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ON QUASICAUSTICS AND THEIR LOGARITHMIC VECTOR FIELDS

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Suppose $F\colon \left(C^{n+1}\times C^p,\,0\right)\to (C,\,0)$ is a germ of a holomorphic function, and $(S,\,0)\subset \left(C^{n+1},\,0\right)$ is a germ of some hypersurface in $\left(C^{n+1},\,0\right)$. The quasicaustic $Q(F)$ of F is defined as $Q(F)=\{a\in C^p; F(\bullet,\,a)$ has a critical point on $S\}$. We investigate the structure of quasicaustics corresponding to boundary singularities. The procedure for calculating the modules of logarithmic vector fields is given. The minimal set of generators for the Whitney's cross-cap singular variety is explicitly calculated.

1. INTRODUCTION

Let $\Pi=\left(M\times\widetilde{M},\,\pi_2^*\widetilde{\omega}-\pi_1^*\widetilde{\omega}\right)$ be a product symplectic space — the phase space of geometrical optics (see [6]), where $(M,\,\omega)$, $\left(\widetilde{M},\,\widetilde{\omega}\right)$ are two copies of the symplectic space of oriented lines in Euclidean space V (see [8]). Geometrically, quasicaustics appear in diffraction on apertures (see [9]). If $A\subset\Pi$ is a Lagrangian subvariety representing an optical instrument (say a halfplane aperture [8]) and L is an incident system of rays, that is, also a Lagrangian subvariety of $(M,\,\omega)$, then the Lagrangian variety of diffracted rays is a symplectic image $A(L)$ (see [7]). Let $\pi_V\colon T^*V\to V$ be the usual projection and $\widetilde{L}$ the canonical representative of $A(L)$ in T^*V (see [6]). Then the caustic of $\widetilde{L}$ is defined to be a hypersurface of V formed by two components: singular values of $\pi_V\,|_{\widetilde{L}-Sing\,\widetilde{L}}$, and $\pi_V\left(Sing\,\widetilde{L}\right)$. The latter one is called the quasicaustic of $\widetilde{L}$ by an optical instrument A. Let $F\colon \left(C^{n+1}\times C^p,\,0\right)\to (C,\,0)$ be a germ of a holomorphic function generating $\widetilde{L}$ (see [7]). By $(S,\,0)\subset\left(C^{n+1},\,0\right)$ we denote a germ of some hypersurface in $\left(C^{n+1},\,0\right)$. The quasicaustic $Q(F)$ of F is defined as

$$Q(F)=\{a\in C^p; F(\bullet,\,a)\text{ has a critical point on }S\}.$$

Let F represent the distance function from the general wavefront in the presence of an obstacle formed by an aperture (see [9, 5]) with boundary S. The corresponding quasicaustic $Q(F)$ is built up from the rays orthogonal to the given wavefront and

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touching the boundary of the aperture. The quasicaustic is a subvariety of the usual caustic (also called the bifurcation set [2, 3])

$$\{a \in C^p; F(\bullet, a) \text{ or } F\,|_{S\times C^p}\,(\bullet, a) \text{ have a critical point}\},$$

and represents the structure of shadows formed by the common, pecular positions of aperture and incident wavefront.

In this paper we investigate the structure of quasicaustics corresponding to simple boundary singularities [1, 2]. We also give, using the methods applied to the usual bifurcation sets [3, 4, 12], the general method for computing the vector fields tangent to the quasicaustic provided by the holomorphic function germs.

2. Vector fields on quasicaustics

Let $\mathcal{O}_{(y,x)}$ denote the ring of holomorphic functions $h\colon (C \times C^n, 0) \to (C, 0)$. The hypersurface $S = \{y = 0\}$ corresponds to the boundary of an aperture. Following the general scheme used in [2] for boundary singularities, we shall consider holomorphic functions $f\colon (C \times C^n, 0) \to (C, 0)$ of finite codimension, that is,

$$\dim_C \mathcal{O}_{(y,x)}/\Delta(f) < \infty,$$

where $\Delta(f) = \langle y\,(\partial f/\partial y), \partial f/\partial x_1, \dots, \partial f/\partial x_n\rangle$ denotes the ideal in $\mathcal{O}_{(y,x)}$ generated by the partial derivatives $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ and $y(\partial f/\partial y)$ (see [1, 10]). Let $g_0, \dots, g_{\mu-1}$ form a basis for $\mathcal{O}_{(y,x)}/\Delta(f)$ with $g_0 = 1$ and $g_i \in \mathcal{M}_{(y,x)}$. Then the miniversal deformation, in the category of deformations of functions on manifolds with boundary, as a Morse family for the corresponding diffracted Lagrangian variety (see [1]) is defined as follows

$$F\colon \left(C \times C^n \times C^{\mu-1}, 0\right) \to (C, 0)$$
$$F(y, x, a) = f(y, x) + \sum_{i=1}^{\mu-1} a_i g_i(y, x).$$

The set-germ

$$(\Sigma_r F, 0) = \left(\{(x, a) \in C^n \times C^p; \frac{\partial F}{\partial y}\,|_S = \frac{\partial F}{\partial x_1}\,|_S = \dots = \frac{\partial F}{\partial x_n}\,|_S = 0\}, 0\right)$$

we call the restricted critical set.

Using the splitting Lemma (see [10]) and the versality property of F we have,

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PROPOSITION 2.1.

A. *The restricted critical set* $(\Sigma_r F, 0)$ *is the germ of a smooth manifold of dimension* $p-1$.

B. *The quasicaustic of* F, $(Q(F), 0)$ *is an image of* $(\Sigma_r F, 0)$ *by the natural projection* $\pi \colon \Sigma_r F, 0 \to C^p, 0$ *to the second factor.*

The set of logarithmic vector fields of $Q(F)$ at 0 is defined (see [**11, 12**]) to be the set of germs of holomorphic vector fields on C^p at 0, tangent to the nonsingular part of $Q(F)$; it is an $\mathcal{O}_{(a)}$-module

PROPOSITION 2.2. *Let* $\xi \in \operatorname{Derlog} Q(F)$; *then it is* π*-liftable, that is, for some germ of a vector field* $\tilde{\xi}$, *on* $C^n \times C^p$, *tangent to* $\Sigma_r F$ *at* 0 *we have*

$$\xi \circ \pi = d\pi \circ \tilde{\xi}.$$

PROOF: ξ lifts uniquely by π at every point $a \in C^p - \Gamma(\pi|_{\Sigma_r F})$. Hence ξ lifts to a holomorphic vector field $\tilde{\xi}_1$ on $C^n \times C^p$, tangent to $\Sigma_r F$ and defined off a set of codimension 2 in $C^n \times C^p$. By Hartog's theorem $\tilde{\xi}_1$ extends to a holomorphic vector field $\tilde{\xi}$ tangent to $\Sigma_r F$. ∎

Now using the π-lowerable vector fields $\tilde{\xi}$ tangent to $\Sigma_r F$ we will construct the module $\operatorname{Derlog} Q(F)$. Letting F be as above, we define the ideal

$$I(F) = \langle \psi(x, a), \frac{\partial \overline{F}}{\partial x_1}(x, a), \dots, \frac{\partial \overline{F}}{\partial x_n}(x, a) \rangle \mathcal{O}_{(x,a)},$$

where ψ and $\overline{F}$ are given by decomposition

$$F(y, x, a) = F(0, x, a) + y\psi(x, a) + y^2 g(y, x, a), \quad \overline{F}(x, a) := F(0, x, a).$$

Let $\tilde{\xi} = \sum_{i=1}^{n} \beta_i (\partial/\partial x_i) + \sum_{i=1}^{p} \gamma_i (\partial/\partial a_i)$, β_i, $\gamma_i \in \mathcal{O}_{(x,a)}$, be the germ of a vector field at $0 \in C^n \times C^p$, tangent to $\Sigma_r F$. Then we have

$$\tilde{\xi}\left(\frac{\partial F}{\partial y}(0, x, a)\right) \in I(F)$$

and

$$\tilde{\xi}\left(\frac{\partial F}{\partial x_i}(0, x, a)\right) \in I(F), i = 1, \dots, n.$$

For our $F(y, x, a) = f(y, x) + \sum_{i=1}^{\mu-1} a_i g_i(y, x)$ we have

$$\psi(x, a) = \frac{\partial f}{\partial y}(0, x) + \sum_{i=1}^{\mu-1} a_i \frac{\partial g_i}{\partial y}(0, x).$$

So we need

$$\sum_{i=1}^{n} \beta_i \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \gamma_i \frac{\partial g_i}{\partial y} \,|_{0\times C^n} \in I(F)$$

and

$$\sum_{i=1}^{n} \beta_i \frac{\partial^2 \overline{F}}{\partial x_i \partial x_j} + \sum_{i=1}^{\mu-1} \gamma_i \frac{\partial \overline{g}_i}{\partial x_j} \in I(F), \quad 1 \leqslant j \leqslant n,$$

where $\overline{g}(x) := g(0, x)$. Thus we obtain

Lemma 2.3. *$\widetilde{\xi}$ is a lifting of $\xi \in \operatorname{Derlog} Q(F)$, $\xi = \sum_{i=1}^{p} \alpha_i(a)(\partial/\partial a_i)$ if and only if for some $\beta_i \in \mathcal{O}_{(x,a)}$, $(i = 1, \dots, n)$ we have*

$$\text{(2.1)} \qquad \begin{aligned} &\sum_{i=1}^{n} \beta_i \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial y} \,|_{0\times C^n} \in I(F), \\ &\sum_{i=1}^{n} \beta_i \frac{\partial^2 \overline{F}}{\partial x_i \partial x_j} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial \overline{g}_i}{\partial x_j} \in I(F). \end{aligned}$$

We have chosen the normal form for F in such a way that the variables $a_\mu, \dots, a_p$ $(p \geqslant \mu - 1)$ do not appear in F. Now following the general scheme used in [**3**, **4**] for ordinary bifurcation sets, we can propose the procedure for constructing the tangent vector fields to quasicaustics.

By the Preparation Theorem (see [**10**]), the module

$$\mathcal{O}_{(y,x,a)}/\overline{\Delta}(F),$$

where $\overline{\Delta}(F) = \langle y - \partial F/\partial y, \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle \mathcal{O}_{(y,x,a)}$, is a free $\mathcal{O}_{(a)}$-module (see [**1**]) generated by $1, g_1, \dots, g_{\mu-1}$. So for any $h \in \mathcal{O}_{(y,x,a)}$ we can write

$$\text{(2.2)} \qquad \begin{aligned} h(y, x, a) = \beta(y, x, a) y \frac{\partial F}{\partial y}(y, x, a) &+ \sum_{i=1}^{n} \beta_i(y, x, a) \frac{\partial F}{\partial x_i}(y, x, a) \\ &+ \sum_{j=1}^{\mu-1} \alpha_j(a) g_j(y, x) + \alpha(a), \end{aligned}$$

for some $\beta_i \in \mathcal{O}_{(y,x,a)}$, $\alpha_j \in \mathcal{O}_{(a)}$, $\alpha \in \mathcal{O}_{(a)}$.

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Proposition 2.4. *Let $h \in \mathcal{O}_{(y,x,a)}$ satisfy*

$$\frac{\partial h}{\partial y}\,|_{0\times C^n\times C^p} \in I(F),\ \frac{\partial h}{\partial x_i}\,|_{0\times C^n\times C^p} \in I(F), \quad i = 1, \ldots, n.$$

Then the vector field $\xi = \sum_{i=1}^{p} \alpha_i(\partial/\partial a_i)$, where α_i, $i = 1, \ldots, \mu - 1$, are defined in (2.2) and α_i, $i = \mu, \ldots, p$ are arbitrary holomorphic functions from $\mathcal{O}_{(a)}$, is tangent to the quasicaustic $Q(F) = \pi(\Sigma_r F)$. Conversely; suppose $\xi = \sum_{i=1}^{p} \alpha_i(\partial/\partial a_i)$ is tangent to $Q(F)$; then there is some $h \in \mathcal{O}_{(y,x,a)}$ as above with

$$h = \sum_{i=1}^{n} \beta_i \frac{\partial F}{\partial x_i} + \beta y \frac{\partial F}{\partial y} + \sum_{i=1}^{\mu-1} \alpha_i g_i + \alpha,$$

and $\partial h/\partial x_i\,|_{0\times C^n\times C^p} \in I(F)$, $\partial h/\partial y\,|_{0\times C^n\times C^p} \in I(F)$.

Proof: For derivatives of h we have

$$\begin{aligned}
\frac{\partial h}{\partial y}\,|_{\overline{S}} &= \beta\psi\,|_{\overline{S}} + \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial y}\,|_{\overline{S}}\, \frac{\partial \overline{F}}{\partial x_i} \\
&\quad + \sum_{i=1}^{n} \beta_i\,|_{\overline{S}}\, \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial y}\,|_{S} \in I(F), \\
\frac{\partial h}{\partial x_j}\,|_{\overline{S}} &= \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial x_j}\,|_{\overline{S}}\, \frac{\partial \overline{F}}{\partial x_i} + \sum_{i=1}^{n} \beta_i\,|_{\overline{S}}\, \frac{\partial^2 \overline{F}}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial x_j}\,|_{S} \in I(F),\ j = 1, \ldots, n,
\end{aligned}$$

where $\overline{S} = \{(y, x, a) \in C \times C^n \times C^p; y = 0\}$, But, on the basis of assumptions, these conditions are equivalent to (2.1), so $\sum_{i=1}^{p} \alpha_i(\partial/\partial a_i)$, is tangent to $Q(F)$. The converse statement is straightforward. ∎

We see that the set of all such h with $\partial h/\partial y\,|_{\overline{S}} \in I(F)$, $\partial h/\partial x_i\,|_{\overline{S}} \in I(F)$, $1 \leqslant i \leqslant n$ form an $\mathcal{O}_{(a)}$-module. In fact it is the kernel of the $\mathcal{O}_{(a)}$-module homomorphism,

$$\Phi : \mathcal{O}_{(y,x,a)} \ni h \to \left(\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right) \in \left(\frac{\mathcal{O}_{(y,x,a)}}{I(F) + \langle y\rangle \mathcal{M}_{(y,x,a)}}\right)^{n+1}.$$

$\overline{\Delta}(F) \subset I(F) + \langle y\rangle \mathcal{M}_{(y,x,a)}$ and clearly the set of tangent vector fields to $Q(F)$ is a finitely generated $\mathcal{O}_{(a)}$-module.

3. Quasicaustics of simple boundary singularities

The simple singularities of functions on the boundary $\{y=0\}$ of a manifold with boundary were classified in [**2**], (see [**1**], p.281). Their miniversal unfoldings are:

$$\begin{aligned}
&\widetilde{A}_\mu: \quad \pm y \pm x^{\mu+1} + \sum_{i=1}^{\mu-1} a_i x^i,\ \mu \geqslant 1,\\
&B_\mu: \quad \pm y^\mu \pm x^2 + \sum_{i=1}^{\mu-1} a_i y^{\mu-i},\ \mu \geqslant 2,\\
&C_\mu: \quad yx \pm x^\mu + \sum_{i=1}^{\mu-1} a_i x^{\mu-i},\ \mu \geqslant 2,\\
&\widetilde{D}_\mu: \quad \pm y + x_1^2 x_2 \pm x_2^{\mu-1} + \sum_{i=1}^{\mu-2} a_i x_2^i + a_{\mu-1} x_1,\ \mu \geqslant 4,\\
&\widetilde{E}_6: \quad \pm y + x_1^3 \pm x_2^4 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_1 x_2^2,\\
&\widetilde{E}_7: \quad \pm y + x_1^3 + x_1 x_2^3 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_2^3 + a_6 x_2^4,\\
&\widetilde{E}_8: \quad \pm y + x_1^3 + x_2^5 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_2^3 + a_6 x_1 x_2^2 + a_7 x_1 x_2^3,\\
&F_4: \quad \pm y^2 + x^3 + a_2 y + a_3 x + a_1 xy.
\end{aligned}$$

Thus we have, after direct checking, the following.

Proposition 3.1. *The quasicaustics for simple boundary singularities are:*

$$\begin{aligned}
&\widetilde{A}_\mu, \widetilde{D}_\mu, \widetilde{E}_k: \quad Q(F) = \emptyset,\\
&B_\mu: \quad Q(F) = \{a \in C^{\mu-1}; a_{\mu-1} = 0\},\\
&C_\mu: \quad Q(F) = \{a \in C^{\mu-1}; a_{\mu-1} = 0\},\\
&F_4: \quad Q(F) = \{a \in C^3; a_2^2 + \tfrac{1}{3} a_1^2 a_3 = 0\}, \ \textit{(that is Whitney's cross-cap).}
\end{aligned}$$

Thus we need to calculate only the module of vector fields tangent to $Q(F_4)$. Let us define the germ, at zero, of the variety $X := Q(F_4) \cup \{a_1 = 0\}$. We see that the vector fields tangent to $(X, 0)$ lie in $\operatorname{Derlog} Q(F_4)$.

Proposition 3.2. *The vector fields*

$$\begin{aligned}
V_1 &= -\frac{1}{6} a_1^2 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_3},\\
V_2 &= a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2},\\
V_3 &= -\frac{1}{3} a_1 \frac{\partial}{\partial a_1} + a_3 \frac{2}{3} \frac{\partial}{\partial a_3},
\end{aligned}$$

form a free basis for the $\mathcal{O}_{(a)}$-module Derlog X.

Before we prove this theorem we need the following.

PROPOSITION 3.3. *For corank two boundary singularities $F\colon (C \times C \times C^p, 0) - (C, 0)$, the space of functions $h \in \mathcal{O}_{(y,x,a)}$ reconstructing the $\mathcal{O}_{(a)}$-module of vector fields tangent to quasicaustic $Q(F)$ has the following form*

$$h(y,\, x,\, a) = \int_0^x \left(\frac{\partial F}{\partial y}(0,\, s,\, a)\psi_1(s,\, a) + \frac{\partial F}{\partial x}(0,\, s,\, a)\psi_2(s,\, a)\right) ds \\ + y^2\xi(y,\, x,\, a),$$

where $\psi_i \in \mathcal{O}_{(x,a)}$, $(i = 1, 2)$, $\xi \in \mathcal{O}_{(y,x,a)}$.

PROOF: Every function $h \in \mathcal{O}_{(y,x,a)}$ can be written in the form

$$h(y,\, x,\, a) = \eta_2(x,\, a) + y\eta_1(x,\, a) + y^2\eta(y,\, x,\, a),$$

and thus $$\frac{\partial h}{\partial y}(0,\, x,\, a) = \eta_1(x,\, a), \quad \frac{\partial h}{\partial x}(0,\, x,\, a) = \frac{\partial \eta_2}{\partial x}(x,\, a)$$

By Proposition 2.4, we can take

$$\eta_1(x,\, a) \in I(F), \text{ and } \eta_2(x,\, a) = \int_0^x g(s,\, a)ds, \quad g \in I(F),$$

obtaining all functions

$$\eta_2(x,\, a) + y\eta_1(x,\, a) + y^2\eta(y,\, x,\, a) \left(\operatorname{mod} \overline{\Delta}(F)\right),$$

defining the $\mathcal{O}_{(a)}$-module of vector fields tangent to $Q(F)$. Now we see that

$$\eta_2(x,\, a) + y\eta_1(x,\, a) + y^2\eta(y,\, x,\, a) = \eta_2(x,\, a) \\ + y^2\xi(y,\, x,\, a)\left(\operatorname{mod} \langle y\frac{\partial F}{\partial y},\, y\frac{\partial F}{\partial x}\rangle\mathcal{O}_{(y,x,a)}\right),$$

where $\xi \in \mathcal{O}_{(y,x,a)}$. Adding an element of $\langle y\rangle\overline{J}(F)$, ($\overline{J}(F)$ is an ideal of $\mathcal{O}_{(y,x,a)}$ generated by: $\partial F/\partial y$, $\partial F/\partial x_1, \ldots, \partial F/\partial x_n$) does preserve the space of functions and does not affect the resulting vector field. $\square$

PROOF OF PROPOSITION 3.2: $I(F_4) = \langle a_1x + a_2,\, 3x^2 + a_3\rangle\mathcal{O}_{(x,a)}$. By Proposition 3.3, taking $\psi_1, \psi_2, \xi \not\equiv 1$, we have

$$\begin{aligned} h_1(x,\, a) &= \frac{1}{2}a_1x^2 + a_2x = -\frac{1}{6}a_1^2y + a_2x - \frac{1}{6}a_1a_3 \left(\operatorname{mod} \overline{\Delta}(F_4)\right), \\ h_2(x,\, a) &= y^2 = -a_1xy - a_2y \left(\operatorname{mod} \overline{\Delta}(F_4)\right), \\ h_3(x,\, a) &= x^3 + xa_3 = -\frac{1}{3}a_1xy + \frac{2}{3}a_3x \left(\operatorname{mod} \overline{\Delta}(F_4)\right). \end{aligned}$$

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Then the corresponding V_i belongs to $\operatorname{Derlog} Q(F_4)$, $(i = 1, 2, 3)$. By simple computation we obtain

$$V_1(a_1) = 0, \quad V_2(a_1) = -a_1, \quad V_3(a_1) = -\frac{1}{3}a_1,$$

so $V_i \in \operatorname{Derlog} X$ as well. We also have

$$\det\left(V_1(a), V_2(a), V_3(a)\right) = -\frac{1}{3}a_1\left(a_2^2 + \frac{1}{3}a_3a_1^2\right)$$

is a reduced equation for $(X, 0)$, so by the results of Saito [11] (see also [4]) we find that $(X, 0)$ is a free divisor. □

We define the following ideals of $\mathcal{O}_{(y,x)}$ and $\mathcal{O}_{(y,x,a)}$ respectively,

$$\Theta(f) = \langle y\rangle J(f) + \langle\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\rangle^2\mathcal{O}_{(y,x)},$$

and

$$\overline{\Theta}(F) = \langle y\rangle \overline{J}(F) + \langle\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\rangle^2\mathcal{O}_{(y,x,a)}.$$

For determining all fields tangent to the quasicaustic we need the following.

Lemma 3.4. *The space $\mathcal{O}_{(y,x)}/\Theta(f)$ is finite dimensional. Its C-basis also generates the quotient space $\mathcal{O}_{(y,x,a)}/\overline{\Theta}(F)$ as an $\mathcal{O}_{(a)}$-module.*

Proof: $\Theta(f) \supset \Delta(f)$ and f is finitely determined as a boundary singularity. Thus $\mathcal{O}_{(y,x)}/\Theta(f)$ is C-finite dimensional with the basis $\{g_1, \dots, g_N\}$. Let us define the mapping

$$\Psi\colon (C \times C^n \times C^p, 0) \to \left(C \times C^n \times C^{\frac{n(n+1)}{2}} \times C^p, 0\right),$$

$$\Psi(y, x, a) = \left(y\frac{\partial F}{\partial y}(y, x, a), y\frac{\partial F}{\partial x_1}(y, x, a), \dots, y\frac{\partial F}{\partial x_n}(y, x, a)\right.,$$
$$\left.\frac{\partial F}{\partial x_i}(y, x, a)\frac{\partial F}{\partial x_j}(y, x, a), a\right),$$

with $1 \leqslant i, j \leqslant n$; $i \leqslant j$, and ordered set of pairs (i, j). Thus we have

$$\mathcal{O}_{(y,x,a)}/\Psi^*\left(\mathcal{M}_{(y,x,a)}\right)\mathcal{O}_{(y,x,a)} \cong \mathcal{O}_{(y,x)}/\Theta(f)\mathcal{O}_{(y,x)}.$$

By the Preparation Theorem (see [10]) every element h of $\mathcal{O}_{(y,x,a)}$ has the form:

$$h(y, x, a) = \sum_{\ell=1}^{N}\phi_i\left(y\frac{\partial F}{\partial y}(y, x, a), y\frac{\partial F}{\partial x_1}(y, x, a), \dots, y\frac{\partial F}{\partial x_n}(y, x, a)\right.,$$
$$\left.\frac{\partial F}{\partial x_i}(y, x, a)\frac{\partial F}{\partial x_j}(y, x, a), a\right) g_\ell(y, x).$$

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Thus

$$\mathcal{O}_{(y,x,a)}/\overline{\Theta}(F) \cong \left\{\sum_{i=1}^{N} \psi_i(a) g_i(y, x)\right\}, \quad \psi_i \in \mathcal{O}_{(a)},$$

which completes the proof of Lemma 3.4. ∎

Let $\{g_1, \ldots, g_N\}$ be a C-basis for $\mathcal{O}_{(y,x)}/\Theta(f)$. In general we have:

PROPOSITION 3.5. *Functions $h \in \mathcal{O}_{(y,x,a)}$ which reconstruct the $\mathcal{O}_{(a)}$-module of vector fields tangent to $Q(F)$, can be written as:*

$$h(y, x, a) = \sum_{i=1}^{N} \alpha_i(a) g_i(y, x),$$

where

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial y}(0, x) \in I(F),$$

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial x_j}(0, x) \in I(F),$$

$1 \leqslant j \leqslant n$.

PROOF: By Lemma 3.4, any $h \in \mathcal{O}_{(y,x,a)}$ can be written as

$$\begin{aligned} h(y, x, a) = \sum_{i=1}^{N} \alpha_i(a) g_i(y, x) &+ \beta(y, x, a) y \frac{\partial F}{\partial y}(y, x, a) \\ &+ \sum_{j=1}^{n} \beta_j(y, x, a) y \frac{\partial F}{\partial x_j}(y, x, a) \\ &+ \sum_{k,l=1}^{n} \beta_{k,\ell}(y, x, a) \frac{\partial F}{\partial x_k}(y, x, a) \frac{\partial F}{\partial x_\ell}(y, x, a), \end{aligned}$$

where $\alpha_i \in \mathcal{O}_{(a)}$, $\beta \ldots \beta_j$, $\beta_{k\ell} \in \mathcal{O}_{(y,x,a)}$. By simply checking the assumptions of Proposition 2.4, we see that the three last terms in the above formula do not affect on the resulting vector field belonging to $\operatorname{Derlog} Q(F)$. This proves Proposition 3.5. ∎

PROPOSITION 3.6. *$\mathcal{O}_{(a)}$-module* $\operatorname{Derlog} Q(F_4)$, *that is, the module of holomorphic vector fields tangent to Whitney's cross-cap, is generated by the following*

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fields:

$$V_1 = -\frac{1}{6}a_1^2\frac{\partial}{\partial a_2} + a_2\frac{\partial}{\partial a_3},$$
$$V_2 = a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2},$$
$$V_3 = -\frac{1}{3}a_1\frac{\partial}{\partial a_1} + \frac{2}{3}a_3\frac{\partial}{\partial a_3},$$
$$V_4 = a_2\frac{\partial}{\partial a_1} - \frac{1}{3}a_1a_3\frac{\partial}{\partial a_2},$$

which satisfy the relation

$$-a_1V_4 + 2a_3V_1 - 3a_2V_3 = 0.$$

PROOF: We have $\Theta(f) = \langle y^2, x^2y, x^4\rangle\mathcal{O}_{(y,x)}$, and

$$\mathcal{O}_{(y,x)}/\Theta(f) \cong [1, x, y, x^2, x^3, xy]_C.$$

By Proposition 3.5, all functions $h \in \mathcal{O}_{(y,x,a)}$ leading to the construction of Derlog $Q(F_4)$ can be written in the form:

$$h(y, x, a) = \alpha_1(a) + \alpha_2(a)x + \alpha_5(a)y + \alpha_6(a)xy + \alpha_3(a)x^2 + \alpha_4(a)x^3,$$

where $\alpha_i \in \mathcal{O}_{(a)}$, $i = 1, \ldots, 6$ are such that

$$\alpha_5(a) + \alpha_6(a)x \in I(F_4),$$
$$\alpha_2(a) + 2\alpha_3(a)x + 3\alpha_4(a)x^2 \in I(F_4).$$

By simple calculations we check that V_i, $i = 1, \ldots, 4$ are tangent to Whitneys's cross-cap. Calculations using power series or a homogeneous filtration show that these are the only vector fields generating Derlog $Q(F_4)$. In fact

$$h = \alpha_1 - \frac{1}{3}\alpha_3a_3 + \left(\alpha_2 - \frac{1}{3}\alpha_4a_3\right)x + \left(\alpha_5 - \frac{1}{3}\alpha_3a_1\right)y + \left(\alpha_6 - \frac{1}{3}\alpha_4a_1\right)xy \; (\text{mod } \overline{\Delta}(F_4)).$$

Hence all vector fields belonging to Derlog $Q(F_4)$ can be written in the form:

$$V = \alpha_6\frac{\partial}{\partial a_1} + \alpha_5\frac{\partial}{\partial a_2} + \alpha_5'\frac{\partial}{\partial a_3} - \frac{1}{6}a_1\alpha_6'\frac{\partial}{\partial a_2} + \alpha_4V_3, \tag{6.3}$$

where α_4, α_5, α_6, α_5', $\alpha_6' \in \mathcal{O}_{(a)}$, satisfy the following equations:

$$\begin{aligned} &\alpha_5 + \alpha_6x \in I(F_4), \\ &\alpha_5' + \alpha_6'x \in I(F_4), \end{aligned} \tag{6.4}$$

which are simple rewritten versions of (6.2). Here we use the formula

$$x^2 = -\frac{1}{3}a_3(\text{mod } I(F_4)).$$

Solving (6.4) using power series, we obtain an expression for (6.3), which involves only V_i, $i = 1, 2, 3, 4$, namely:

$$\begin{aligned} V = {} & A_0V_2 + A_1V_4 + V_2\sum_{i=1}^{\infty} A_{2i}\left(-\frac{a_3}{3}\right)^i \\ & + V_4\sum_{i=1}^{\infty} A_{2i+1}\left(-\frac{a_3}{3}\right)^i + C_0V_1 - \frac{1}{2}a_1C_1\left(V_3 + \frac{1}{3}V_2\right) \\ & + V_1\sum_{i=1}^{\infty} C_{2i}\left(-\frac{a_3}{3}\right)^i + -\frac{1}{2}\left(V_3 + \frac{1}{3}V_2\right)\sum_{i=1}^{\infty} C_{2i+1}\left(-\frac{a_3}{3}\right)^i + \alpha_4V_3, \end{aligned}$$

where A_i, C_i, $\alpha_4 \in \mathcal{O}_{(a)}$. □

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