that $C_F(Q^*) > C_F(Q) \geq \{1\}$. Since V is a p'-group (see (4.4) and (4.5)), it follows then that some conjugate of $C_F(Q^*) \times Q^*$ is contained in W , whence $WL \cap N > L$.

Now, since $|N/L| = q^2$, we have that G/C is isomorphic to a subgroup of $\mathrm{GL}(2, q)$. Thus, since $C \leq R$, it follows from (4.13) that Z/C is the centre of G/C . But $O_q(G/C) = \{1\}$. Therefore, Z/C is a q'-group. Hence, from (4.9), (4.12) and (4.13), we get that G/C is a q'-group, and so, in view of (4.16), $V \leq C$. Thus the maximal nilpotent subgroups of G/C are just the conjugates of PC/C and WC/C , so that

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$WC/C \geq Z/C$. Since Z/C is a q'-group, in fact, $W_{q'}C/C \geq Z/C$. But $W_{q'}C$ acts trivially on $(WL\cap N)/L$; so Z, too, must act trivially on $WL\cap N/L$. Consequently, $\{1\} < (WL\cap N)/L \leq C_{N/L}(Z/L) \triangleleft G/L$, and hence $C_{N/L}(Z/L) = N/L$ as N/L is a chief factor. It follows thus that Z acts trivially on the whole of N/L, whence $Z = C$ and *a fortiori* $Z = R$, contrary to (4.13). The proof of Proposition 4.1 is now complete.

Let H be the binary octahedral group, that is, the group defined on the generators a, b, c by the relations

$$a^2 = b^3 = c^4 = abc .$$

As is well-known, H has just one element z of order 2, the centre Z of H is generated by z and $H/Z \cong S_4$. Let U be a vector space over $GF(3)$ which affords the representation of H induced from a non-trivial one-dimensional representation of Z and let U^* be any non-zero H-invariant subspace of U. It is easy to see that, for the split extension G of U^* by H, $\nu(G) = 3$ and $h(G) = 4$. (In fact, U^* can be chosen to be of order 3^4, so that G is of order $2^4.3^5$.) Thus, the bound obtained in Proposition 4.1 is the best possible.

We conclude this paper by mentioning that we have examples of groups G for which $h(G) \geq \log_3\left(\log_3\left(\nu(G)\right)\right)$.

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