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TWO THEOREMS ON THE CLASS NUMBER OF POSITIVE DEFINITE QUADRATIC FORMS

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0. In this note we study the estimate from above and below and the asymptotic behaviour of the class number of positive definite integral quadratic forms.

1. Let S_1, S_2 be positive definite matrices of degree m; then S_1, S_2 are called equivalent (resp. equivalent in the narrow sense) if $S_1 = {}^tTS_2T$ for some T in $GL(m, \boldsymbol{Z})$ (resp. $SL(m, \boldsymbol{Z})$). By definition $E(S)$ is the order of the unit group of S, i.e., the number of matrices in $GL(m, \boldsymbol{Z})$ such that ${}^tTST = S$. Let m, D be natural numbers; by $H_m(D)$ (resp. $h_m(D)$) we denote the number of equivalence classes (resp. equivalence classes in the narrow sense) in positive definite integral matrices of degree m and determinant D.

THEOREM 1. *Let m be a natural number larger than 2, and ε be any positive number. Then we have*

$$c_1(m)D^{(m-1)/2} \leq H_m(D) \leq c_2(m, \varepsilon)D^{(m-1)/2+\varepsilon},$$

where $c_1(m)$ is a positive constant depending on m, and $c_2(m, \varepsilon)$ is a positive constant depending on m and ε. Moreover we can take 0 instead of ε if we consider cases of square-free D.

COROLLARY. *For even m we have*

$$h_m(D) \sim^* 2H_m(D) \qquad as\ D \to \infty .$$

THEOREM 2. *Let m be a natural number; then*

$$H_m(D) \sim 2 \sum \frac{1}{E(S)} \qquad as\ D \to \infty ,$$

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*) $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

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where S runs over a set of representatives of different equivalence classes in positive definite integral matrices of degree m and determinant D.

COROLLARY. *Let m be an odd natural number. Then we have*

$$\lim_{\substack{D\to\infty \\ D:\ \text{odd} \\ \text{square-free}}} \frac{H_m(D)}{D^{(m-1)/2}} = \pi^{-m(m+1)/4}\prod_{k=1}^{m}\Gamma\left(\frac{k}{2}\right)\prod_{k=1}^{(m-1)/2}\zeta(2k)\,,$$

where $\zeta(s)$ is the Riemann zeta-function.

Remark. It is possible that we obtain the similar result to Theorem 2 for the number of classes in a genus on some assumptions (for example, on the assumption that D is square-free).

2. LEMMA 1. *The number of groups of finite order in $GL(m,\boldsymbol{Z})$ is finite up to conjugacy.*

Proof. Let G be a group of finite order in $GL(m,\boldsymbol{Z})$ and S be the positive definite matrix $\sum_{A\in G}{}^tAA$. Then there exists an element U in $GL(m,\boldsymbol{Z})$ such that ${}^tUSU$ is reduced in the sense of Minkowski and the integral orthogonal group of ${}^tUSU$ contains $U^{-1}GU$. From Satz 4 in [8], absolute values of all entries of $U^{-1}MU(M\in G)$ are not larger than some constant depending on m.

3. *Proof of Theorem* 1.

Let S be a positive definite integral matrix of degree m and determinant D. Then the mass $M(S)$ of S is by definition

$$\sum\frac{1}{E(S_k)}\,,$$

where S_k runs over the representatives of equivalence classes in the genus of S, and it is well known ([7])

$$M(S)=\frac{2\Gamma(1/2)\Gamma(2/2)\cdots\Gamma(m/2)}{\pi^{m(m+1)/4}\prod_p\alpha_p}\cdot D^{(m+1)/2}\qquad(m>1)\,,$$

where $\alpha_p=\alpha_p(S)$ is the density of S at the prime p and it is defined by

$$\frac{1}{2}\lim_{\ell\to\infty}(p^\ell)^{-m(m-1)/2}M(S\,;p^\ell)\,,$$

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where $M(S; p^\ell)$ is the number of integral matrices T mod p^ℓ such that ${}^tTST \equiv S \bmod p^\ell$.

If p does not divide $2D$, then we have ([3], [7])

$$\alpha_p = \begin{cases} \prod_{k=1}^{(m-1)/2} (1 - p^{-2k}) & m\colon \text{odd}, \\ \left(1 - \left(\dfrac{(-1)^{m/2} D}{p}\right) p^{-m/2}\right) \prod_{k=1}^{(m/2)-1} (1 - p^{-2k}) & m\colon \text{even}. \end{cases}$$

If

$$(1) \qquad S \cong \begin{pmatrix} 1_{m-2} & & \\ & \varepsilon_p & \\ & & D\varepsilon_p^{-1} \end{pmatrix} \quad \text{over } \boldsymbol{Z}_p \text{ for } p|D \text{ and } p \neq 2,$$

where ε_p is a unit of $\boldsymbol{Z}_p$,
then we have ([3])

$$\alpha_p = 2D^{(p)} \begin{cases} \left(1 - \left(\dfrac{(-1)^{(m-1)/2} \varepsilon_p}{p}\right) p^{-(m-1)/2}\right) \prod_{k=1}^{(m-1)/2-1} (1 - p^{-2k}) & m\colon \text{odd}, \\ \prod_{k=1}^{(m/2)-1} (1 - p^{-2k}) & m\colon \text{even}, \end{cases}$$

where $D^{(p)}$ represents the p-part of D.

If $8|D$, and

$$(2) \qquad S \cong \begin{pmatrix} A & \\ & D \end{pmatrix} \quad \text{over } \boldsymbol{Z}_2,$$

where A is unimodular over $\boldsymbol{Z}_2$ with determinant 1, then by the similar proof to Hilfssatz 10, 11 in [3] we have

$$M(S; 2^\ell) = 2^{\ell(m-1)} M(A; 2^\ell) M(D; 2^\ell),$$

and so

$$\alpha_2(S) = 4D^{(2)} \alpha_2(A),$$

where $D^{(2)}$ represents the 2-part of D. Thus, on the assumption (2) if $8|D$, we have

$$\alpha_2(S)/D^{(2)} \leq c_1,$$

where c_1 depends on only m. From now on, c_i represents a positive constant depending on only m, and $c_i(\varepsilon)$ depends on m and ε.

If S satisfies the above condition (1) for any odd prime p, then we have

$$\prod_{p\neq 2}\alpha_p^{-1}=\begin{cases}\dfrac{D^{(2)}}{D}\displaystyle\prod_{k=1}^{(m-1)/2}\zeta(2k)\prod_{k=1}^{(m-1)/2}(1-2^{-2k})\prod_{\substack{p\mid D\\ p\neq 2}}2^{-1}(1-p^{-(m-1)}) \\ \quad\times\left(1-\left(\dfrac{(-1)^{(m-1)/2}\varepsilon_p}{p}\right)p^{-(m-1)/2}\right)^{-1} & m:\ \text{odd}\,,\\ \dfrac{D^{(2)}}{D}\dfrac{1}{\prod_{\substack{p\mid D\\ p\neq 2}}2}\displaystyle\prod_{k=1}^{(m/2)-1}\zeta(2k)\cdot L\left(\dfrac{m}{2},\left(\dfrac{(-1)^{m/2}D}{*}\right)\right)\prod_{k=1}^{(m/2)-1}(1-2^{-2k}) \\ \quad\times\left(1-\left(\dfrac{(-1)^{m/2}D}{2}\right)2^{-m/2}\right) & m:\ \text{even}\,.\end{cases}$$

Thus on the assumptions (1), and (2) if $8|D$, the mass $M(S)$ satisfies

$$M(S)\geq c_2D^{(m-1)/2}\prod_{\substack{p\mid D\\ p\neq 2}}2^{-1}\begin{cases}\displaystyle\prod_{\substack{p\mid D\\ p\neq 2}}\left(1+\left(\dfrac{-\varepsilon_p}{p}\right)p^{-1}\right) & m=3\,,\\ 1 & m\geq 4\,.\end{cases}$$

Therefore if the number of odd primes dividing D is zero or one, and S satisfies above conditions (1) and (2) if $8|D$ $\left(\text{for example, } S=\begin{pmatrix}1_{m-1} & \\ & D\end{pmatrix}\right)$, then

$$H_m(D)\geq M(S)\geq c_3D^{(m-1)/2}\qquad \text{for } m\geq 3\,.$$

Suppose that odd primes dividing D are $p_1, p_2, \cdots, p_t (t\geq 2)$, and put the p-part of $D=p^{u_p}$. If there exists j such that u_{p_j} is odd, then for any given unit ε_{p_i} of $Z_{p_i}(i\neq j)$ there exist a unit ε_{p_j} of Z_{p_j} and a positive definite integral matrix S with $|S|=D$ such that S satisfies the condition (1) and

$$S\cong\begin{pmatrix}1_{m-1} & \\ & D\end{pmatrix}\quad \text{over } Z_2\,.$$

If any u_{p_i} is even, then for any given unit ε_{p_i} of Z_{p_i} there exist a unit ε_2 of Z_2 and a positive definite integral matrix S with $|S|=D$ such that S satisfies the condition (1) and

$$S\cong\begin{pmatrix}1_{m-3} & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_2^{-1} & \\ & & & D\end{pmatrix}\quad \text{over } Z_2\,.$$

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Hence we obtain

$$H_m(D) \geq \sum_{\substack{\left(\frac{\varepsilon_{p_i}}{p_i}\right)=\pm 1 \\ i \neq j}} M(S) \geq \frac{1}{2} c_2 D^{(m-1)/2} \qquad \text{for } m \geq 4,$$

and for $m = 3$

$$\begin{aligned} H_m(D) &\geq \sum_{\substack{\left(\frac{\varepsilon_{p_i}}{p_i}\right)=\pm 1 \\ i \neq j}} M(S) \geq c_2 D 2^{-t} \sum \prod_{i=1}^{t} \left(1 + \left(\frac{-\varepsilon_{p_i}}{p_i}\right) p_i^{-1}\right) \\ &\geq c_2 D 2^{-t-1} \sum \prod_{\substack{i=1 \\ i \neq j}}^{t} \left(1 + \left(\frac{-\varepsilon_{p_i}}{p_i}\right) p_i^{-1}\right) \\ &= 2^{-2} c_2 D . \end{aligned}$$

Thus, we have proved $H_m(D) \geq c_4 D^{(m-1)/2}$.

Let c_5 be the maximal order of groups of finite order in $GL(m, \boldsymbol{Z})$. Then we have

$$H_m(D) \leq c_5 \sum M(S) ,$$

where S runs over the representatives of genera of positive definite integral matrices of degree m and determinant D. This implies

$$H_m(D) \leq c_6 D^{(m+1)/2} \prod_{p \nmid 2D} \alpha_p^{-1} \prod_{p \mid 2D} \left(\sum \alpha_p^{-1}\right) , \tag{3}$$

where $\sum \alpha_p^{-1}$ is the sum of the inverses of densities of matrices, up to equivalence, over $\boldsymbol{Z}_p$ of degree m and determinant D. On the other hand, we have

$$\prod_{p \nmid 2D} \alpha_p^{-1} = \begin{cases} \displaystyle\prod_{p \nmid 2D} \prod_{k=1}^{(m-1)/2} (1 - p^{-2k})^{-1} & m\colon \text{ odd} , \\ \displaystyle\prod_{p \nmid 2D} \left(1 - \left(\frac{(-1)^{m/2} D}{p}\right) p^{-m/2}\right)^{-1} \prod_{k=1}^{(m/2)-1} (1 - p^{-2k})^{-1} & m\colon \text{ even} , \end{cases}$$

$$\leq c_7 .$$

Let

$$S \cong \begin{pmatrix} p^{t_1} S_1 & & \\ & \ddots & \\ & & p^{t_s} S_s \end{pmatrix} \quad \text{over } \boldsymbol{Z}_p , \qquad (p \neq 2)$$

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where S_i are unimodular and $0 \le t_1 < t_2 < \cdots < t_s$, and put $n_i =$ degree of S_i, $m_i = \sum_{k=i}^{s} n_k$. Then we get

$$\alpha_p(S) = 2^{s-1} p^{\omega(t_i, n_i)} \prod_{i=1}^{s} \alpha_p(S_i) \qquad \text{for odd prime } p,$$

where $\omega(t_i, n_i) = \sum_{k=1}^{s} t_k n_k (m_k - (n_k - 1)/2)$, and the sum $\sum \alpha_p^{-1}$ in (3) is

$$\sum{}' \alpha_p^{-1} = \sum_{n_k, t_k} \sum_{\substack{\deg S_i = n_i \\ \prod_i |S_i| = D/D^{(p)}}} \alpha_p^{-1}$$

$$= \sum_{n_k, t_k} \frac{2^{1-s}}{p^{\omega(t_k, n_k)}} \sum \prod_{k=1}^{s} \alpha_p(S_k)^{-1}.$$

We, now, estimate $\sum \prod_{k=1}^{s} \alpha_p(S_k)^{-1}$:

$$\sum \prod_{k=1}^{s} \alpha_p(S_k)^{-1}$$

$$= \sum \prod_{n_k=2} \alpha_p(S_k)^{-1} \prod_{n_k \neq 2} \alpha_p(S_k)^{-1}$$

$$= \sum \prod_{n_k=2} \left(1 - \left(\frac{-|S_k|}{p}\right) p^{-1}\right)^{-1} \prod_{n_k \neq 2} \alpha_p(S_k)^{-1}$$

$$= \sum \prod_{n_k=2} \left(1 + \left(\frac{-|S_k|}{p}\right) p^{-1}\right) \prod_{n_k=2} (1 - p^{-2})^{-1} \prod_{n_k \neq 2} \alpha_p(S_k)^{-1}$$

$$\le \left\{\prod_{k=2}^{m} (1 - p^{-k})^{-1}\right\}^{c_8} \sum \prod_{n_k=2} \left(1 + \left(\frac{-|S_k|}{p}\right) p^{-1}\right).$$

If some n_k is not 2, then we can take any unit of Z_p as $|S_k|$ for k satisfying $n_k = 2$, and $\sum \prod_{n_k=2} \left(1 + \left(\frac{-|S_k|}{p}\right) p^{-1}\right) = 2^{s-1}$. If all n_k are 2, then $\sum \prod_{k=1}^{s} \left(1 + \left(\frac{-|S_k|}{p}\right) p^{-1}\right) = 2^{s-1} \left(1 + \left(\frac{(-1)^{m/2} D/D^{(p)}}{p}\right) p^{-m/2}\right)$. This implies

$$\sum \alpha_p^{-1} \le \left\{\prod_{k=2}^{m} (1 - p^{-k})^{-1}\right\}^{c_9} \sum_{n_k, t_k} \frac{1}{p^{\omega(t_k, n_k)}} \qquad \text{for odd } p,$$

Put $D^{(p)} = p^{u_p}$, then $u_p = \sum n_k t_k$ and $\omega(t_k, n_k) \ge u_p$ and the equality arises if and only if $n_1 = m - 1$, $n_2 = 1$, $t_1 = 0$ and $t_2 = u_p$.

If we confine ourselves to the case of square-free D, then we have $n_1 = m - 1$, $n_2 = 1$, $t_1 = 0$ and $t_2 = u_p$ $(= 1)$. Hence in this case, we have

$$\prod_{\substack{p \mid D \\ p \neq 2}} \sum \alpha_p^{-1} \le c_{10} D^{(2)}/D.$$

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We come back to the case of general D. Let β_s be the number of partitions $m = \sum_{i=1}^{s} n_i, n_i > 0$, and put $\ell = \omega(t_k, n_k) - u_p = t_s n_s(n_s - 1)/2 + \sum_{k=1}^{s-1} t_k n_k(m_k - (n_k + 1)/2)$; then in case of $s > 1$, we have $t_{s-1} \leq \ell$ and $0 \leq t_{s-i} \leq \ell - i + 1$. This implies that the number of systems $\{t_k\}_{k=1}^{s}$ such that $\ell = \omega(t_k, n_k) - u_p$ for some n_k satisfying $\sum_{k=1}^{s} n_k = m, n_k > 0$, $\sum n_k t_k = u_p$, and $0 \leq t_1 < t_2 < \cdots < t_s$ is at most $(\ell + 1)\ell(\ell - 1)\cdots(\ell - s + 3)$. Therefore we get

$$\sum_{n_k, t_k} p^{-\omega(t_k, n_k)} \leq \frac{1}{D^{(p)}}\left\{\sum_{s=2}^{m} \beta_s \sum_{\ell=s-2}^{\infty} \frac{(\ell+1)\ell\cdots(\ell - s + 3)}{p^\ell}\right\} + p^{-u_p(m+1)/2}$$
$$= \frac{1}{D^{(p)}}\left\{\sum_{s=2}^{m} \beta_s \frac{(s-1)!}{(p-1)^s} p^2 + p^{-u_p(m-1)/2}\right\},$$

and finally we have

$$\prod_{\substack{p \mid D \\ p \neq 2}} \sum \alpha_p^{-1} \leq c_{10}(\varepsilon)\left(\frac{D^{(2)}}{D}\right)^{1-\varepsilon}.$$

Now we estimate $\sum \alpha_2^{-1}$:
Let $S \cong \begin{pmatrix} S_1 & \\ & S_2 \end{pmatrix}$ over Z_2 and S_1 is unimodular of degree n and $S_2 \equiv 0(2)$; then from the similar proof of Hilfssatz 10, 11 in [3] it follows that

$$M(S; 2^\ell) \geq (2^{\ell-1})^{(m-n)n} M(S_1; 2^\ell) M(S_2; 2^\ell)$$

and so $\alpha_2(S) \geq 2^{1-(m-n)n}\alpha_2(S_1)\alpha_2(S_2)$. Let

$$S \cong \begin{pmatrix} 2^{t_1}S_1 & & \\ & \ddots & \\ & & 2^{t_s}S_s \end{pmatrix} \quad \text{over } Z_2,$$

where S_i are unimodular and $0 \leq t_1 < \cdots < t_s$ and put $n_i =$ degree of S_i and $m_i = \sum_{k=i}^{s} n_k$; then we get

$$\alpha_2(S)^{-1} \leq 2^{-(s-1)-\omega(t_k, n_k)+\sum_{k=1}^{s-1} n_k m_{k+1}} \prod \alpha_2(S_i)^{-1}.$$

The number of unimodular matrices, up to equivalence, of degree $\leq m$, and the number of partitions $\sum_{i=1}^{s} n_i = m$, are finite, hence we get

$$\sum \alpha_2(S)^{-1} \leq c_{11} \sum 2^{-\omega(t_k, n_k)}$$
$$\leq c_{12} \frac{1}{D^{(2)}}.$$

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From these we have

$$H_m(D) \leq c_{13}(\varepsilon) D^{(m-1)/2+\varepsilon}.$$

4. LEMMA 2. *Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$, and suppose that there is a non-trivial isometry σ of L such that σ has 1 as an eigenvalue of σ. Then there exist non-zero two sublattices L_1, L_2 such that*

$$L \supset L_1 \perp L_2 \supset c_{14} L ,$$

where c_{14} is a natural number depending on the rank of L.

Proof. Let n be the order of σ. Then n is not larger than some constant depending on the rank of L. The assumption implies $\sum_{i=1}^{n} \sigma^i \neq 0$. Put $L_0 = \{x \in L ; \sigma x = x\}$. Then $L_0 \neq 0$, since there exists some x in L such that $\sum_{i=1}^{n} \sigma^i x \neq 0$, and the rank of L_0 is not equal to the rank of L. For any element x in L, $\sum_{i=1}^{n} \sigma^i x$ is in L_0, and $nx - \sum_{i=1}^{n} \sigma^i x$ is in $L_0^{\perp}$. This means

$$L \supset L_0 \perp L_0^{\perp} \supset nL .$$

Remark. $L \supset L_1 \perp L_2 \supset c_{14} L$ is equivalent to

$$L_1 \perp L_2 \supset c_{14} L \supset c_{14}(L_1 \perp L_2) .$$

5. LEMMA 3. *By $H_m^0(D)$ we denote the number of equivalence classes of positive definite integral matrices of degree m and determinant D which have a non-trivial unit with* 1 *as an eigenvalue. Then we have*

$$H_m^0(D) \leq c_{15}(\varepsilon) D^{(m-2)/2+\varepsilon} \qquad \textit{for any } \varepsilon > 0 .$$

Proof. For $m = 2$, $c_{16}(\varepsilon) D^{1/2-\varepsilon} \leq H_2(D) \leq c_{17}(\varepsilon) D^{1/2+\varepsilon}$ for any $\varepsilon > 0$ is proved by Siegel. From Lemma 2 it follows

$$\begin{aligned} H_m^0(D) &\leq c_{14}^m \sum_{a=1}^{c_{14}^{2m}} \sum_{b=1}^{[m/2]} \sum_{c \mid aD} H_b(c) H_{m-b}(aD/c) \\ &\leq c_{18}(\varepsilon) \sum_{a=1}^{c_{14}^{2m}} \sum_{b=1}^{[m/2]} (aD)^{(m-b-1)/2+\varepsilon} \sum_{c \mid aD} c^{(2b\cdot m)/2} \\ &\leq c_{19}(\varepsilon) \sum_{a=1}^{c_{14}^{2m}} a^{(m-2)/2+2\varepsilon} D^{(m-2)/2+2\varepsilon} \\ &\leq c_{20}(\varepsilon) D^{(m-2)/2+2\varepsilon} . \end{aligned}$$

6. *Proof of Corollary of Theorem* 1.

Let S be a positive definite integral matrix of even degree m and

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determinant D. Suppose that any matrix which is equivalent to S is always equivalent to S in the narrow sense; then the unit group of S contains a unit of whose determinant is -1. This implies that the difference $2H_m(D) - h_m(D)$ is at most the number of equivalence classes which have a unit of determinant -1. From Lemma 3 and Theorem 1 follows our corollary.

7. *Proof of Theorem* 2

In case of $m = 2$, let $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $D = ac - b^2$ and $c \geq a \geq 2|b|$. Since $E(S) > 2$ implies $c = a$ or $a = |2b|$, the number of equivalence classes which have a non-trivial unit is at most $c_{21}(\varepsilon)D^{\varepsilon}$ for any $\varepsilon > 0$. This completes the proof in case of $m = 2$. From Lemma 3 it is sufficient to prove Theorem 2 that we estimate the number of equivalence classes such that they have a non-trivial unit and any non-trivial unit has not 1 as an eigenvalue. Let S be such a matrix, and L be a lattice over $\boldsymbol{Z}$ corresponding to S. We denote the orthogonal group of L (= the unit group of S) by G. From the assumption, we see that G contains a unit σ such that σ has not 1 as an eigenvalue and the order q of σ is an odd prime or 4. If $q = 4$, then $\sigma^2 = -1$. If $q \neq 4$, then $\sigma + \cdots + \sigma^q = 0$. Hence the ring $\boldsymbol{Z}[\sigma]$ is isomorphic to the maximal order O of $Q(\sqrt[q]{1})$. Since, then, L is a torsion-free O-module, from the theory of modules over Dedekind domain it follows that L is O-isomorphic to a direct sum of ideals of $Q(\sqrt[q]{1})$:

$$L \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n ,$$

where $A_1 = \cdots = A_{n-1} = O$, and the ideal A_n is a (fixed) representative of some ideal class. (This ideal class is uniquely determined by L.) This identification transforms S to a totally positive definite Hermitian matrix $H(S) = (h_{ij})$ with h_{ij} in $(A_i\overline{A}_j\theta)^{-1}$, where the bar denotes the complex conjugate and θ is the different of $Q(\sqrt[q]{1})$. Moreover if S_1, S_2 are equivalent and have σ as a unit and $S_1 = S_2[T]$ for some T in $GL(m, \boldsymbol{Z})$ satisfying $\sigma T = \sigma T$, then for corresponding Hermitian forms $H(S_1)$, $H(S_2)$ there exists a matrix $X = (x_{ij})$ such that

$$H(S_1) = XH(S_2)^t\overline{X}\,, \quad \text{and} \quad x_{ij}, x'_{ij} \in A_i^{-1}A_j\,,$$

where $(x'_{ij}) = X^{-1}$. We remark that there is a natural number c such that all entries of $cH(S)$ are integers in $Q(\sqrt[q]{1})$, and the group $G = \{X$

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$= (x_{ij})\,;\, x_{ij}, x'_{ij} \in A_i^{-1}A_j$, where $(x'_{ij}) = X^{-1}\}$ and $GL(n, O)$ are commensurable. On the other hand, any totally positive definite Hermitian matrix is equivalent (with respect to $GL(n, O)$) to some element in $\bigcup_{i=1}^{d} S\{X_i\}$, where S is a sufficiently large Siegel domain and X_i is a non-singular integral matrix. (S, X_i, d depend on only q and n.) This implies that the class number of positive definite Hermitian forms with the norm of determinant $\leq D$ is at most $c(q)D^{n/2}$, where the constant $c(q)$ depends on only q. From these it follows that the number of equivalence classes in which there is some positive definite matrix S such that S has σ as a unit and $|S| \leq D$ is at most $c_{22}D^{n/2}$. Since $m > 2$ implies $n < m - 1$, we have proved Theorem 2.

7. *Proof of Corollary of Theorem* 2.

It is easy to calculate the mass of square-free and odd determinant by using [3], [6]:

$$\sum_S \frac{1}{E(S)} = \frac{D^{(m-1)/2}}{4\pi^{m(m+1)/4}} \prod_{k=1}^{m} \Gamma\left(\frac{k}{2}\right) \prod_{k=1}^{(m-1)/2} \zeta(2k)$$
$$\times \left\{(1 + 2^{-(m-1)/2})\left(1 + \delta\left(\frac{-1}{D}\right)^{\frac{m+1}{2}} D^{-(m-1)/2}\right)\right.$$
$$\left. + (1 - 2^{-(m-1)/2})\left(1 - \delta\left(\frac{-1}{D}\right)^{\frac{m+1}{2}} D^{-(m-1)/2}\right)\right\},$$

where S runs over a set of representatives of classes of positive definite integral matricies of odd degree $m \geq 3$ and of square-free and odd determinant D, and $\delta = (-1)^{(n+1)(n+2)/2 + ((D-1)/2)n}(n = (m-3)/2)$. Corollary follows from this.

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