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WEAK CONVERGENCE AND ONE-SAMPLE RANK STATISTICS UNDER Φ-MIXING*

BY
K. L. MEHRA

1. **Introduction.** Let $\{X_i: i=1, 2, \ldots\}$ be a real strictly stationary process (defined on a probability space $(\Omega, \mathscr{A}, P)$) which has absolutely continuous finite dimensional distributions (with respect to Lebesgue measure) and satisfies the ϕ-mixing condition: Let M_1^k and M_{k+n}^∞ denote the sub-σ-fields generated, respectively, by $\{X_i: i\leq k\}$ and $\{X_i: i\geq k+n\}$; then, for each $k\geq 1$ and $n\geq 1$, $E_1\in M_1^k$ and $E_2 \in M_{k+n}^\infty$ together imply

$$|P(E_1\cap E_2)-P(E_1)\cdot P(E_2)|\leq \phi(n)P(E_1), \tag{1.1}$$

where ϕ, $0\leq\phi\leq 1$, is a non-increasing function of positive integers which approaches 0 as $n\to\infty$. In [3], Fears and Mehra proved the Chernoff-Savage Theorem [2] concerning the asymptotic normality of two-sample linear rank statistics for sequences of observations which satisfy the above ϕ-mixing dependence. The proof uses the weak convergence approach of Pyke and Shorack [4] and is based on a Hájek-Rényi type inequality for one-sample empirical processes under ϕ-mixing, which enables one to study weak convergence properties of the one and two sample empirical processes for ϕ-mixing sequences. The object of the present paper is to establish similar results for the one-sample linear rank statistics under ϕ-mixing, viz., the statistics of the type

$$T_N = \frac{1}{N}\sum_{i=1}^{N} c_{Ni}^* \tau_{Ni}, \tag{1.2}$$

where $\tau_{Ni}=1, 0, -1$ according as the ith order statistics $|X|^{(i)}$, $1\leq i\leq N$, in an ordering of $|X_k|$, $k=1, 2, \ldots, N$, corresponds to a positive, zero or negative X and $\{c_{Ni}^*: 1\leq i\leq N\}$ is a certain appropriate double sequence of scores. In the process we establish a Hájek-Rényi type inequality (see (2.9)) for the one-sample signed empirical process $V_N(t)$, defined by (2.6) below, which should be of interest per se. The results of this paper are related to those of Pyke and Shorack [5] and are employed in a separate paper to study the asymptotic relative efficiency of Hodges-Lehmann type estimates of location and related rank tests for sequences of dependent observations satisfying 'mixing' conditions.

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In section 2, some notation and preliminary results concerning the weak convergence of one-sample signed empirical processes are described. Section 3 contains an identity relating the signed empirical processes $\{L_N(t):0\leq t\leq 1\}$ and $\{V_N(t):0\leq t\leq 1\}$ (see (2.4) and (2.6) for definitions) and the main theorem concerning the weak convergence of L_N and T_N^*. In the last section 4, a convenient Chernoff-Savage type theorem for the one-sample linear rank statistics T_N is given.

2. **Notation and Preliminary Results.** Let $H_0(F)$ denote the distribution function (d.f.) of $|X_1|$ (X_1) and $H_N(F_N)$ the empirical d.f. corresponding to the first N $|X|$'s (X's) and let G_N denote the empirical function

$$G_N(x) = \frac{1}{N}\sum_{i=1}^{N} I_{[|X_i|<x]}\,\mathrm{sgn}(X_i), \tag{2.1}$$

where $sgn(X_i)=1, 0$ or -1 according as X_i is positive, zero or negative. Let $R_{Ni}(S_{Ni})$ stand for the number of positive (negative) X's whose absolute values do not exceed $|X|^{(i)}$, $1\leq i\leq N$. Then $R_{Ni}-S_{Ni}=NG_NH_N^{-1}(i/N)$, where the inverse function $H_N^{-1}(t)$, $0\leq t\leq 1$, is defined by $H_N^{-1}(t)=\inf\{x:H_N(x)\geq t\}$ (similarly H_0^{-1}, H^{-1} etc.) so that as in Pyke and Shorack [4] using summation by parts and the relations $\tau_{N1}=R_{N1}-S_{N1}$ and $\tau_{Nk}=(R_{Nk}-S_{Nk})-(R_{N(k-1)}-S_{N(k-1)})$, $1<k\leq N$, the statistic T_N is expressible as

$$T_N = \frac{1}{N}\sum_{i=1}^{N} c_{Ni}(R_{Ni}-S_{Ni}) = \int_0^1 G_NH_N^{-1}(t)\,d\nu_N(t), \tag{2.2}$$

where c_{Ni}'s are related to c_{Ni}^*'s by $c_{Ni}^*=\sum_{k\geq i} c_{Nk}$, $1\leq i\leq N$ and ν_N denotes the (signed) measure giving weight c_{Ni} to (i/N) $1\leq i\leq N$. Assuming that $0<F(0)<1$, denote by $m(n)$ the number of positive (negative) X's, $\lambda_N=(m/N)$, $F^+(F^-)$ the conditional d.f. of $|X_1|$ given $X_1>0$ $(X_1<0)$ and

$$\begin{aligned} H &= H_{\lambda_N} = \lambda_NF^+ + (1-\lambda_N)F^- \\ G &= G_{\lambda_N} = \lambda_NF^+ - (1-\lambda_N)F^- \end{aligned} \tag{2.3}$$

(H and G are both random and depend on N, but this fact is suppressed in the notation). Note that if we set $\lambda_0=1-F(0)$, then $H_0(x)=H_{\lambda_0}(x)=F(x)-F(-x)$ and $G_0(x)=G_{\lambda_0}(x)=F(x)+F(-x)-2F(0)$ are the d.f.'s of $|X_1|$ and $|X_1|\,\mathrm{sgn}(X_1)$ respectively. Further also note that on account of the absolute continuity assumption of section 1, $(n/N)=1-\lambda_N$ with probability one. Define now the empirical process $\{L_N(t):0\leq t\leq 1\}$ by

$$L_N(t) = N^{1/2}[G_NH_N^{-1}(t)-GH^{-1}(t)]; \tag{2.4}$$

then setting $\eta_N=\int_0^1 GH^{-1}(t)\,d\nu_N(t)$, we obtain from (2.2) that

$$T_N^* = N^{1/2}(T_N-\eta_N) = \int_0^1 L_N\,d\nu_N(t). \tag{2.5}$$

To study the asymptotic distribution of T_N^*, as $N\to\infty$, under suitable conditions on the measures ν_N and the sequence $\{X_i : i\geq 1\}$, we shall study in section 3 the weak convergence of the process L_N relative to various metrics. For this we need to study the weak convergence of the one-sample signed empirical processes $\{V_N(t):0\leq t\leq 1\}$ and $\{V_N^*(t):0\leq t\leq 1\}$, where

$$V_N(t) = N^{1/2}[G_N H_0^{-1}(t)-GH_0^{-1}(t)] \quad \text{and} \quad V_N^*(t)-N^{1/2}[H_N H_0^{-1}(t)-HH_0^{-1}(t)]. \tag{2.6}$$

We shall now prove a result similar to Lemma 2.2 of Pyke and Shorack [4] (see also Lemma 2.1 of Fears and Mehra [3]).

LEMMA 2.1. *Assume that the ϕ-mixing sequence $\{X_i\}$ satisfies the conditions imposed in section* 1, *with $\sum_{n=1}^{\infty} n^2\phi_n^{1/2}<\infty$. Then given $\varepsilon>0$, there exists a θ, $0<\theta<\frac{1}{2}$, depending on ε alone and an integer $N_0=N_0(\varepsilon,\phi)$ (N_0 depends on $\{X_1\}$ through ϕ alone) such that for $N\geq N_0$*

$$P\left[\sup_{0\leq t\leq\theta}|V_N(t)/q(t)|\geq\varepsilon\right]\leq\varepsilon, \tag{2.7}$$

where $q(t)=[t(1-t)]^{(1/2)-\delta}$, $0\leq t\leq 1$, for some δ, $0<\delta<\frac{1}{2}$. The same result holds for V_N^* in place of V_N.

Proof. The proof is similar to Lemma 2.1 of [3]. Let

$$g_t(x)=[I_{[|x|\leq H_0^{-1}(t)]}\operatorname{sgn}(x)-(I_{[x>0]}F^+H_0^{-1}(t)-I_{[x<0]}F^-H_0^{-1}(t))], \quad 0\leq t\leq 1,$$

and consider M real points $0<s_1<s_2<\cdots<s_M=\theta<\frac{1}{2}$, with $s_\ell=(\ell\theta/M)$, $1\leq\ell\leq M$. Since $Eg_t(X_1)/I_{[X_1>0]}]=0$ a.s., it follows that for any $1<j<k\leq M$,

$$\begin{aligned} E\left[\frac{g_{s_k}(X_1)}{q(s_{k-1})}-\frac{g_{s_j}(X_1)}{q(s_{j-1})}\right]^2 &= E\left\{E\left[\left(\frac{g_{s_k}(X_1)}{q(s_{k-1})}-\frac{g_{s_j}(X_1)}{q(s_{j-1})}\right)^2\Big/ I_{[X_1>0]}\right]\right\} \\ &\leq \frac{s_k}{q^2(s_{k-1})}+\frac{s_j}{q^2(s_{j-1})}-\frac{2s_j}{q(s_{k-1})q(s_{j-1})} \\ &\leq \frac{4\theta}{M}\sum_{j<l\leq k}(1/q^2(s_{l-1})), \end{aligned} \tag{2.8}$$

the last inequality in (2.8) following from (2.3) to (2.6) of [3]. Now proceeding exactly as in [3] with $\xi_1=[V_N(s_1)/q(s_1)]$, $\xi_i=[V_N(s_{i+1})/q(s_i)]-[V_N(s_i)/q(s_{i-1})]$, $1<i<M$, and using Lemma 22.1 and Theorem 12.2 of [1] and the inequality

$$[q^2(s_l)/q^2(s_{l-1})]\leq 2 \quad \text{for} \quad 1<l\leq M, \tag{2.9}$$

we obtain

$$\mathrm{P}\left[\max_{1\leq i\leq M}\left|\frac{V_N(s_i)}{q(s_i)}\right|\geq\varepsilon\right]\leq\frac{K_\phi}{\varepsilon^4}\left[1+\frac{4M}{N}\right]\left[\frac{\theta}{M}\sum_{l=1}^{M=1}(1/q^2(s_l))\right]^2; \tag{2.10}$$

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(K, K_ϕ, K', etc. are generic constants throughout). Now since $|FH_0^{-1}(t)-FH_0^{-1}(s)|+|F(-H_0^{-1}(s))-F(-H_0^{-1}(t))|=|t-s|$, we have for $0\leq s<t\leq 1$

$$(2.11)\qquad |V_N(t)-V_N(s)|\leq |Y_N(t)-Y_N(s)|+\left(1+\frac{\lambda_N}{\lambda_0}+\frac{1-\lambda_N}{1-\lambda_0}\right)N^{1/2}(t-s),$$

where $Y_N(t)=N^{1/2}[H_NH_0^{-1}(t)-t]$. Further from (22.17) of Billingsley [1], we have for $s_i\leq t\leq s_{i+1}$

$$(2.12)\qquad |Y_N(t)-Y_N(s_i)|\leq |Y_N(s_{i+1})|+|Y_N(s_i)+N^{1/2}(s_{i+1}-s_i),$$

so that from (2.9), (2.11), (2.12) and the monotonicity of q, we obtain after some manipulation

$$(2.13)\qquad \begin{aligned}\sup_{(\theta/M)\leq t\leq\theta}\left|\frac{V_N(t)}{q(t)}\right| &\leq 2\max_{1\leq i\leq M}\frac{|V_N(s_i)|}{q(s_i)}<4\max_{1\leq i\leq M}\frac{|Y_N(s_i)|}{q(s_i)}\\ &\quad+\left(2+\frac{\lambda_N}{\lambda_0}+\frac{1-\lambda_N}{1-\lambda_0}\right)[(2N\theta)^{1/2}/M^{(1/2)+\delta}].\end{aligned}$$

Now for given ε and θ choose M and N sufficiently large, say $N\geq N_0(\varepsilon,\theta,\phi)$, such that

$$(2.14)\qquad \frac{4N\theta}{\varepsilon}>M>\frac{2N\theta}{\varepsilon}\quad\text{and}\quad P\left[\left(2+\frac{\lambda_N}{\lambda_0}+\frac{1-\lambda_N}{1-\lambda_0}\right)\frac{\varepsilon^{1/2}}{M^\delta}\geq\frac{\varepsilon}{3}\right]\leq\frac{\varepsilon}{6};$$

(for large enough N (2.14) is clearly possible since $\lambda_N\to_p 0$, as $N\to\infty$, uniformly in mixing sequences $\{X_i\}$). Using the inequality (2.14) of [3] and (2.10) above, it follows from (2.13) and (2.14) that

$$(2.15)\qquad P\left[\sup_{(\theta/M)\leq t\leq\theta}|V_N(t)/q(t)|\geq(2\varepsilon/3)\right]\leq\frac{K_\phi}{\varepsilon^5}\left(\int_0^\theta q^{-2}(t)\,dt\right)^2+\frac{\varepsilon}{6}.$$

Further note that since $H_N(H_0^{-1}(\theta/M))=0$ implies that

$$V_N(t)\leq N^{1/2}[(\lambda_N/\lambda_0)+((1-\lambda_N)/(1-\lambda_0))]t\quad\text{for}\quad 0\leq t\leq(\theta/M),$$

we have from (2.14)

$$(2.16)\qquad \begin{aligned}&P\left[\sup_{0\leq t<(\theta/M)}\frac{|V_N(t)|}{q(t)}<\frac{\varepsilon}{3}\right]\\ &\geq P\left[\left\{\left[\left(\frac{\lambda_N}{\lambda_0}\right)+\left(\frac{1-\lambda_N}{1-\lambda_0}\right)\right]\frac{\varepsilon^{1/2}}{M^\delta}<\frac{\varepsilon}{3}\right\}\cap\{H_NH_0^{-1}(\theta/M)=0\}\right]\geq 1-\frac{2\varepsilon}{3}\end{aligned}$$

The desired result follows from (2.15) and (2.16) if we choose θ so small that the first term on the right in (2.15) is less than $\varepsilon/6$. The proof of the inequality (2.7) for $\{V_N^*:0<t<1\}$ is similar. □

Let $C=C[0,1]$ be the space of continuous functions on [0, 1] and $D=D[0,1]$ the space of right-continuous functions on [0, 1] that have left-hand limits. Let ρ

and d denote, respectively, the uniform and the Skorohod metrics (see Billingsley (1968) p. 115). Both (C, ρ) and (D, d) are complete separable metric spaces. Now let F_N denote the empirical d.f. of $X_1, X_2, \ldots, X_N$ and

$$F_N^+(s) = \frac{1}{N}\sum_{i=1}^{N} I_{[0<X_i\le x]},\ F_N^-(x) = \frac{1}{N}\sum_{i=1}^{N} I_{[0<-X_i\le x]}; \tag{2.17}$$

then setting $V_N^+(t)=N^{1/2}[F_N^+H_0^{-1}(t)-\lambda_N F^+H_0^{-1}(t)]$ and $V_N^-(t)=N^{1/2}[F_N^-H_0^{-1}(t)-(1-\lambda_N)F^-H_0^{-1}(t)]$, it can be easily seen that

$$\begin{cases} V_N^+(t) = U_N(FH_0^{-1}(t))-U_N(F(0))[1-F^+H_0^{-1}(t)] \quad \text{and} \\ V_N^-(t) = \bar{U}_N(F(0))[1-F^-H_0^{-1}(t)]-\bar{U}_N(F(-H_0^{-1}(t))), \end{cases} \tag{2.18}$$

where $U_N(t)$ and $\bar{U}_N(t)$ are the one-sample empirical processes defined by $U_N(t)=N^{1/2}[F_NF^{-1}(t)-t]$ and $\bar{U}_N(t)=N^{1/2}[F_N(F^{-1}(t)-)-t]$. Define now the processes $\{W_N(u):0\le u\le 1\}$, for $N\ge 0$, by

$$\begin{aligned} W_N(u) &= V_N^-(2u) \quad &\text{if } 0\le u<\tfrac{1}{2} \\ &= V_N^+(2u-1) \quad &\text{if } \tfrac{1}{2}\le u\le 1, \end{aligned} \tag{2.19}$$

where the processes V_0^+ and V_0^- are defined by

$$\begin{aligned} V_0^+(t) &= U_0(FH_0^{-1}(t))-U_0(F(0))[1-F^+H_0^{-1}(t)] \\ V_0^-(t) &= U_0(F(0))[1-F^-(H_0^{-1}(t))]-U_0F(-H_0^{-1}(t)) \end{aligned} \tag{2.20}$$

and U_0 is the a.s. continuous Gaussian process given by (2.21) of [3]. (See also Theorem 22.1 of [1]).

LEMMA 2.2. *Let the function q and the sequence $\{X_n\}$ be as in Lemma 2.1. Then, as $N\to\infty$, (i) $W_N\to_L W_0$ relative to (D, d), and (ii) $(W_N/q^*)\to_L (W_0/q^*)$ relative to (D, d), where $q^*(u)$, $0\le u\le 1$, is defined by $q^*(u)=q(2u)$ for $0\le u<\frac{1}{2}$ and $q^*(u)=q(2u-1)$ for $\frac{1}{2}\le u\le 1$. Also note that W_0-process is a.s. continuous.*

Proof. First note that due to the assumed continuity of F, both processes U_N and $\bar{U}_N$ converge weakly, relative to (D, d), to the U_0-process (by Theorem 22.1 of [1]). Therefore it follows from (2.18) that the finite dimensional distributions of W_N-process converge to those of W_0-process and that condition (i) of Theorem 15.2 of [1] is satisfied. Now for a given function f on $[0, 1]$, let $\omega_\delta(f)$, $0<\delta<1$, be the modulus of continuity of f. Then using (2.21) of [1] and the equality

$$|F(H_0^{-1}(t))-F(H_0^{-1}(s))|+|F(-H_0^{-1}(s))-F(-H_0^{-1}(t))| = |t-s| \tag{2.21}$$

for $s, t\in[0, 1]$, it follows from (2.18) that $\omega_\delta(V_N^+)$ and $\omega_\delta(V_N^-)$ can be made arbitrarily small in probability for sufficiently small δ and sufficiently large N. Since $V_N^+(t)\to_p 0$ and $V_N^-(t)\to_p 0$, as $t\to 0$ or 1, it follows that condition (ii) of Theorem 15.2 of [1] is also satisfied for the W_N-processes. Thus part (i) of this lemma follows from Theorem 15.1 of [1]. For the proof of part (ii), first note that since

$$V_N(t) = V_N^+(t)-V_N^-(t) \quad \text{and} \quad V_N^*(t) = V_N^+(t)+V_N^-(t), \tag{2.22}$$

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$0 \leq t \leq 1$, the conclusion of Lemma 2.1 holds for V_N^+ or V_N^- in place of V_N. In view of this last assertion, (2.21) and the fact that $(V_N^+(t)/q(t))$ and $(V_N^-(t)/q(t))$ also converges to 0 in probability, as $t \to 0$ or 1, part (ii) follows by using the result and arguments of part (i) as done in the proof of Theorem 2.1 of [3].

REMARK 2.1. Consider the process $\{W_N^*(t): 0 \leq t \leq 1\}$, $N \geq 0$, with $W_N^* = \ell(W_N)$ obtained through a linear transformation $\ell: D \to D$ and defined by

$$
\begin{aligned}
l(g(t)) &= g\left(\frac{2t+1}{2}\right) - g(t) \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{2} \\
&= g\left(\frac{2t-1}{2}\right) + g(t) \quad \text{for} \quad \tfrac{1}{2} \leq t \leq 1;
\end{aligned}
\tag{2.23}
$$

for the transformation ℓ, defined by (2.23), note that $g \in D' = \{f: f \in D, f(0) = f(\frac{1}{2}) = f(1) = 0\}$ implies $\ell(g) \in D'$. Further for the process W_N^*, we have $W_N^*(t) = V_N(2t)$ if $0 \leq t \leq \frac{1}{2}$ and $W_N^*(t) = V_N^*(2t-1)$ for $(\frac{1}{2}) \leq t \leq 1$; consequently W_N^* and (W_N^*/q) $(N \geq 0)$ satisfy, respectively, the conclusions (i) and (ii) of Lemma 2.2, where we have set $V_0(t) = V_0^+(t) - V_0^-(t)$ and $V_0^*(t) = V_0^+(t) + V_0^-(t)$. This is because ℓ satisfies the conditions of Theorem 5.1 of [1]. Also $\ell: C' \to C'$, where $C' = \{f: f \in C, f(0) = (\frac{1}{2}) = f(1) = 0\}$, so that $P[W_0^* \in C] = 1$.

Now define the processes $\{X_N(t): 0 \leq t \leq 1\}$, $N \geq 0$, by

$$
\begin{aligned}
X_N(t) &= \lambda_N && \text{for} \quad 0 \leq t < \tfrac{1}{3} \\
&= V_N^-(3t-1)/q(3t-1) && \text{for} \quad \tfrac{1}{3} \leq t < \tfrac{2}{3} \\
&= V_N^+(3t-2)/q(3t-2) && \text{for} \quad \tfrac{2}{3} \leq t \leq 1.
\end{aligned}
$$

The same arguments as in Lemma 2.2 show that $X_N \to_L X_0$, as $N \to \infty$, relative to (D, d). Thus using item 3.1.1 of Skorohod we can construct processes $\tilde{X}_N$, $N \geq 0$, on a single probability space $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{p})$, which have the same finite dimensional distributions as their counterparts X_N, $N \geq 0$, defined on $(\Omega, \mathscr{A}, p)$ and which satisfy $d(\tilde{X}_N, \tilde{X}_0) \to_{a.s.} 0$, as $N \to \infty$. Defining now, as in Pyke and Shorack [5],

$$
\tilde{m} = N\tilde{X}_N(0), \quad \tilde{n} = N - \tilde{m} \quad \text{for} \quad N \geq 1 \quad \text{and}
$$

$$
\tilde{V}_N^-(t) = q(t)\tilde{X}_N((t+1)/3), \; \tilde{V}_N^+(t) = q(t)\tilde{X}_N((t+2)/3) \quad \text{for} \quad N \geq 0 \qquad (0 \leq t \leq 1),
$$

we have that (i) $(\tilde{\lambda}_N, \tilde{V}_N^-, \tilde{V}_N^+)$ have the same finite dimensional distributions as $(\lambda_N, V_N^-, V_N^+)$, (ii) that the processes $\tilde{V}_0^+$ and $\tilde{V}_0^-$ are a.s. continuous and (iii) with probability 1, the processes $\tilde{V}_N^-$ and $\tilde{V}_N^+$ have jumps of size $N^{-1/2}$ and are otherwise continuous for $N \geq 1$. If we set $\tilde{V}_N = \tilde{V}_N^+ - \tilde{V}_N^-$ and $\tilde{V}_N = \tilde{V}_N^+ + \tilde{V}_N^-$ $(N \geq 0)$, it follows that

$$
\tilde{\lambda}_N \to_{a.s.} 0 \quad \text{and} \quad (\tilde{V}_N, \tilde{V}_N^*) \to_{a.s.} (\tilde{V}_0, \tilde{V}_0^*), (\tilde{V}_N^+, \tilde{V}_N^-) \to_{a.s.} (\tilde{V}_0^+, \tilde{V}_0^-), \quad \text{as} \quad N \to \infty
\tag{2.24}
$$

(relative to the product (Skorohod) topology of the space $D \times D$).

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From now onward we shall work with the space $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{p})$ with the symbol $\sim$ dropped from all subsequent notation. The results asserted below, as pointed out by Pyke and Shorack [4], are generally valid only for the specially constructed processes, except for the implied weak convergence results which are valid for the original processes.

Let the metrices d_q and ρ_q be defined by $d_q(f, g)=d(f/q, g/q)$ and similarly for ρ_q, and Q denote the class of functions q' on [0, 1] defined by $Q=\{q'$: there exists positive numbers K, δ, ε $(0<\delta, \varepsilon<\frac{1}{2})$ such that $q'(t)\geq K[t(1-t)]^{(1/2)-\delta}$ on $[0, \varepsilon]$ and $[1-\varepsilon, 1]$ are bounded away from zero on $[\varepsilon, 1-\varepsilon]\}$.

Now since the processes V_0 and V_0^* are a.s. continuous, one can conclude from (2.24) as in Fears and Mehra [3] (see the proof of Theorem 3.1 of [3]) that V_- and V_0^+ satisfy the conclusions of Lemma 2.1 and as $N\to\infty$,

$$(2.25) \qquad \rho_q(V_N, V_0)\to_{\text{a.s.}} 0 \quad \text{and} \quad \rho_q(V_N^*, V_0^*)\to_{\text{a.s.}} 0 \quad \text{for} \quad q\in Q.$$

For studying the weak convergence of the empirical processes L_N and L_N^* in section 3, we need to prove Theorem 2.1 below, the counterpart of Theorem 2.2 of [4]. To accomplish this, let $K_N=H_0H_N^{-1}$, $K=H_0H^{-1}$ and I as the identity function on [0, 1], and note that under the conditions of section 1, $\rho(K_N, I)\to_{a.s.} 0$ (see Lemma 2.3 of [4] and the proof of Theorem 3.1 of [3]), so that

$$(2.26) \qquad \rho(V_N(K_N), V_0)\leq \rho(V_N, V_0)+(V_0(K_N), V_0)\to_{\text{a.s.}} 0,$$

using (2.25) and the a.s. continuity of V_0 on [0, 1]. In view of (2.26), Theorem 2.1 can be proved with exactly the same arguments as for Theorem 2.2 of [4], provided we first prove the following counterpart of Lemma 2.5 of [4] (c.f., Theorem 3.1 of [3]):

LEMMA 2.3. *Under the conditions of Lemma* 2.1, *for given* $\varepsilon, \tau>0$ $(\varepsilon, \tau<\frac{1}{2})$, *there exists a* $b>0$ *and an* N_0 *such that for* $N\geq N_0$

$$P\left[K_N(t)\leq bt^{1-\tau} \quad \text{for} \quad t\geq\frac{1}{N}\right]\geq 1-\varepsilon.$$

Proof. Since $\rho(K_N, I)\to_{a.s.} 0$, for given $\varepsilon>0$ there exists an $N_0'=N_0'(\varepsilon)$ such that $K_N(t)<t+\varepsilon$ a.s. for $N\geq N_0'$. Since it is possible to choose a $b=b(\varepsilon)$ and a $\theta=\theta(\varepsilon)$ such that $t+\varepsilon\leq bt^{1-\tau}$ for all $t>\theta$, the problem reduces to the consideration of the interval $[0, \theta]$ for sufficiently small θ by choosing an appropriately large b. We need to consider only the interval $[1/N, \theta]$. Now using Lemma 2.1, choose θ and N_0'' such that for $N\geq N_0''$

$$(2.27) \qquad P[E_N]\geq 1-\varepsilon \quad \text{where} \quad E_N=\{V_N\leq q(t) \quad \text{for} \quad 0\leq t\leq\theta\},$$

with $q(t)=[t(1-t)]^{1/2-\delta}$ and $\delta=\tau/2(1-\tau)$. Now on E_N

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$$K_N(t) = H_N H_N^{-1}(t) - N^{1/2} Y_N(K_N(t))$$
$$\leq \left(t+\frac{1}{N}\right) + N^{-1/2} q(K_N(t)) \tag{2.28}$$
$$\leq 2t + t^{-1/2} q(K_N(t)), \quad \text{for} \quad \frac{1}{N} \leq t \leq \theta,$$

which yields $K_N(t) \leq b t^{1-r}$ for $1/N \leq t \leq \theta$ as shown in the proof of (3.7) of [3]. The result, therefore, follows from (2.27) for $N_0 = \max(N_0', N_0'')$.

We thus have Theorem 2.1 below, for which we define

$$V_N'(K_N(t)) = V_N(K_N(t)) \quad \text{for} \quad \frac{1}{N} \leq t \leq 1-\frac{1}{N} \tag{2.29}$$
$$= 0 \quad \text{otherwise.}$$

THEOREM 2.1. *Under the conditions of Lemma* 2.1 *and for* $q \in Q$,

$$\rho_q(V_N'(K_N), V_0) \to_p 0, \quad \text{as} \quad N \to \infty. \tag{2.30}$$

The convergence (2.30) *also holds for* V_N^*, V_0^*, *or* V_N^+, V_0^+ *or* V_N^-, V_0^- *in place of* V_N, V_0.

3. **Weak Convergence of the Signed Empirical Process L_N.** The basic identity relating the signed empirical process L_N with the processes V_N and V_N^* which enables us to study the weak convergence of L_N (relative to various metrics) from that of V_N and V_N^*, is given by Lemma 3.1 below. Using Theorem 2.1 above, this identity and arguments similar to those used in Pyke and Shorack [4], one can deduce Theorem 3.1 below which gives sufficient conditions (on ν_N, F etc.) for the asymptotic normality of T_N^*.

On account of the absolute continuity assumption for the finite dimensional distributions of the process $\{X_N\}$, the distribution of order statistics $(|X|^{(1)}, |X|^{(2)}, \ldots, |X|^{(N)})$ is also absolutely continuous. It follows as in [4] that, for each $0 \leq k \leq N$, $P[HH_N^{-1}(t) \neq t$ at all t except the points $t = (i/N)$, $0 \leq i \leq N/m = k] = 1$. Thus, except at these finite number of points, $L_N(t)$ can be expressed a.s. as

$$L_N(t) = V_N(K_N(t)) + \frac{GH^{-1}(u_t) - GH^{-1}(t)}{u_t - t}(u_t - t)N^{1/2},$$

where $u_t = HH_N^{-1}(t)$. Further

$$u_t - t = (H_N H_N^{-1}(t) - t) - N^{-1/2} V_N^* K_N(t)),$$

so we obtain

$$L_N(t) = V_N(K_N(t)) - A_N(t) V_N^*(K_N(t)) + \delta_N(t), \tag{3.1}$$

where

$$\begin{cases} A_N(t) = \dfrac{GH^{-1}(u_t) - GH^{-1}(t)}{u_t - t} \quad \text{and} \\ \delta_N(t) = A_N(t) N^{1/2} [H_N H_N^{-1}(t) - t]. \end{cases} \tag{3.2}$$

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Since for $t \in [0, 1]$

$$|GH_N^{-1}(t)-GH^{-1}(t)| \leq \lambda_N |F^+H_N^{-1}(t)-F^+H^{-1}(t)| \\ +(1-\lambda_N)|F^-H_N^{-1}(t)-F^-H^{-1}(t)| = |HH_N^{-1}(t)-t|,$$

it follows from (3.2) that $|A_N| \leq 1$ and $|\delta_N| \leq N^{-1/2}$. Also for points t at which $HH_N^{-1}(t)=t$, $L_N(t)=V_N^*(K(t))$. Defining $L_N(t)$ by left continuity at undefined points in (3.1), we obtain

LEMMA 3.1. *With probability* 1,

$$L_N(t) = V_N(K_N(t))-A_N(t)V_N^*(K_N(t))+\delta_N(t)$$

for all $t \in (0, 1)$, *where* A_N *and* δ_N *are given by* (3.2).

Since $\lambda F^+H_\lambda^{-1}(t)+(1-\lambda)F^-H_\lambda^{-1}(t)=t$, both $F^+H_\lambda^{-1}$ and $F^-H_\lambda^{-1}$ are absolutely continuous; let $a_N^+(a_N^-)$ and $a_0^+(a_0^-)$ denote the derivatives of $F^+H^{-1}(F^-H^{-1})$ and $F^+H_0^{-1}(F^-H_0^{-1})$, respectively. Now set

$$L_0(t) = V_0(t)-a_0(t)V_0^*(t),\ a_0(t) = \lambda_0 a_0^+(t)-(1-\lambda_0)a_0^-(t) \tag{3.3}$$

and, as in Pyke and Shorack [4], $L_N'=L_N(\delta_N'=\delta_N)$ on $[1/N, 1]$ (on $[1/N, 1-(1/N)]$) and zero elsewhere. Then we have from (2.29)

$$\rho_q(L_N', L_0) \leq \rho_q(V_N'(K_N), V_0)+\rho(A_N, 0)\rho_q(V_N^{*\prime}(K_N), V_0^*) \\ +\rho(A_N, a_0)\rho_q(V_0^*, 0)+\sup_{1-(1/N)<t\leq 1}\left|\frac{L_N(t)}{q(t)}\right|+N^{-1/2},$$

so that in view of Theorem 2.1, $|A_N|\leq 1$ and the assertion about V_0^* just before (2.25), it follows that for $q \in Q$, $\rho_q(L_N', L_0)\to 0$, as $N\to\infty$, provided we show that $\rho(A_N, a_0)=o_p(1)$ and $\sup_{1-(1/N)\leq t<1}|L_N(t)/q(t)|=o(1)$, as $N\to\infty$. The second requirement follows since in the interval $[1/N, 1]$,

$$|L_N(t)| = N^{1/2}|\lambda_N(1-F-H^{-1}(t))-(1-\lambda_N)(1-F^-H^{-1}(t))| \leq N^{1/2}(1-t);$$

the first one follows, as in Pyke and Shorack [4], under the additional assumption 3.1 below: (see Lemmas 4.1 and 4.2 of [4]).

ASSUMPTION 3.1. The functions FH^{-1} have derivatives a_λ^* for all $t \in (0, 1)$ and for some λ', a_λ^*, is continuous on (0, 1) and has one-sided limits at 0 and 1.

Let $\bar{D}$ denote the set of left-continuous functions on [0, 1] that have only jump discontinuities. Then from $\rho_q(L_N', L_0)\to_p 0$, it follows that $L_N' \to_L L_0$, relative to $(\bar{D}, \rho_q)$, as $N\to\infty$. The same holds for d_q in place of ρ_q in above. We can now conclude

THEOREM 3.1. (i) *Suppose that the* ϕ*-mixing process* $\{X_n\}$ *satisfies the conditions of Lemma* 2.1, $0<F(0)<1$ *and Assumption* 3.1 *holds. If* (ii) *for a Lebesgue-Stieltjes measure* ν *on* (0, 1), $\int_0^1 q\,d|\nu|<\infty$ *for some* $q \in Q$ *and* (iii)

$$\int_{1/N}^1 L_N\,d(\nu_N-\nu)\to 0, \quad \text{as} \quad N\to\infty, \tag{3.4}$$

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then $T_N^* \to_p \int_0^1 L_0\, dv$, *which is a* $N(0, \sigma_0^2)$ r.v. *with* $\sigma_0^2 < \infty$ *given by*

$$\sigma_0^2 = 8\int_0^1 \int_0^t E[(1-b_0(t))V_0^+(t) - b_0(t)V_0^-(t)] \times [(1-b_0(s))V_0^+(s) - b_0(s)V_0^-(s)] \cdot dv(s)\, dv(t), \tag{3.5}$$

where $b_0(t) = d(FH_0^{-1}(t))/dt$ *and* V_0^+, V_0^- *are as in* (2.20).

Proof. Since $\rho_q(L_N', L_0) \to_p 0$, the result follows from the inequality

$$| T_N^* - \int_0^1 L_0\, dv | \leq | \int_0^1 L_N'\, d(\nu_N - \nu) | + \rho_q(L_N', L_0) \int_0^1 q d\, |\nu|,$$

(2.20) and (3.2), provided we show the finiteness of σ_0^2. For this it would suffice to show the finiteness of one of the four terms, say

$$\int_0^1 \int_0^t E[V_0^+(t)V_0^-(s)]\, dv(s)\, dv(t); \tag{3.6}$$

for the remaining the same arguments are applicable. Now setting $c(s, t)$ as the covariance function of the U_0-process, we obtain from (2.20) that (3.6) equals

$$\begin{aligned}\int_0^1 \int_0^t &[(1-F^-H_0^{-1}(s))c(F(0), FH_0^{-1}(t)) \\ &-(1-F^+H_0^{-1}(t))(1-F^-H_0^{-1}(s)) \cdot c(F(0), F(0)) - c(F(-H_0^{-1}(s)), FH_0^{-1}(t)) \\ &+(1-F^+H_0^{-1}(t))c(F(-H_0^{-1}(s)), F(0))] \cdot dv(s)\, dv(t) \\ = \int_0^1 \int_0^t &E[\xi(X_1)\eta(X_k) + \xi(X_k)\eta(X_1)]\, dv(s)\, dv(t),\end{aligned} \tag{3.7}$$

where

$$\xi(x) = g^*_{FH_0^{-1}(t)}(x) - (1-F^+H_0^{-1}(t))g^*_{F(0)}(x),$$

$$\eta(x) = (1-F^-H_0^{-1}(s))g^*_{F(0)}(x) - g^*_{F(-H_0^{-1}(s))}(x)$$

and

$$g_t^*(x) = I_{(-\infty, F^{-1}(t)]}(x) - t.$$

Using $F^+H_0^{-1}(t) \leq \lambda_0^{-1}(t)$, $1-F^+H_0^{-1}(t) \leq \lambda_0^{-1}(1-t)$ (similarly for $F^-H_0^{-1}(s)$) and $E\,|g_s(X_1)g_t(X_k)| < 2\phi_{k-1}^{1/2}[s(1-s)t(1-t)]^{1/2}$, we obtain that there exists a constant K_3 such that (3.7) does not exceed

$$K_3 \int_0^1 \int_0^t \{[s(1-s)t(1-t)]^\delta\} q(s)q(t)\, d\,|\nu|\,(s)\, d\,|\nu|\,(t),$$

which is finite on account of the assumption $\int_0^1 q(t)\, d\,|\nu|\,(t) < \infty$.

REMARK 3.1. It can be easily shown (See corollary 4.1 of [4] that Assumption 3.1 above is satisfied if either (i) $f = F'$ is symmetric about zero or (ii) f is continuous, H_0 is strictly increasing and the limits $\text{Lim}_{x\to\pm\infty} [f(x)/f(-x)]$ exist. In case of symmetry of f, $FH_0^{-1}(t) = (1+t)/2$ so that $c_0(t) = \frac{1}{2}$ and the variance (3.5) takes a much simpler form in this case.

4. A Chernoff-Savage Theorem. Let ν be induced by a non-constant function $-J$, of bounded variation on $(\varepsilon, 1-\varepsilon)$ for every $\varepsilon>0$, and let $J_N(t)=c^*_{Ni}$ on $(i-1/N, i/N]$ for $1\leq i\leq N$ and $J_N(0)=J_N(0+)$. Then we can write

$$N^{1/2}\left[T_N-\int_0^1 J(H)\,dG\right] = T^*_N+\gamma_N, \quad \text{where} \quad \gamma_N = N^{1/2}\int_0^1 [J_N(H)-J(H)]\,dG.$$

It can be shown under the conditions of Proposition 5.1 of [4], that $\gamma_N=o_p(1)$ and (3.4) holds, as $N\to\infty$. Consequently, we obtain under the additional hypothesis (i) of Theorem 3.1 that

$$N^{1/2}\left[T_N-\int_0^1 J(H)\,dG\right] \to_L N(0, \sigma_0^2), \tag{4.1}$$

as $N\to\infty$, with σ_0^2 given by (3.5). We can, however, further improve this result by replacing in (4.1) the random quantity $\int_0^1 J(H)\,dG$ by the fixed quantity $\int_0^1 J(H_0)\,dG_0$. The following theorem can be compiled by following the arguments of Theorem 1 of Pyke and Shorack [6].

THEOREM 4.1. *Suppose the hypothesis* (i) *and* (ii) *of Theorem* 3.1 *hold and*

$$N^{-1/2}\sum_{i=1}^{N} |c^*_{Ni}-J((i/N)\wedge(N-1/N))| < \delta_N$$

with $\delta_N=o(1)$ *as* $N\to\infty$. *Then the statistic*

$$\tilde{T}_N = N^{1/2}\left[T_N-\int_0^1 J(H_0)\,dG_0\right] \to_L \int_0^1 L_0\,dv,$$

a $N(0, \sigma_0^2)$ *r.v. with* σ_0^2 *given by* (3.5).

Proof. Similar to that of Theorem 1 of [6].

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