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NECESSARY OPTIMALITY CONDITIONS FOR A PROBLEM WITH COSTS OF RAPID VARIATION OF CONTROL

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Abstract

In this paper a control problem with a cost functional depending on the number of switchings and on the speed of alterations of control is considered. Necessary conditions for the existence of an optimal solution are given.

1. Introduction

Denote by U a set of piecewise continuous vector functions $u(\cdot) = (u^1(\cdot),\dots,u^r(\cdot))$ defined on the interval $[0, T]$, whose values at the instant t belong to a non-empty compact set M contained in $\mathbf{R}^r$. Besides, let us assume that, for the functions $u(\cdot)$ belonging to U, the condition: $S(u) < \infty$ is satisfied, where $S(u)$ is defined by the formula

$$S(u) = \sup\left\{\frac{|u(t) - u(t')|}{|t - t'|};\ t \neq t',\ u \text{ continuous on } [t, t']\right\}. \tag{1}$$

The set U will be called a set of admissible controls and its elements admissible controls.

Consider the following

Problem I. Determine the minimal value of the functional

$$I(x, u) = \int_0^T f_0(t, x(t), u(t))\, dt + \Phi(u) \tag{2}$$

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under the conditions

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{3}$$

$$x(0) = x_0, \tag{4}$$

$$u(\cdot) \in U, \tag{5}$$

where f_0: $[0, T] \times \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}$, f: $[0, T] \times \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^n$, $x(\cdot) \in W_{1,1}^n([0, T])$, $x_0 \in \mathbf{R}^n$, $W_{1,1}^n([0, T])$ stands for the space of absolutely continuous functions with norm

$$\|x\| = |x(0)| + \int_0^T |\dot{x}(t)|\, dt.$$

The functional $\Phi(u)$ is defined by the formula

$$\Phi(u) = \gamma_1 N(u) + \gamma_2 S(u) \tag{6}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, $N(u)$ denotes the number of points of discontinuity of the function $u(\cdot)$ on $[0, T]$, and $S(u)$ is defined by (1).

A problem like this was formulated by E. S. Noussair [3]. In the paper just mentioned the author proved a theorem on the existence of a solution to Problem I in the class of piecewise continuous functions. The present paper includes a necessary condition for optimality which has been obtained on the basis of the Ioffe-Tikhomirov extremum principle ([2], Section 1.1).

2. A necessary condition for optimality

Let (x_*, u_*) be a solution to Problem I. Such a solution exists if the assumptions of Theorem 3 are satisfied (*cf.* [3]). Assume what follows:

1° there exists a summable function $\alpha(\cdot)$ such that $f_0(t, x(t), u(t)) \geqslant \alpha(t)$ for any pair (x, u) satisfying (3)–(5) and for almost all $t \in [0, T]$,

2° the functions $f_0(t, x, u)$, $f(t, x, u)$ are continuous with respect to the group of variables (t, x, u), continuously differentiable with respect to x and differentiable with respect to u,

3° f_0 is convex with respect to u with any x and $t \in [0, T]$,

4° there exists a neighbourhood $V \subset W_{1,1}^n([0, T])$ of the point x_* such that for any $x \in V$, any $u_1, u_2 \in U_p$ and any $\alpha \in [0, 1]$ there exists some $u \in U_p$ such that

$$f(t, x(t), u(t)) = \alpha f(t, x(t), u_1(t)) + (1 - \alpha) f(t, x(t), u_2(t))$$

for each $t \in [0, T]$, where $U_p \subset U$ is a set of functions that have p fixed points of discontinuity,

5° the set M, occurring in the definition of the set of admissible controls U, is a perpendicular parallelepiped contained in $\mathbf{R}^r$, defined as follows:

$$M = \{z = (z^1, \ldots, z^r) \in \mathbf{R}^r;\ z^i \in [\alpha^i, \beta^i],\ i = 1, \ldots, r\},$$

where $-\infty < \alpha_i < \beta_i < \infty$ for $i = 1, 2, \ldots, r$.

REMARK 1. *Conditions 1°, 2°, 3° and 5° are typical assumptions which occur in many theorems of optimization theory. Condition 4° has been introduced here in order that the Ioffe-Tikhomirov extremum principle could be applied. It is easily noticed that this assumption is satisfied by all systems linear with respect to the control, i.e. systems of the form*

$$f(t, x, u) = g_1(t, x)u + g_2(t, x),$$

where $u \in U$, $u(t) \in M$, and M is a convex set.

Denote by H a Hamilton function

$$H(\lambda_0, \psi, x, u, t) = (\psi(t), f(t, x(t), u(t)) - \lambda_0 f_0(t, x(t), u(t)), \tag{7}$$

where $\psi(t)$ is an absolutely continuous function on $[0, T]$ satisfying the conjugate equation

$$\frac{d\psi(t)}{dt} = -f_x^*(t, x(t), u(t))\psi(t) + \lambda_0 f_{0_x}(t, x(t), u(t))$$

and the condition $\psi(T) = 0$, while λ_0 is some non-negative constant.

In the first place, let us consider the case when u is a scalar function, that is, $r = 1$.

We shall prove the following

THEOREM 1. *If $u_*(\cdot)$ is an optimal control in Problem* I *and*

1. *conditions* (1°–5°) *are satisfied,*

2. $H_u(\lambda_0, \psi(t), x_*(t), u_*(t), t) \neq 0$ *for almost all $t \in [0, T]$, then the optimal control $u_*(\cdot)$ is a piecewise constant function assuming values equal to α^1 or β^1 only, and the number of points of discontinuity $N(u_*)$ can be estimated as follows*

$$N(u_*) \leqslant N(u_0) + \frac{c - \alpha_0}{\gamma_1}, \tag{8}$$

where $u_0(\cdot)$ is some admissible control,

$$c = \int_0^T f_0(t, x_0(t), u_0(t))\, dt + \gamma_2 S(u_0), \quad \alpha_0 = \int_0^T \alpha(t)\, dt.$$

PROOF. Denote $X = W_{1,1}^n([0, T])$, $Y = L_1^n([0, T]) \times \mathbf{R}^n$. Let $F\colon X \times U \to Y$ be an operator defined by the equality

$$[F(x(\cdot), u(\cdot))](t) = (\dot{x}(t) - f(t, x(t), u(t)), x(0) - x_0).$$

Then Problem I can be formulated in the equivalent form of:

Problem II. Determine the minimal value of the functional

$$I(x, u) = \int_0^T f_0(t, x(t), u(t))\, dt + \Phi(u),$$

in the set $X \times U$ under the conditions

$$F(x, u) = 0, \qquad u(\cdot) \in U.$$

Let $u_*(\cdot)$ be an optimal control in Problem I (Problem II). Denote by $t_1, t_2, \ldots, t_z$ points of discontinuity of the function $u_*(\cdot)$ on $[0, T]$. Note that Z is a finite number. Indeed, from (2) we have

$$\begin{aligned} I(x_*, u_*) &= \int_0^T f_0(t, x_*, u_*)\, dt + \gamma_1 N(u_*) + \gamma_2 S(u_*) \\ &\leqslant I(0,0) = \int_0^T f_0(t, 0, 0)\, dt. \end{aligned}$$

Making use of assumption 1°, we obtain the following estimate

$$0 \leqslant Z = N(u_*) \leqslant \frac{1}{\gamma_1}\left[a + \alpha_0 - \gamma_2 S(u_*)\right] \leqslant \frac{1}{\gamma_1}[a + \alpha_0] < \infty,$$

where

$$a = \int_0^T f_0(t, 0, 0)\, dt, \qquad \alpha_0 = \int_0^T \alpha(t)\, dt.$$

Let U_z stand for a subset of U, composed of functions possessing exactly Z fixed points of discontinuity $t_1, t_2, \ldots, t_z$.

It is easily seen that, if $u_*(\cdot)$ is an optimal control in Problem II, it is also optimal for the following Problem III (but not inversely, in general).

Problem III. Find the minimal value of the functional

$$I(x, u) = \int_0^T f_0(t, x(t), u(t))\, dt + \gamma_1 N(u) + \gamma_2 S(u)$$

under the conditions

$$F(x, u) = 0, \qquad u(\cdot) \in U_z.$$

For $u(\cdot)$ belonging to the set U_z, the $N(u)$ is constant and equal to Z; consequently, $u_*(\cdot)$ is also a solution to

Problem IV. Find the minimum of the functional

$$\bar{I}(x, u) = \int_0^T f_0(t, x(t), u(t))\, dt + \gamma_2 S(u)$$

under the conditions

$$F(x, u) = 0, \qquad u(\cdot) \in U_z.$$

And thus, if $u_*(\cdot)$ is an optimal control in Problem I, it is such for Problem IV.

To Problem IV we shall now apply the extremum principle ([2], Chapter 1).

The functional $S(u)$ is convex on U_z. Indeed, let t, t' be any points belonging to the interval of continuity of the functions $u_1(\cdot)$ and $u_2(\cdot)$. We have

$$\begin{aligned} S(\lambda u_1 + (1-\lambda)u_2) &= \sup\{|\lambda u_1(t) + (1-\lambda)u_2(t) \\ &\qquad -\lambda u_1(t') - (1-\lambda)u_2(t')|/|t-t'|\} \\ &\leqslant \lambda \sup\{|u_1(t) - u_1(t')|/|t-t'|\} \\ &\quad + (1-\lambda)\sup\{|u_2(t) - u_2(t')|/|t-t'|\} \\ &= \lambda S(u_1) + (1-\lambda)S(u_2), \end{aligned}$$

where $\lambda \in [0, 1]$. Consequently, the functional $\bar{I}(x, u)$ is convex with respect to u with any $x \in V$.

The mapping F is decomposed into mappings F_1: $X \times U \to Y_1 = L_1^n[0, T]$ and h_1: $X \times U \to \mathbf{R}^n$, where

$$[F_1(x(\cdot), u(\cdot))](t) = \dot{x}(t) - f(t, x(t), u(t)), \qquad h_1(x(\cdot), u(\cdot)) = x(0) - x_0.$$

The mapping F_1 possesses a continuous Fréchet derivative with respect to x for any $u(\cdot) \in U$, equal at the point (x_*, u_*) to

$$\left[F_{1_x}\big(x_*(\cdot), u_*(\cdot)\big)\bar{x}(\cdot)\right](t) = \dot{\bar{x}}(t) - f_x\big(t, x_*(t), u_*(t)\big)\bar{x}(t),$$

where $\bar{x}(\cdot) \in W_{1,1}^n([0, T])$. The mapping F_1 is regular at the point x_* because the equation

$$\dot{\bar{x}}(t) - f_x\big(t, x_*(t), u_*(t)\big)\bar{x}(t) = y(t)$$

possesses the solution $\bar{x}(t) \in W_{1,1}^n([0, T])$ for each

$$y(\cdot) \in L_1^n([0, T]), \quad ([2], \text{Section } 0.2).$$

The functional $\bar{I}(x, u) = \int_0^T f_0(t, x(t), u(t))\,dt + \gamma_2 S(u)$ is also continuously Fréchet differentiable with respect to x with each fixed $u \in U$, and its derivative at the point (x_*, u_*) is expressed by the formula

$$\bar{I}_x\big(x_*(\cdot), u_*(\cdot)\big)\bar{x}(\cdot)(t) = \int_0^T \Big(f_{0_x}\big(t, x_*(t), u_*(t)\big), \bar{x}(t)\Big)\,dt.$$

In view of the above, all assumptions of the extremum principle are satisfied.

The Lagrange function for Problem IV has the form

$$\mathfrak{L}(x, u, \lambda_0, \lambda_1, y^*) = \lambda_0 \bar{I}(x, u) + (\lambda_1, h_1) + \langle y^*, F_1(x, u)\rangle,$$

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where $\lambda_0 \in \mathbf{R}$, $\lambda_1 \in \mathbf{R}^n$, $y^* \in Y_1^*$. The function $\mathfrak{L}$ can be represented in the form

$$\mathfrak{L}(x, u, \lambda_0, \lambda_1, y^*) = \lambda_0\Big(\int_0^T f_0(t, x(t), u(t))\,dt + \gamma_2 S(u)\Big) + (\lambda_1, x(0) - x_0)$$
$$+ \int_0^T (\psi(t), \dot{x}(t) - f(t, x(t), u(t))\,dt,$$

where $\psi(\cdot) \in L_\infty^n([0, T])$.

Applying the extremum principle, we obtain that a necessary condition for the point (x_*, u_*) to be a solution to Problem IV is the existence of Lagrange multipliers $\lambda_0 \geqslant 0$, λ_1, y^*, not vanishing simultaneously, for which

$$\mathfrak{L}_x(x_*, u_*, \lambda_0, \lambda_1, y^*) = 0,$$
$$\mathfrak{L}(x_*, u_*, \lambda_0, \lambda_1, y^*) = \min_{u \in U_z} \mathfrak{L}(x_*, u, \lambda_0, \lambda_1, y^*).$$

Hence

$$\lambda_0 \int_0^T \big(f_{0x}(t, x_*(t), u_*(t)), \bar{x}(t)\big)\,dt + (\lambda_1, \bar{x}(0))$$
$$+ \int_0^T \big(\psi(t), \dot{\bar{x}}(t) - f_x(t, x_*(t), u_*(t))\bar{x}(t)\big)\,dt = 0 \tag{9}$$

for any $\bar{x}(\cdot) \in W_{1,1}^n([0, T])$, and

$$\int_0^T \Big[\lambda_0 f_0(t, x_*(t), u_*(t)) - \big(\psi(t), f(t, x_*(t), u_*(t))\big)\Big]\,dt + \lambda_0 \gamma_2 S(u_*)$$
$$= \min_{u \in U_z} \Big\{ \int_0^T \Big[\lambda_0 f_0(t, x_*(t), u(t))$$
$$- \big(\psi(t), f(t, x_*(t), u(t))\big)\Big]\,dt + \lambda_0 \gamma_2 S(u) \Big\}. \tag{10}$$

Making use of (7), we may write condition (9) in the form

$$\int_0^T (\psi(t), \dot{\bar{x}}(t))\,dt - \int_0^T \big(H_x(\lambda_0, \psi, x_*, u_*, t), \bar{x}(t)\big)\,dt + (\lambda_1, \bar{x}(0)) = 0. \tag{11}$$

Integrating the second addendum of (11) by parts, we get

$$\int_0^T \Big(\psi(t) - \int_t^T H_x\,dt, \dot{\bar{x}}(t)\Big)\,dt + \Big(-\int_0^T H_x\,dt + \lambda_1, \bar{x}(0)\Big) = 0. \tag{12}$$

Since the last equality holds for an arbitrary absolutely continuous function $\bar{x}(\cdot)$ therefore

$$\psi(t) - \int_t^T H_x\,dt = \text{const}, \qquad t \in [0, T].$$

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Hence it appears that $\psi(t)$ is absolutely continuous on $[0, T]$ and satisfies the differential equation

$$\frac{d\psi}{dt} = -H_x(\lambda_0, \psi, x_*, u_*, t) = -f_x(t, x_*, u_*)\psi(t) + \lambda_0 f_{0_x}(t, x_*, u_*).$$

Let us now come back to equality (12). It holds for any $\bar{x}(\cdot) \in W^n_{1,1}([0, T])$, in particular, for such that $\bar{x}(0) = 0$. Then (12) has the form

$$\int_0^T \left(\psi(t) - \int_t^T H_x dt, \dot{\bar{x}}(t) \right) dt = 0.$$

Integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \left(\psi(t) - \int_t^T H_x dt, \dot{\bar{x}}(t) \right) dt \\ &= \left(\psi(t) - \int_t^T H_x dt, \bar{x}(t) \right) \Big|_0^T - \int_0^T (\dot{\psi}(t) + H_x, \bar{x}(t)) \, dt \\ &= (\psi(T), \bar{x}(T)). \end{aligned}$$

Since $\bar{x}(T)$ may assume any values, therefore $\psi(T) = 0$. Consequently, the function ψ satisfies the equation $\dot{\psi} = -H_x$ with the condition $\psi(T) = 0$.

Condition (10) has the form

$$\begin{aligned} &\int_0^T - H(\lambda_0, \psi, x_*, u_*, t) \, dt + \lambda_0 \gamma_2 S(u_*) \\ &\qquad = \min_{u \in U_z} \left\{ \int_0^T - H(\lambda_0, \psi, x_*, u, t) \, dt + \lambda_0 \gamma_2 S(u) \right\}. \qquad (13) \end{aligned}$$

It follows from condition (13), assumption (2°) and the definition of the function S that the function u_* takes the values:

$$u_*(t) = \begin{cases} \alpha^1 & \text{when } H_u(\lambda_0, \psi(t), x_*(t), u_*(t), t) > 0, \\ \beta^1 & \text{when } H_u(\lambda_0, \psi(t), x_*(t), u_*(t), t) < 0. \end{cases}$$

Since in Problem IV under consideration $u \in U_z$, the optimal control u_* is a piecewise constant function.

Let us now estimate the number of points of discontinuity of the optimal control $u_*(\cdot)$.

Let $u_0(\cdot)$ be some admissible control. Thus the inequality

$$I(x_*, u_*) \leqslant I(x_0, u_0)$$

holds, where $x_0(\cdot)$ is a trajectory of system (3)–(4), corresponding to the control $u_0(\cdot)$. Hence

$$\int_0^T f_0\big(t, x_*(t), u_*(t)\big)\, dt + \gamma_1 N\big(u_*\big) + \gamma_2 S\big(u_*\big)$$
$$\leqslant \int_0^T f_0(t, x_0(t), u_0(t))\, dt + \gamma_1 N(u_0) + \gamma_2 S(u_0). \tag{14}$$

Since $u_*(\cdot)$ is piecewise constant, therefore $S(u_*) = 0$.

It follows from assumption (1°) that

$$\alpha_0 = \int_0^T \alpha(t)\, dt \leqslant \int_0^T f_0(t, x(t), u(t))\, dt$$

for an arbitrary admissible process (x, u). Hence from (14) we obtain

$$\gamma_1 N\big(u_*\big) + \alpha_0 \leqslant \int_0^T f_0(t, x_0(t), u_0(t))\, dt + \gamma_1 N(u_0) + \gamma_2 S(u_0).$$

By c let us denote a number equalling

$$c = \int_0^T f_0(t, x_0(t), u_0(t))\, dt + \gamma_2 S(u_0).$$

We then obtain the following estimate of the number of points of discontinuity of $u_*(\cdot)$

$$N\big(u_*\big) \leqslant N(u_0) + \frac{c - \alpha_0}{\gamma_1}.$$

We have thus obtained the complete proof of Theorem 1.

Let us now consider the case of a vector control. It is easy to prove the following.

LEMMA 1. *If $u_*(\cdot) = (u_*^1(\cdot), u_*^2(\cdot), \dots, u_*^r(\cdot))$ is an optimal control in Problem* I, *then each of the functions u_*^i, $i = 1, \dots, r$, is an optimal control in a scalar problem of the form*

$$I_1(u^i) = I\big(x, u_*^1, u_*^2, \dots, u_*^{i-1}, u^i, u_*^{i+1}, \dots, u_*^r\big) \to \inf,$$
$$\dot{x}(t) = f\big(t, x(t), u_*^1(t), u_*^2(t), \dots, u^i(t), \dots, u_*^r(t)\big),$$

where $u^i(\cdot) \in U^i$, and U^i is a set defined as follows:

$$U^i = \big\{u^i(\cdot) \text{ piecewise continuous on } [0, T],\ u^i(t) \in \big[\alpha^i, \beta^i\big],\ S(u^i) < \infty\big\}.$$

Repeating, for each $i = 1, 2, \dots, r$, the proof of Theorem 1, we obtain the following;

THEOREM 2. *Let assumptions* (1°)–(5°) *be satisfied. If, for each* $i = 1, 2, \ldots, r$, $H_{u^i} \neq 0$ *almost everywhere on* $[0, T]$, *then the optimal control for problem I*, $u_*(\cdot) = (u_*^1(\cdot), u_*^2(\cdot), \ldots, u_*^r(\cdot))$ *is a piecewise constant function whose components* $u_*^i(\cdot)$ *assume the values* α^i *or* β^i *only, and the number of points of discontinuity is estimated by formula* (8).

We shall now consider the case of a linear problem. Namely, assume that the functions f and f_0 are of the form

$$f = Ax + Bu, \qquad f_0 = ax + bu,$$

where A is a matrix of dimension $n \times n$ with constant coefficients, B is a constant matrix of dimension $n \times r$, a and b are, respectively, n- and r-dimensional vectors. It is easy to notice that, in this case, all earlier assumptions (1°)–(5°) are satisfied.

For a linear problem, we shall now give a sufficient condition for $H_{u^i} \neq 0$, $i = 1, \ldots, r$, to be satisfied.

THEOREM 3. *If a system*

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \tag{15}$$

where $\tilde{x} = (x^0, x^1, \ldots, x^n)$, $\dot{x}^0(t) = ax(t) + bu(t)$, $\tilde{A}$ *is a matrix of degree* $(n+1) \times (n+1)$ *of the form*

$$\tilde{A} = \begin{bmatrix} 0 & a^1, a^2, \ldots, a^n \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix},$$

$\tilde{B}$ *is a matrix of degree* $(n+1) \times r$ *of the form*

$$\tilde{B} = \begin{bmatrix} b^1 \ldots . b^r \\ B \end{bmatrix},$$

is regularly controllable (*cf.* [1]), *then, for each* $i = 1, 2, \ldots, r$, *the condition* $H_{u^i} \neq 0$ *is satisfied.*

PROOF. We have

$$H = (\psi, Ax + Bu) + \lambda_0(ax + bu),$$

where $\lambda_0 \leqslant 0$, $\psi(t)$ is an absolutely continuous function and λ_0 and ψ do not vanish simultaneously.

Let $\tilde{\psi} = (\lambda_0, \psi^1, \ldots, \psi^n)$. Then H has the form

$$H = (\tilde{\psi}, \tilde{A}\tilde{x} + \tilde{B}u) = (\tilde{\psi}, \tilde{A}\tilde{x}) + (\tilde{\psi}, \tilde{B}u).$$

The function $\tilde{\psi}$ is a solution to the equation

$$\dot{\tilde{\psi}}(t) = -H_x = -\tilde{A}^*\tilde{\psi}(t). \tag{16}$$

Since A is a constant matrix, therefore $\tilde{\psi}$ is an analytic function of the variable t. The derivative of the function H with respect to u has the form

$$H_u = \tilde{B}^*\tilde{\psi}(t).$$

Let us now suppose that there exists $j \in \{1, 2, \ldots, r\}$ such that $H_{u^j} = 0$ on set $Q \subset [0, T]$ of positive measure, *i.e.*

$$\left(\tilde{b}_j, \tilde{\psi}(t)\right) = 0 \quad \text{for } t \in Q,$$

where $\tilde{b}_j$ is the jth column of the matrix $\tilde{B}$. The function $(\tilde{b}_j, \tilde{\psi}(t))$ is analytic and vanishes on the set of positive measure, so

$$\left(\tilde{b}_j, \tilde{\psi}(t)\right) = 0 \quad \text{for each } t \in [0, T].$$

Differentiating with respect to t, we have

$$\left(\tilde{b}_j, \dot{\tilde{\psi}}(t)\right) = 0.$$

Taking account of (16), we get

$$\left(\tilde{b}_j, -\tilde{A}^*\tilde{\psi}(t)\right) = 0,$$

that is,

$$\left(\tilde{A}\tilde{b}_j, \tilde{\psi}(t)\right) = 0.$$

Further, proceeding analogously as in the proof of Theorem 6.5 (cf. [1]), we come to a conclusion that system (15) is not regularly controllable. This contradicts the assumption of the theorem. Consequently, $H_{u^i} \neq 0$ for $i = 1, 2, \ldots, r$.

To close with, we shall give an example illustrating the theorem we have proved.

EXAMPLE. In the space $\mathbf{R}^2$ let us consider a control system of the form

$$\begin{aligned} \dot{x}^1 &= \tfrac{1}{2}x^2 + \tfrac{1}{3}u^1, \\ \dot{x}^2 &= \tfrac{1}{3}x^1 + \tfrac{1}{2}x^2 + \tfrac{1}{4}u^2, \\ x^1(0) &= x^2(0) = 0, \qquad |u^1| \leqslant 1, |u^2| \leqslant 1, \end{aligned} \tag{17}$$

with a cost functional defined by the formula

$$I(x, u) = \int_0^1 \left(-x^1 - x^2 + \tfrac{1}{3}u^1\right) dt + N(u) + S(u), \tag{18}$$

where $x = (x^1, x^2)$, $u = (u^1, u^2)$.

It is easy to see that in this example the system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$ is of the form (*cf.* Theorem 3)

$$\begin{bmatrix} \dot{x}^0 \\ \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}. \tag{19}$$

By putting, successively, $u^1 = 0$ and $u^2 = 0$, it can easily be checked that system (19) is regularly controllable. From Theorem 3 it follows that in this example the assumptions of Theorem 2 are satisfied. Making use of the Cauchy formula, we obtain

$$\begin{aligned} |x(t)| &= \left| \int_0^t e^{A(\tau - t)} Bu(\tau)\, d\tau \right| \\ &\leqslant e^{\|A\|} \|B\| \|u\| \leqslant e \cdot \frac{1}{5} \cdot \sqrt{2} \leqslant 1 \quad \text{for } t \in [0, 1], \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Hence

$$f_0(t, x, u) = -x^1 - x^2 + \tfrac{1}{3}u^1 \geqslant -2\tfrac{1}{3}.$$

And consequently, for α_0 in formula (8) we may take $\alpha_0 = -2\frac{1}{3}$. Put $u_0 \equiv 0$. It follows from (17) and (19) that

$$c = \int_0^1 f_0(t, x_0(t), u_0(t))\, dt + \gamma_2 S(u_0) = 0.$$

Using Theorem 2, we infer that each component of the optimal control $u_* = (u_*^1, u_*^2)$ is piecewise constant, assumes the values ± 1 and possesses at most two jumps.

References

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