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A Geometric Extension of Schwarz's Lemma and Applications

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Abstract. Let f be a holomorphic function of the unit disc $\mathbb{D}$, preserving the origin. According to Schwarz's Lemma, $|f'(0)| \le 1$, provided that $f(\mathbb{D}) \subset \mathbb{D}$. We prove that this bound still holds, assuming only that $f(\mathbb{D})$ does not contain any closed rectilinear segment $[0, e^{i\phi}]$, $\phi \in [0, 2\pi]$, *i.e.*, does not contain any entire radius of the closed unit disc. Furthermore, we apply this result to the hyperbolic density and give a covering theorem.

1 Introduction and Statement of Results

Let $f: \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$. The classical Scwharz Lemma asserts that

$$|f'(0)| \le 1. \tag{1.1}$$

Numerous geometric variations and extensions of Schwarz's Lemma have been proved; see, for example, [2–6, 8, 14] and [11, Chapter 4].

Here we will prove a geometric extension of Schwarz's Lemma, inspired by a recent theorem of Solynin [14, Theorem 4].

Let A_ϕ be the rectilinear segment $[0, e^{i\phi}]$, $\phi \in [0, 2\pi]$. Our purpose is to prove that the bound (1.1) still holds under the assumption $A_\phi \setminus f(\mathbb{D}) \neq \varnothing$, for every $\phi \in [0, 2\pi]$. This hypothesis is, of course, weaker than $f(\mathbb{D}) \subset \mathbb{D}$ and geometrically means that the image $f(\mathbb{D})$ does not contain any of the closed radii $[0, e^{i\phi}]$, $\phi \in [0, 2\pi]$, of the unit disc.

Theorem 1.1 *Let $f: \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. Assume that $A_\phi \setminus f(\mathbb{D}) \neq \varnothing$, for all $\phi \in [0, 2\pi]$. Then*

$$|f'(0)| \le 1. \tag{1.2}$$

Further, equality holds in (1.2) if and only if f has the form $f(z) = cz$, where $c \in \mathbb{C}$ and $|c| = 1$.

The main vehicles for the proof are polarization with respect to circles and the hyperbolic density (see Section 2).

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As Solynin did in [14], we will present two equivalent formulations of Theorem 1.1 (*cf.* [14, Corollaries 1 and 2]). The first one involves the density of the hyperbolic metric, which is presented in Section 2.

Corollary 1.2 *Let Ω be a hyperbolic domain in $\mathbb{C}$. Suppose that there exists a point $z_0 \in \Omega$ for which $\lambda(z_0, \Omega) \le k$, for some $k > 0$. Then Ω either contains a closed segment with one endpoint at z_0 and length $2/k$, or it coincides with the disk of radius $2/k$ and center z_0.*

This is proved by applying Theorem 1.1 to the function $f(z) = \frac{k}{2}(G(z) - z_0)$, where $G: \mathbb{D} \to \Omega$ is a universal covering map of Ω with $G(0) = z_0$.

Furthermore, Theorem 1.1 can be adapted to a covering theorem for radial segments.

Corollary 1.3 *Let $f: \mathbb{D} \to \mathbb{C}$ be holomorphic with $f(0) = 0$. If $|f'(0)| \ge 1$, then either $f(\mathbb{D}) = \mathbb{D}$, or $f(\mathbb{D})$ contains a closed segment with one endpoint at the origin and length 1.*

Covering properties of holomorphic functions are a classical subject in geometric function theory. We refer to [7, §§10-11] and references therein for more information.

The article is organized as follows. In Section 2 we present the basic tools of our proofs: the hyperbolic density and polarization with respect to circles. In Section 3 we prove Theorem 1.1. Throughout this article we will denote by $D(z_0, r)$ the disc of radius $r > 0$ centred at $z_0 \in \mathbb{C}$, by $r\mathbb{D}$ the disc $D(0, r)$, and by C_r its boundary.

2 Preliminaries

2.1 Hyperbolic Density

Let Ω be a hyperbolic domain in the extended complex plane $\mathbb{C}_\infty$; that is, the complement $\mathbb{C}_\infty \setminus \Omega$ of Ω contains at least three points. Then the hyperbolic density $\lambda(\cdot, \Omega)$ (the density of the Hyperbolic or Poincaré metric for Ω) is defined as follows. Let $h: \mathbb{D} \to \Omega$ be a holomorphic universal covering map (see *e.g.*, [1, p. 41], [10, p. 680]). Then

$$\lambda(h(z), \Omega)|h'(z)| = \frac{2}{1 - |z|^2}, \quad \text{for every } z \in \mathbb{D}. \tag{2.1}$$

For example if $\Omega = \mathbb{D}$, then (2.1) gives

$$\lambda(z, \mathbb{D}) = \frac{2}{1 - |z|^2}, \quad \text{for every } z \in \mathbb{D}. \tag{2.2}$$

The Principle of the Hyperbolic metric (see [10, p. 682], [12, p. 49]) implies that if D, Ω are hyperbolic domains and $f: D \to \Omega$ is a holomorphic function, then

$$\lambda(f(z), \Omega)|f'(z)| \le \lambda(z, D), \quad \text{for every } z \in D, \tag{2.3}$$

with equality if and only if f is a covering map (this result can be found also in [1, p. 43] as the general version of the Schwarz–Pick lemma).

The inequality (2.3) easily implies that for hyperbolic domains $D \subset \Omega$,

$$\lambda(z,\Omega) \le \lambda(z,D), \quad \text{for every } z \in D. \tag{2.4}$$

Equality occurs if and only if $D = \Omega$.

For more information about the hyperbolic density, we refer the reader to [1] and [10, Chapter 9].

2.2 Polarization with Respect to Circles

Let $r > 0$ and C_r be the circle with radius r and center at the origin. Let also $z \in \mathbb{C}$, $z \ne 0$. The symmetric point of z with respect to the circle C_r, is the point $\widetilde{z} = \frac{r^2}{\overline{z}}$. We also set $\widetilde{0} = \infty$, $\widetilde{\infty} = 0$.

The polarization of a set $\Omega \subset \mathbb{C}$ with respect to the circle C_r is defined as

$$P_{C_r}(\Omega) = \Big((\Omega \cup \widetilde{\Omega}) \cap \overline{r\mathbb{D}} \Big) \cup \Big((\Omega \cap \widetilde{\Omega}) \cap (\mathbb{C} \smallsetminus r\mathbb{D}) \Big),$$

where $\widetilde{\Omega} = \{\widetilde{z} : z \in \Omega\}$, is the reflection of the set Ω with respect to C_r.

Remark 2.1 By describing the polarization of Ω with respect to C_r we have that a point z belongs to $P_{C_r}\Omega$ if at least one of the followings holds:

(i) $z \in \Omega$ and $|z| \le r$,
(ii) $\widetilde{z} \in \Omega$ and $|z| \le r$,
(iii) $z, \widetilde{z} \in \Omega$.

The next result follows by a theorem of Solynin [13], which gives the behaviour of hyperbolic density under polarization with respect to circles. Let Ω be a hyperbolic domain containing the origin and C_r the circle as above. Then

$$\lambda(0, P_{C_r}\Omega) \le \lambda(0,\Omega). \tag{2.5}$$

Equality holds in (2.5) if and only if $\Omega = P_{C_r}\Omega$ or $\Omega = \widetilde{P_{C_r}\Omega}$.

We mention here that the hyperbolic density $\lambda(z, P_{C_r}\Omega)$ of $P_{C_r}\Omega$ is defined for every connected component of $P_{C_r}\Omega$.

For more information about polarization, we refer the reader to [7, 13] and the references therein.

3 Proof of Theorem 1.1

We consider the family $\mathcal{F}$ of holomorphic functions $f{:}\,\mathbb{D} \to \mathbb{C}$, with $f(0) = 0$ and $A_\phi \smallsetminus f(\mathbb{D}) \ne \varnothing$, for all $\phi \in [0, 2\pi]$.

By applying Montel's normality criterion, we see that $\mathcal{F}$ is a normal family (*cf.* [14]).

Lemma 3.1 *The family $\mathcal{F}$ is compact.*

Proof As $\mathcal{F}$ is a normal family we only have to prove that the limit of every locally uniformly convergent subsequence belongs to $\mathcal{F}$. Let $\{f_n\}_{n\ge1} \subset \mathcal{F}$ be a sequence that converges locally uniformly to a function f. The function f is holomorphic in $\mathbb{D}$ with $f(0) = \lim_{n\to\infty} f_n(0) = 0$. It remains to show that for all $\phi \in [0, 2\pi]$, $A_\phi \smallsetminus f(\mathbb{D}) \ne \varnothing$.

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Suppose that there exists $\phi \in [0, 2\pi]$ such that $A_\phi \setminus f(\mathbb{D}) = \varnothing$. But $f_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, so for all $n \in \mathbb{N}$ there exists $w_n \in A_\phi \setminus f_n(\mathbb{D})$. Since A_ϕ is compact, there exists a subsequence w_{n_k} converging to a point $w_0 \in A_\phi$. Also, $A_\phi \subset f(\mathbb{D})$; so there exists $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$.

Since z_0 is a root of the nonconstant holomorphic function $f(z) - w_0$, there exists $r > 0$ such that $f(z) \neq w_0$ for all $z \in \overline{D(z_0, r)} \setminus \{z_0\}$, where $\overline{D(z_0, r)} \subset \mathbb{D}$. Let

$$m = \min\{|f(z) - w_0| : |z - z_0| = r\}.$$

As f_n converges to f uniformly in $\overline{D(z_0, r)}$, there exists $k_1 \in \mathbb{N}$ such that

$$|f_{n_k}(z) - f(z)| < \frac{m}{2}, \quad \text{for all } k \geq k_1 \quad \text{and for all } z \in \overline{D(z_0, r)}.$$

Also, as $w_{n_k} \to w_0$, there exists $k_2 \in \mathbb{N}$ such that

$$|w_{n_k} - w_0| < \frac{m}{2}, \quad \text{for all } k \geq k_2.$$

Let $k_0 = \max\{k_1, k_2\}$. Then for all z with $|z - z_0| = r$ and for all $k \geq k_0$,

$$\begin{aligned} \left| \left(f_{n_k}(z) - w_{n_k} \right) - \left(f(z) - w_0 \right) \right| &\leq \left| f_{n_k}(z) - f(z) \right| + |w_0 - w_{n_k}| < \frac{m}{2} + \frac{m}{2} \\ &\leq \left| f(z) - w_0 \right|. \end{aligned}$$

Therefore, by Rouche's theorem, for k sufficiently large, the function $f_{n_k}(z) - w_{n_k}$ has zero in $D(z_0, r)$, a contradiction. ∎

We are now ready to proceed with the proof of our main result.

Proof of Theorem 1.1 Since $\mathcal{F}$ is a normal and compact family, there exists $F \in \mathcal{F}$ such that

$$|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|.$$

As the function $h(z) = z$ belongs to the family $\mathcal{F}$, we deduce that

$$|F'(0)| \geq 1. \tag{3.1}$$

Let $\Omega = F(\mathbb{D})$ and let $G : \mathbb{D} \to \Omega$ be the universal covering map of Ω, with $G(0) = 0$ and $G'(0) > 0$ (see *e.g.*, [1, p. 41]). The function G belongs to the family $\mathcal{F}$, because $G(\mathbb{D}) = \Omega$. The general analytic function $\mathbf{G^{-1}}$ maps Ω into $\mathbb{D}$, and hence by [9, Theorem 2.20], F is subordinate to G. By the theorem of subordination [9, Theorem 2.21], $|F'(0)| \leq |G'(0)|$, and since F is the maximal function for the family $\mathcal{F}$, we have the equality $|F'(0)| = G'(0)$. By (2.3) and (2.2)

$$\lambda(0, \Omega)|F'(0)| = \lambda(0, \Omega)G'(0) = \lambda(0, \mathbb{D}) = 2.$$

Hence, by the equality case of relation (2.3), F is a holomorphic covering of $\mathbb{D}$ to Ω with $F(0) = 0$ and

$$|F'(0)| = \frac{2}{\lambda(0, \Omega)}. \tag{3.2}$$

Let $M = \overline{\mathbb{D}} \setminus \Omega$ and

$$\alpha = \inf\{|z| : z \in M\}, \quad \beta = \sup\{|z| : z \in M\}.$$

Since $F(0) = 0$, we have $\alpha > 0$.

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We consider the following cases.

Case 1: $\alpha = \beta$. Then for all $z \in M$, $|z| = \alpha$ and hence $M \subseteq C_\alpha$. We claim that $M = C_\alpha$. Suppose that there exists $z_0 = \alpha e^{i\phi_0} \notin M$. Then $z_0 \in \Omega$ and as $A_{\phi_0} \setminus \Omega \neq \varnothing$, there exists $r \in [0,1] \setminus \{\alpha\}$ such that $z_1 = re^{i\phi_0} \notin \Omega$ and therefore $z_1 \in M$. But if $|z_1| = r < \alpha$, then $\inf_{z\in M} |z| \leq r < \alpha$; a contradiction. In the same way, if $|z_1| = r > \alpha$, then $\sup_{z\in M} |z| \geq r > \alpha$, which also gives a contradiction. Hence, $M = C_\alpha$.

If $\alpha \in (0,1)$, then there exists $z \in \Omega$ with $|z| > \alpha$. This is absurd, because Ω is connected, $C_\alpha \cap \Omega = \varnothing$ and $0 \in \Omega$. Therefore, $\alpha = 1$.

As Ω is connected, we conclude that $\Omega \subset \mathbb{D}$. Hence, by Schwarz's Lemma, $|F'(0)| \leq 1$. By (3.1), $|F'(0)| = 1$. So we have equality in Schwarz's Lemma. Therefore, $F(z) = cz$, where $c \in \mathbb{C}$ with $|c| = 1$ and $\Omega = \mathbb{D}$.

Case 2: $0 < \alpha < \beta \leq 1$. We are going to show that this case cannot occur.

We set $\gamma = \sqrt{\alpha\beta}$. Note that $\alpha < \gamma < \beta$, and so $0 < \gamma < 1$.

Let C_γ be the circle with center at the origin of radius γ and let Ω_1 be the connected component containing 0 of the polarization of Ω with respect to the circle C_γ.

Let $F_1 : \mathbb{D} \to \Omega_1$ be the holomorphic universal covering of Ω_1 with $F_1(0) = 0$ and $F_1'(0) > 0$. We show that $F_1 \in \mathcal{F}$.

Let $\phi \in [0, 2\pi]$. It suffices to prove that $A_\phi \setminus \Omega_1 \neq \varnothing$. Since $F \in \mathcal{F}$, there exists $z_\phi \in A_\phi \setminus \Omega$. Let $\widetilde{z_\phi}$ be the symmetric of the point z_ϕ with respect to the circle C_γ.

- If $\widetilde{z_\phi} \notin \Omega$, then $z_\phi \notin P_{C_\gamma}\Omega \supset \Omega_1$, so $A_\phi \setminus \Omega_1 \neq \varnothing$.
- If $\widetilde{z_\phi} \in \Omega$ and z_ϕ is in the exterior of the circle C_γ, then $z_\phi \notin P_{C_\gamma}\Omega \supset \Omega_1$, and as before $A_\phi \setminus \Omega_1 \neq \varnothing$.
- If $\widetilde{z_\phi} \in \Omega$ and z_ϕ is in the interior of the circle C_γ, then $\widetilde{z_\phi} \notin P_{C_\gamma}\Omega \supset \Omega_1$. It remains to show that $0 < |\widetilde{z_\phi}| \leq 1$. But $\alpha \leq |z_\phi| \leq \beta$; hence

$$0 < \alpha = \frac{\alpha\beta}{\beta} \leq |\widetilde{z_\phi}| = \frac{\gamma^2}{|\overline{z_\phi}|} = \frac{\gamma^2}{|z_\phi|} \leq \frac{\alpha\beta}{\alpha} = \beta \leq 1.$$

So in all cases, $A_\phi \setminus \Omega_1 \neq \varnothing$, which gives $F_1 \in \mathcal{F}$.

Since $F_1 : \mathbb{D} \to \Omega_1$ by (2.3) and (2.2), we get

$$F_1'(0) = \frac{2}{\lambda(0, \Omega_1)}. \tag{3.3}$$

But from (2.5),

$$\lambda(0, \Omega_1) \leq \lambda(0, \Omega). \tag{3.4}$$

So combining (3.2), (3.3), and (3.4) we have that $F_1'(0) \geq |F'(0)|$, and as F is a maximal function for the family $\mathcal{F}$, we have $F_1'(0) = |F'(0)|$. Therefore, we have equality in (3.4), and hence by the equality case of (2.5), we have either $\Omega = \Omega_1$ or $\Omega = \widetilde{\Omega_1}$. The latter case is rejected because $\widetilde{\Omega_1}$ contains ∞ and F is holomorphic, hence $\Omega = \Omega_1$.

We now consider the set $\Omega_2 = \Omega \cup \gamma\mathbb{D}$. Since $\alpha < \gamma$, there exists $z_0 \in M$ with $|z_0| < \gamma$ and hence $\Omega \neq \Omega_2$. Therefore, (2.4) gives

$$\lambda(0, \Omega_2) < \lambda(0, \Omega). \tag{3.5}$$

We will prove that Ω_2 has the geometric property $A_\phi \setminus \Omega_2 \neq \varnothing$, for every $\phi \in [0, 2\pi]$. We assume conversely that there exists a $\phi \in [0, 2\pi]$ such that $A_\phi \setminus \Omega_2 = \varnothing$.

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This means that Ω_2 contains the set $B_\phi = \{re^{i\phi} : \gamma \le r \le 1\}$. But since $\Omega = \Omega_1$ is polarized with respect to C_γ and B_ϕ lies in the exterior of C_γ, we have that $P_{C_\gamma} B_\phi \subset \Omega$ and so $\{re^{i\phi} : \gamma^2 \le r \le 1\} \subset \Omega$. By the fact that $A_\phi \smallsetminus \Omega \ne \varnothing$, there exists a $z_0 \in A_\phi \smallsetminus \Omega$, with modulus $|z_0| < \gamma^2 \le \alpha$. But this means that $z_0 \in M$ and $|z_0| < \alpha$, which is a contradiction. So $A_\phi \smallsetminus \Omega_2 \ne \varnothing$ for every $\phi \in [0, 2\pi]$.

We consider the holomorphic universal covering $F_2 : \mathbb{D} \to \Omega_2$ with $F_2(0) = 0$ and $F_2'(0) > 0$. Then $F_2 \in \mathcal{F}$ and therefore by (2.3), (2.2), (3.3), and the fact that F_1 is a maximal function

$$\frac{2}{\lambda(0, \Omega_2)} = F_2'(0) \le F_1'(0) = \frac{2}{\lambda(0, \Omega_1)},$$

which contradicts (3.5). So Case 2 cannot occur.

Therefore, for every $f \in \mathcal{F}$, $|f'(0)| \le |F'(0)| = 1$.

If $|f'(0)| = 1$ for some $f \in \mathcal{F}$, then f is a holomorphic covering of $f(\mathbb{D})$. If we consider again the set M and the cases $\alpha = \beta$ and $\alpha < \beta$ as above, we conclude that $f(z) = cz$ for a constant $c \in \mathbb{C}$ with $|c| = 1$ and the proof is complete. ∎

References

[1] A. F. Beardon and D. Minda, *The hyperbolic metric and geometric function theory*. In: Quasiconformal mappings and their applications, Narosa Publishing House, New Delhi, India, 2007, pp. 9–56.

[2] D. Betsakos, *Geometric versions of Schwarz's lemma for quasiregular mappings*. Proc. Amer. Math. Soc. **139**(2011), 1397–1407. http://dx.doi.org/10.1090/S0002-9939-2010-10604-4

[3] A. Bermant, *On certain generalizations of E. Lindelöf's principle and their applications*. Mat. Sb. **20(62)**(1947), 55–112.

[4] R. B. Burckel, D. E. Marshall, D. Minda, P. Poggi-Corradini, and T. J. Ransford, *Area, capacity and diameter versions of Schwarz's lemma*. Conform. Geom. Dyn. **12**(2008), 133–152. http://dx.doi.org/10.1090/S1088-4173-08-00181-1

[5] G. Cleanthous, *Monotonicity theorems for analytic functions centered at infinity*. Proc. Amer. Math. Soc. **142**(2014), 3545-3551. http://dx.doi.org/10.1090/S0002-9939-2014-12084-3

[6] G. Cleanthous and A. G. Georgiadis, *Multi-point bounds for analytic functions under measure conditions*. Complex Var. Elliptic Equ. **60**(2015), 470–477. http://dx.doi.org/10.1080/17476933.2014.944864

[7] V. N. Dubinin, *Symmetrization in the geometric theory of functions of a complex variable*. (Russian) Uspekhi Math. Nauk. **49**(1994), 3–76; translation in Russian Math. Surveys **49**(1994), 1-79. http://dx.doi.org/10.1070/RM1994v049n01ABEH002002

[8] ———, *Geometric versions of Schwarz's lemma and symmetrization*. J. Math. Sci. (N. Y.) **178**(2011), 150–157.

[9] W. K. Hayman and P. B. Kennedy, *Subharmonic functions*. Vol. I., London Mathematical Society Monographs, 9, Academic Press, London-New York, 1976.

[10] W. K. Hayman, *Subharmonic functions*. Vol. II., London Mathematical Society Monographs, 20, Ademic Press, London, 1989.

[11] ———, *Multivalent functions*. Second ed., Cambridge Tracts in Mathematics, 110, Cambridge University Press, Cambridge, 1994. http://dx.doi.org/10.1017/CBO9780511526268

[12] R. Nevanlinna, *Analytic functions*. Springer-Verlag, New York-Berlin, 1970.

[13] A. Yu. Solynin, *Polarization and functional inequalities*. Algebra i Analiz **8**(1996), 148–185 (Russian); English translation in St. Petersburg Math. J. **8**(1997), 1015–1038.

[14] ———, *A Schwarz lemma for meromorphic functions and estimates for the hyperbolic metric*. Proc. Amer. Math. Soc. **136**(2008), 3133–3143. http://dx.doi.org/10.1090/S0002-9939-08-09309-X

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