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Introduction

1.1 Function-Theoretic Operator Theory on Vectorial Hardy Spaces, Reproducing Kernel Hilbert Spaces, and Discrete-Time Linear Systems: Background

Arguably the synthesis of Hardy space function theory with operator theory begins with the famous paper of Beurling [50] making the connection between invariant subspaces for the shift operator on the Hardy space H^2 and inner functions, including the canonical factorization of any H^2-function as the product of an outer function, a Blaschke product, and a singular inner function. Around the same time appeared work of Livšic [122], obtaining triangular models for operators close to self-adjoint (see Kriete [117] and Vinnikov [174] for updates) and finding a characteristic function as a unitary invariant for a class of operators close to being unitary [122]. Ensuing work of Sz.-Nagy–Foias–Bercovici-Kércy [171] and of de Branges–Rovnyak [58, 59] further developed a model theory based on a characteristic function for operators close to being unitary. The work of Sz.-Nagy–Foias made an explicit connection with dilation theory, while that of de Branges–Rovnyak went beyond Hardy spaces by involving more general reproducing kernel Hilbert spaces only contractively included in a larger ambient reproducing kernel Hilbert space. However, as emphasized by Helton [99, 100], Fuhrmann [87], and others, at least implicit in a lot of this work in function-theoretic operator theory were connections with systems theory. In particular, the explicit formula for the Sz.-Nagy–Foias characteristic function is recognizable as having the form of a transfer function for a conservative discrete-time linear system; the fact that a rational inner function has such a realization can be traced to the engineering circuit-theory literature from the 1950s (see [99]), and the de Branges–Rovnyak model theory can be developed from a system-theory perspective (see [34]). For a thorough overview of all these connections between Hardy-space

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function theory, operator theory, and systems theory and connections with still other applications in engineering and harmonic analysis as of 2002, we refer to the two-volume treatise of Nikolski [132, 133].

In the ensuing decades, there has been much work extending these approaches to the context of multivariable function theory synthesized with multivariable operator theory. In particular, there have been contributions to multivariable operator theory with a distinctive reproducing kernel flavor, in both concrete and abstract commutative settings (Arveson [19] and Bhattacharyya, Eschmeier, Sarkar, and collaborators [51, 53–55, 80, 91, 163, 164], freely noncommutative settings (Ball–Bolotnikov–Fang [30, 33]), and sometimes with interplay between the two settings (Davidson–Pitts [69], Ball–Bolotnikov [22], Jury–Martin [108], Salomon–Shalit–Shamovich [160], Hartz [94]). For the free noncommutative setting, there is now a notion of reproducing kernel and associated reproducing kernel Hilbert space on a noncommutative Reinhardt-domain setting (Ball-Vinnikov and collaborators [43, 45]) as well as on more general free noncommutative domains (Ball–Marx–Vinnikov [42]), which fits into the framework of a general noncommutative function theory [9, 109]. There has also been work using system-theory ideas to push multivariable operator theory in new directions (Ball, Bolotnikov, Vinnikov, and collaborators [22–24, 44, 46] and Olofsson [134, 136]). Of course there is some overlap between the systems-theory approach and the reproducing kernel approach, Let us mention one instance of such an overlap: What we have called *observability operator* here and elsewhere in our system-theory approach is essentially the same as what is known as *Gelu Poisson kernel* in the terminology of Gelu Popescu (see e.g. [147, 151–153]).

Apart from the connection with dilation theory, characteristic functions, and operator model theory which we develop here, the multi-shift setting for the study of operator tuples has been a core area of study in operator theory, beginning with the work of Shields [166] and culminating in the recent beefy papers of Chavan, Trivedi, and collaborators [62, 92]. Our goal here is to lay out systematically the free noncommutative function theory for a class of weighted Bergman spaces on a full free Fock space and the associated Sz.-Nagy–Foias style model theory for the class of operators which can be modeled as the compression to a joint $*$-invariant subspace of the shift operator tuples on such a space.

Our primary tool is the system-theory approach outlined above, but there will also be a nontrivial use of reproducing kernel techniques, specifically of the notion of formal noncommutative reproducing kernel Hilbert space developed in [43, 45]. In fact, we shall see that most of the basic results can be derived via either approach, but there is at least one instance (see

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Theorem 9.2.20) where the systems-theory approach leads to some additional information not attainable via the purely reproducing kernel approach.

1.2 The Synthesis of the Systems-Theory and Reproducing Kernel Approaches

1.2.1 The Systems-Theory Approach

By way of motivation for the more general noncommutative, multivariable settings to come, we now illustrate in some detail the system-theory approach to function-theoretic operator theory for the classical setting.

For $\mathcal{X}$ and $\mathcal{Y}$, any pair of Hilbert spaces, we use the notation $\mathcal{L}(\mathcal{X},\mathcal{Y})$ to denote the space of bounded, linear operators from $\mathcal{X}$ to $\mathcal{Y}$, shortening the notation $\mathcal{L}(\mathcal{X},\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. We start with the classical discrete-time linear system

$$\Sigma(\mathbf{U}) : \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad\;\; y(k) = Cx(k) + Du(k) \end{cases} \tag{1.2.1}$$

with $x(k)$ taking values in the *state space* $\mathcal{X}$, $u(k)$ taking values in the *input space* $\mathcal{U}$, and $y(k)$ taking values in the *output space* $\mathcal{Y}$, where $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{X}$ are given Hilbert spaces and where the *system matrix* (sometimes also called *colligation matrix* or *connection matrix*) of the system

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

is a given bounded linear operator. If we let the system evolve on the nonnegative integers $n \in \mathbb{Z}_+$, then the whole trajectory $\{u(n), x(n), y(n)\}_{n\in\mathbb{Z}_+}$ is determined from the input signal $\{u(n)\}_{n\in\mathbb{Z}_+}$ and the initial state $x(0) = x$, according to the formulas

$$\begin{aligned} x(k) &= A^k x + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j), \\ y(k) &= CA^k x + \sum_{j=0}^{k-1} CA^{k-1-j} Bu(k) + Du(k). \end{aligned} \tag{1.2.2}$$

Application of the Z-transform

$$\{f(k)\}_{k\in\mathbb{Z}_+} \mapsto \widehat{f}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

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to the system equations (1.2.1) converts the expressions (1.2.2) to the so-called frequency-domain formulas

$$\begin{aligned}\widehat{x}(\lambda) &= (I-\lambda A)^{-1}x + \lambda(I-\lambda A)^{-1}B\widehat{u}(\lambda),\\ \widehat{y}(\lambda) &= C(I-\lambda A)^{-1}x + \big[D + \lambda C(I-\lambda A)^{-1}B\big]\widehat{u}(\lambda)\\ &= \mathcal{O}_{C,A}x + \Theta_{\mathbf{U}}(\lambda)\widehat{u}(\lambda),\end{aligned} \tag{1.2.3}$$

where

$$\mathcal{O}_{C,A}\colon\ x \mapsto \sum_{k=0}^{\infty}(CA^k x)\,\lambda^k = C(I-\lambda A)^{-1}x \tag{1.2.4}$$

is the observability operator and where

$$\Theta_{\mathbf{U}}(\lambda) = D + \lambda C(I-\lambda A)^{-1}B$$

is the *transfer function* of the system Σ given by (1.2.1). In particular, if the input signal $\{u(n)\}_{n\in\mathbb{Z}_+}$ is taken to be zero, the resulting output $\{y(n)\}_{n\in\mathbb{Z}_+}$ is given by $y = \mathcal{O}_{C,A}x(0)$. If $\mathcal{O}_{C,A}$ is injective, i.e., if (C,A) satisfies the *observability condition*

$$\bigcap_{k=0}^{\infty}\operatorname{Ker} CA^k = \{0\}, \tag{1.2.5}$$

we say that the output pair (C,A) is *observable*. In case $\mathcal{O}_{C,A}$ is bounded as an operator from $\mathcal{X}$ into the standard vector-valued Hardy space of the unit disk

$$H^2_{\mathcal{Y}} = \left\{ f(\lambda) = \sum_{k\ge 0} f_k\lambda^k\colon\ \sum_{k\ge 0}\|f_k\|^2_{\mathcal{Y}} < \infty \right\},$$

we say that the pair (C,A) is *output stable*. Let us mention that it is possible to give a complete characterization as to when a given output pair (C,A) is output stable in terms of the existence of a positive-semidefinite solution of a linear-matrix-inequality (here actually a Stein inequality) determined uniquely by the pair (C,A) (see Theorem 4.0.1 for the precise statement).

The case where the operator system matrix $\mathbf{U}$ is *isometric*, or more generally just *contractive*, is of special interest. In system-theoretic terms, the isometric property of $\mathbf{U}$ has the interpretation that the system $\Sigma(\mathbf{U})$ is *conservative* in the sense that the energy stored by the state at time k ($\|x(k+1)\|^2 - \|x(k)\|^2$) is exactly compensated by the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$). In case $\mathbf{U}$ is contractive, the system $\Sigma(\mathbf{U})$ is said to be *dissipative* in the sense that the net energy ($\|x(k+1)\|^2 - \|x(k)\|^2$) stored by the state at time k is no more than the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$) at time k. In case the system is dissipative (i.e., $\|\mathbf{U}\| \leq 1$), the transfer function $\Theta_{\mathbf{U}}$ is in the *Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ (the class of contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued analytic functions Θ on the open unit disk $\mathbb{D}$), and moreover the observability operator $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ is contractive. Conversely, if Θ is in the Schur class, then Θ has a realization as $\Theta = \Theta_{\mathbf{U}}$ as in (1.2.1) with $\Sigma(\mathbf{U})$ dissipative (in fact, even conservative).

Given any holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function Θ on the unit disk, we associate the multiplication operator $M_\Theta \colon f(z) \mapsto \Theta(z) f(z)$ (or $f \mapsto \Theta \cdot f$ for short). Then, the operator-theoretic significance of the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is that the multiplication operator M_Θ is a contraction from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$ exactly when Θ is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$.

If $\mathbf{U}$ is isometric and, in addition, the state-space operator A is *strongly stable* in the sense that $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$, then the observability operator is a partial isometry (even an isometry in case (C, A) is observable) and the transfer function $\Theta_{\mathbf{U}}$ is *inner* (the boundary values $\Theta_{\mathbf{U}}(\zeta)$ existing as strong radial limits from inside $\mathbb{D}$ for almost every ζ on the unit circle $\mathbb{T}$ are isometric operators from $\mathcal{U}$ to $\mathcal{Y}$), or equivalently, the multiplication operator $M_\Theta \colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ is isometric. Conversely: *any inner function Θ arises in this way as $\Theta = \Theta_{\mathbf{U}}$ with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ isometric with A strongly stable.*

We say that a subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is *shift-invariant* if $f \in \mathcal{M} \Rightarrow S_{\mathcal{Y}} f \in \mathcal{M}$, where $S_{\mathcal{Y}}$ is the shift operator given as the coordinate multiplication operator on $H^2_{\mathcal{Y}}$

$$S_{\mathcal{Y}} = M_\lambda \colon f(\lambda) \mapsto \lambda f(\lambda).$$

Note that if Θ is inner, then $\mathcal{M} := M_\Theta H^2_{\mathcal{U}} = \Theta \cdot H^2_{\mathcal{U}}$ is a shift-invariant subspace for $S_{\mathcal{Y}}$; the content of the Beurling–Lax theorem is that conversely, any such invariant subspace can be represented in this way. Similarly, we say that the subspace $\mathcal{N} \subset H^2_{\mathcal{Y}}$ is *backward-shift-invariant* if $f \in \mathcal{N} \Rightarrow S^*_{\mathcal{Y}} f \in \mathcal{N}$, where the backward-shift operator $S^*_{\mathcal{Y}}$, the Hilbert-space adjoint of the forward-shift operator $S_{\mathcal{Y}}$, works out to be

$$S^*_{\mathcal{Y}} \colon f(\lambda) \mapsto [f(\lambda) - f(0)]/\lambda.$$

The computation

$$S^*_{\mathcal{Y}} \colon C(I - \lambda A)^{-1} x \mapsto \lambda^{-1}[C(I - \lambda A)^{-1} - C]x = C(I - \lambda A)^{-1} A x$$

shows that, for any output-stable pair (C, A), the space $\operatorname{Ran} \mathcal{O}_{C,A}$ is $S^*_{\mathcal{Y}}$-invariant. Conversely, *if $\mathcal{M}^\perp \subset H^2_{\mathcal{Y}}$ is $S^*_{\mathcal{Y}}$-invariant, then there is an output pair* (C, A) (with $C^*C = I - A^*A$) *so that* $\mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{C,A}$. Furthermore, in case $\mathbf{U}$ is unitary with A strongly stable, then the set of possible Z-transformed

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output signals $\widehat{y}(\lambda)$ appearing in the form in the transformed system equations as in (1.2.3) is all of $H^2_{\mathcal{Y}}$ and the additive decomposition of $\widehat{y}(\lambda)$ appearing in (1.2.3) is orthogonal:

$$H^2_{\mathcal{Y}} = \operatorname{Ran} \mathcal{O}_{C,A} \oplus \operatorname{Ran} M_{\Theta_{\mathbf{U}}}.$$

1.2.2 Realization Formulas for Reproducing Kernels

More generally, following the reproducing kernel approach of de Branges–Rovnyak [58, 59], as enhanced in the work of the authors and collaborators [30–34], it is of interest to consider also the case where the $S_{\mathcal{Y}}$-invariant subspace $\mathcal{M}$ carries its own norm distinct from the norm inherited from the ambient space $H^2_{\mathcal{Y}}$ but with the prescription that the inclusion map $\iota\colon \mathcal{M} \to H^2_{\mathcal{Y}}$ be contractive. Then a generalization of the Beurling–Lax theorem due to de Branges–Rovnyak says that one can always find a contractive multiplier (i.e., a Schur-class function, not necessarily inner) Θ so that $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$, with *lifted norm* given by

$$\|\Theta f\| = \inf\{\|g\| \colon g \in H^2_{\mathcal{U}} \text{ such that } \Theta \cdot g = \Theta \cdot f\}.$$

In this context, there is a generalization of orthogonal complement denoted by $\mathcal{M}^{[\perp]}$, which we call the *Brangesian complement* (see Section 3.1.1 for details), which is also contractively included in $H^2_{\mathcal{Y}}$ and provides a linear decomposition

$$H^2_{\mathcal{Y}} = \mathcal{M} + \mathcal{M}^{[\perp]}$$

that is neither orthogonal nor even a direct-sum decomposition but does have a canonical minimality property, making the space $\mathcal{M}^{[\perp]}$ uniquely determined by $\mathcal{M}$ (see [28, 162] or Section 3.1.1). Here $\mathcal{M}^{[\perp]}$ also carries its own norm with the inclusion map into $H^2_{\mathcal{Y}}$ contractive. Furthermore, for the case where $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$ for a Schur-class function Θ, $\mathcal{M}^{[\perp]}$ is a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{M}^{[\perp]}}$ given by

$$K_{\mathcal{M}^{[\perp]}}(\lambda, \mu) = \frac{I_{\mathcal{Y}} - \Theta(\lambda)\Theta(\mu)^*}{1 - \lambda\overline{\mu}}$$

and as a lifted-norm space is induced by the operator $(I - M_\Theta M_\Theta^*)^{\frac{1}{2}}$,

$$\mathcal{M}^{[\perp]} = \operatorname{Ran}(I - M_\Theta M_\Theta^*)^{\frac{1}{2}}$$

with

$$\|(I - M_\Theta M_\Theta^*)^{\frac{1}{2}} f\| = \min\left\{\|g\|_{H^2_{\mathcal{Y}}} \colon (I - M_\Theta M_\Theta^*)^{\frac{1}{2}} g = (I - M_\Theta M_\Theta^*)^{\frac{1}{2}} f\right\}.$$

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Alternatively, one can start with an $S_{\mathcal{Y}}^*$-invariant subspace $\mathcal{N}$ contractively included in $H^2_{\mathcal{Y}}$ and find a contractive output pair (C, A) (so $A^*A + C^*C \preceq I_{\mathcal{X}}$ where $A \in \mathcal{L}(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$) so that $\mathcal{N}$ is the range of the observability operator $\mathcal{O}_{C,A}$. Then, $\mathcal{N}$ is itself a reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathcal{N}}(\lambda, \mu) = C(I_{\mathcal{X}} - \lambda A)^{-1}(I_{\mathcal{X}} - \overline{\mu} A^*)^{-1} C^*.$$

If one then solves the factorization problem for injective $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$,

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} A^* & C^* \end{bmatrix} \tag{1.2.6}$$

and then lets $\mathbf{U}$ be the system matrix $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then $\mathbf{U}$ is unitary with associated transfer function $\Theta_{\mathbf{U}}(\lambda) = D + \lambda C(I - \lambda A)^{-1} B$ giving rise to the contractive multiplier $\Theta_{\mathbf{U}}$ generating the Brangesian complement of $\mathcal{N}$,

$$\mathcal{N}^{[\perp]} = \Theta_{\mathbf{U}} \cdot H^2_{\mathcal{U}},$$

or equivalently, solving the kernel factorization problem

$$\frac{I_{\mathcal{Y}}}{1 - \lambda \overline{\mu}} - C(I - \lambda A)^{-1}(I - \overline{\mu} A^*)^{-1} C^* = \frac{\Theta(\lambda)\Theta(\mu)^*}{1 - \lambda \overline{\mu}}. \tag{1.2.7}$$

Then, the space $H^2_{\mathcal{Y}}$ has an additive decomposition

$$H^2_{\mathcal{Y}} = \mathcal{N} + \mathcal{N}^{[\perp]} = \operatorname{Ran} \mathcal{O}_{C,A} + M_{\Theta_{\mathbf{U}}} H^2_{\mathcal{U}} \tag{1.2.8}$$

corresponding again to the additive decomposition of $\widehat{y} \in H^2_{\mathcal{Y}}$ in (1.2.3), but this time not orthogonal nor a direct sum but rather a Brangesian minimal decomposition. In case one of $\operatorname{Ran} \mathcal{O}_{C,A}$ or $\operatorname{Ran} M_{\Theta_{\mathbf{U}}}$ is contained in $H^2_{\mathcal{Y}}$ isometrically, then they both are isometrically included and the decomposition (1.2.8) is orthogonal, and we recover most of the results discussed above derived via the systems theory approach.

1.2.3 Connections with Operator Model Theory

If we start with a contraction operator T on a Hilbert space $\mathcal{X}$, we can always form the isometric output pair $(C, A) := (D_{T^*}, T^*)$, where $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ is the *defect operator* of T^*, here viewed as an operator from $\mathcal{X}$ to the *defect space* $\mathcal{Y} := \mathcal{D}_{T^*} = \overline{\operatorname{Ran}}\,(I - TT^*)^{\frac{1}{2}}$. Then, we may form the observability operator $\mathcal{O}_{D_{T^*}, T^*} \colon X \to H^2_{\mathcal{D}_{T^*}}$. If we assume that T is *completely noncoisometric* (c.n.c. for short), then $\mathcal{O}_{D_{T^*}, T^*}$ is one-to-one. Since (D_T^*, T^*) is an isometric output pair, one can show that the observability

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operator is isometric exactly when T is *pure* (i.e., T^* is strongly stable). Then, the solution of the factorization problem (1.2.6) with $(C, A) = (D_{T^*}, T^*)$ leads to

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} D_T \\ -T \end{bmatrix} : \mathcal{D}_T \to \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_{T^*} \end{bmatrix}$$

giving rise to the unitary system matrix

$$\mathbf{U}_T = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_T \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_{T^*} \end{bmatrix}$$

with associated transfer function

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T]|_{\mathcal{D}_T} : \ \mathcal{D}_T \to \mathcal{D}_{T^*} \tag{1.2.9}$$

equal to the Sz.-Nagy–Foias as well as the de Branges–Rovnyak characteristic function for the c.n.c. contraction operator T. Furthermore, the observability operator $\mathcal{O}_{D_{T^*}, T^*}$ is isometric exactly when T^* is strongly stable, or equivalently, when Θ_T is inner. In this case, T^* is unitarily equivalent to the restriction of the backward shift $S^*_{\mathcal{D}_{T^*}}$ to its invariant subspace $\mathcal{N} = \operatorname{Ran} \mathcal{O}_{D_{T^*}, T^*} \subset H^2_{\mathcal{D}_{T^*}}$. In case T^* is not strongly stable, it is still the case that T^* is unitarily equivalent to $S_{\mathcal{D}_{T^*}}$ restricted to an invariant subspace $\mathcal{N} = \operatorname{Ran} \mathcal{O}_{D_{T^*}, T^*} \subset H^2_{\mathcal{D}_{T^*}}$, but in this case we have only a contractive containment of $\mathcal{N}$ in the ambient space $H^2_{\mathcal{D}_{T^*}}$. In this case, we can still see that T dilates to an isometry $S_{\mathcal{D}_{T^*}} \oplus V$ on a space $H^2_{\mathcal{D}_{T^*}} \oplus \mathcal{W}$, where V is a unitary operator on the Hilbert space $\mathcal{W}$, i.e., there is a subspace $\widetilde{\mathcal{N}}$ of $H^2_{\mathcal{D}_{T^*}} \oplus \mathcal{W}$ so that T^* is unitarily equivalent to $(S_{\mathcal{D}_{T^*}} \oplus V)^*|_{\mathcal{N}}$. The model theory of Sz.-Nagy–Foias–Bercovici–Kércy [171] gives a functional model for T that is embedded isometrically in a functional model for $(S_{\mathcal{D}_{T^*}} \oplus V)^*|_{\mathcal{N}}$ via a somewhat different approach, whereby one first proves the Sz.-Nagy dilation theorem and finds a model for T inside the geometry of a functional model for the unitary dilation of T. Section 1.5 of the paper of Douglas [73] obtains a model for the isometric (and then by further extension unitary) dilation $S_{\mathcal{D}_{T^*}} \oplus V$ of T by finding a complementary embedding operator Q so that the operator $\begin{bmatrix} \mathcal{O}_{D_{T^*}, T^*} \\ Q \end{bmatrix}$ defines an embedding of $\mathcal{X}$ into a direct-sum space $\begin{bmatrix} H^2_{\mathcal{D}_{T^*}} \\ \overline{\operatorname{Ran}} \, Q \end{bmatrix}$.

We should point out that the Sz.-Nagy–Foias model theory actually applies to the more general situation of a *completely non-unitary* (c.n.u. for short) contraction operator, but explaining this additional feature does not fit into

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our narrative here; for a sample of the difficulties in handling the c.n.u. class in more general multivariable settings, we refer to work of the authors and Vinnikov [22, 46].

1.2.4 Summary

In summary, we have the following themes connecting vectorial Hardy-space function theory, conservative/dissipative discrete-time linear systems, and model theory for Hilbert-space contraction operators:

1. **Backward-shift-invariant subspaces and ranges of observability operators:** A backward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ arises as the range of some isometric observability operator. More generally, a contractively included backward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ arises as the lifted-norm space associated with a contractive observability operator. Moreover, it is possible to characterize in terms of existence of a solution to a certain linear-matrix-inequality when a given output pair (C, A) gives rise to an observability operator $\mathcal{O}_{C,A}$ mapping $\mathcal{X}$ boundedly into $H^2_{\mathcal{Y}}$. The special case $C = I_{\mathcal{X}}$ corresponds to exponential stability for A.
2. **Forward-shift-invariant subspaces and contractive multipliers:** A forward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ has Beurling–Lax representation $\mathcal{M} = M_\Theta \cdot H^2_{\mathcal{U}}$ for some inner multiplier Θ from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$. More generally, a contractively included forward-shift-invariant subspaces of $H^2_{\mathcal{Y}}$ has a lifted-norm Beurling–Lax representation $\Theta \cdot H^2_{\mathcal{U}}$ for a contractive multiplier Θ from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$.
3. **Hardy-space decompositions in backward- and forward-shift-invariant subspaces:** In the case of a conservative linear system with strongly stable state operator A (i.e., $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is isometric and also $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$), the observability operator $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ and the transfer-function multiplier operator $M_{\Theta_{\mathbf{U}}}$ are isometric, and one has the orthogonal decomposition of the form
$$H^2_{\mathcal{Y}} = \operatorname{Ran} \mathcal{O}_{C,A} \oplus \operatorname{Ran} M_{\Theta_{\mathbf{U}}} \oplus \mathcal{W} \tag{1.2.10}$$
for a shift-invariant subspace $\mathcal{W}$; if it is the case that $\mathbf{U}$ is also coisometric, then $\mathcal{W} = \{0\}$. Conversely, if $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is $S_{\mathcal{Y}}$-invariant (and hence $\mathcal{M}^\perp \subset H^2_{\mathcal{Y}}$ is $S^*_{\mathcal{Y}}$-invariant), then there is a unitary $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with A strongly stable such that $\mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{C,A}$ and $\mathcal{M} = \Theta_{\mathbf{U}} \cdot H^2_{\mathcal{U}}$.
   More generally, if $\mathbf{U}$ is merely contractive (rather than isometric or unitary) and/or A is not strongly stable, then a linear decomposition of the form (1.2.10) holds but as a Brangesian minimal decomposition rather

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than as a Hilbert-space orthogonal decomposition. If **U** is coisometric, it is again the case that the space $\mathcal{W}$ is zero.

4. **Model theory for Hilbert-space contraction operators:** An inner (and more generally, contractive) multiplier $M_\Theta \colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ arises as the Sz.-Nagy–Foias/de Branges–Rovnyak characteristic function for some c.n.c. Hilbert-space contraction operator T that in turn induces a canonical functional model for the operator T which also exhibits a unitary dilation $S \oplus V$ for T.

Much work has been done to extend this set of ideas, particularly themes #2 and #4 (the operator-model theory aspects without the system-theoretic connections) to more general settings, e.g.,

(i) to Bergman spaces and hypercontraction operators; see Agler [4], Müller [130], Müller–Vasilescu [131], Hedenmalm–Korenblum–Zhu [97], Duren–Schuster [76]),
(ii) to the Drury–Arveson space and commutative row-contractive operator tuples; see Bhattacharyya–Eschmeier–Sarkar [53, 54], Bhattacharyya–Sarkar [55], and Ball–Bolotnikov [22],
(iii) to more general domains in $\mathbb{C}^d$ than the ball and associated more general commutative operator tuples; see Athavale [20], Curto–Vasilescu [67, 68], Timotin [173], Pott [155], Ambrozie–Engliš–Müller [16], Arazy–Engliš [18].
(iv) to the full Fock space and freely noncommutative row-contractive operator tuples, possibly also constrained to lie in a prescribed noncommutative operator variety; see Bunce [60], Frazho [85], Popescu [141–144, 147, 148],
(v) to a more general formalism of representations of certain operator algebras based on tensor-algebra constructions; see Muhly–Solel [126–129], and
(vi) to noncommutative hypercontractive operator tuples modeled on noncommutative varieties (see Popescu [151–153]) as well as a weighted version of the tensor-algebra context (see Muhly–Solel [129]).

Identification of a characteristic function defined by a formula of the Sz.-Nagy–Foias type (1.2.9) (the main thrust of theme #4 above) can be found (i) for the Bergman space setting first in the work of Olofsson [134–136] and then followed up by the authors [23, 24] and Eschmeier [79], (ii) for the Drury–Arveson space setting earlier in the work of Bhattacharyya et al. [53–55], (iii) for the full Fock space in the work of

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Popescu [143], Ball–Bolotnikov–Fang [30, 33] and Ball–Vinnikov [46], for the tensor-algebra context in Muhly–Solel [127].

Here we focus on a general setting of model spaces, forward and backward shift-operator tuples, and their joint invariant subspaces that simultaneously contain as special cases the Bergman setting (i) and the full Fock space setting (iii). Before plunging into the most general setting, we next sketch how the system theory setup (1.2.1)–(1.2.5) adapts to these motivating special cases.

1.3 Standard Weighted Bergman Spaces

For a Hilbert space $\mathcal{Y}$ and an integer $n \geq 1$, we denote by $\mathcal{A}_{n,\mathcal{Y}}$ the Hilbert space of $\mathcal{Y}$-valued functions f analytic in the open unit disk $\mathbb{D}$ and with finite norm $\|f\|_{A_{n,\mathcal{Y}}}$:

$$\mathcal{A}_{n,\mathcal{Y}} = \left\{ f(\lambda) = \sum_{j\geq 0} f_j \lambda^j : \; \|f\|^2_{\mathcal{A}_{n,\mathcal{Y}}} := \sum_{j\geq 0} \mu_{n,j} \cdot \|f_j\|^2_{\mathcal{Y}} < \infty \right\},$$

where the weights $\mu_{n,j}$'s are defined by

$$\mu_{n,j} := \frac{1}{\binom{j+n-1}{j}} = \frac{j!\,(n-1)!}{(j+n-1)!}. \tag{1.3.1}$$

The space $\mathcal{A}_{n,\mathcal{Y}}$ can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel $k_n(\lambda,\zeta) I_{\mathcal{Y}}$, where

$$k_n(\lambda,\zeta) = (1-\lambda\overline{\zeta})^{-n}. \tag{1.3.2}$$

We introduce the function R_n and its shifted counterparts $R_{n,k}$ by the formulas

$$R_n(\lambda) := (1-\lambda)^{-n} = \sum_{j=0}^{\infty} \mu_{n,j}^{-1} \lambda^j \quad \text{and} \quad R_{n,k}(\lambda) = \sum_{j=0}^{\infty} \mu_{n,j+k}^{-1} \lambda^j, \tag{1.3.3}$$

so that $R_{n,0} = R_n$ and $k_n(\lambda,\zeta) = R_n(\lambda\overline{\zeta})$. Directly from the definitions we see that

$$R_{n,k}(\lambda) = \binom{n+k-1}{k} + \lambda R_{n,k+1}(\lambda), \tag{1.3.4}$$

and, as is proved in [23, Section 2] by making use of the Chu–Vandermonde identity for binomial coefficients, we also have

$$R_{n,k}(\lambda) = \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} R_{n-\ell+1}(\lambda) \quad \text{for} \quad k \geq 1. \tag{1.3.5}$$

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We also record that for any operator $A \in \mathcal{L}(\mathcal{X})$ with spectral radius ρ_A, the operator-valued functions

$$R_{n,k}(\lambda A) = \sum_{j=0}^{\infty} \mu_{n,k+j}^{-1} A^j \lambda^j \tag{1.3.6}$$

are analytic on the disk $\{\lambda : |\lambda| < 1/\rho_A\}$ for any $k \in \mathbb{Z}_+$.

In [23], we considered the following discrete-time time-varying linear system:

$$\Sigma_n \left(\left\{ \begin{bmatrix} A & B_j \\ C & D_j \end{bmatrix} \right\}_{j \in \mathbb{Z}_+} \right) : \begin{cases} x(j+1) = \frac{j+n}{j+1} \cdot Ax(j) + \binom{j+n}{j+1} \cdot B_j u(j) \\ y(j) = Cx(j) + \binom{j+n-1}{j} \cdot D_j u(j) \end{cases} \tag{1.3.7}$$

where $A \in \mathcal{L}(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{X})$, and $D_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{Y})$ are given bounded linear operators acting between given Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{U}_k$ ($k \geq 0$). We note that the case where $n = 1$ and the operators $B_k = B$ and $D_k = D$ are taken independent of the time parameter $k \in \mathbb{Z}_+$ reduces to the classical time-invariant case (1.2.1). If we let the system (1.3.7) evolve on $\mathbb{Z}_+$, then the whole trajectory $\{u(j), x(j), y(j)\}_{j \in \mathbb{Z}_+}$ is determined from the input signal $\{u(j)\}_{j \in \mathbb{Z}_+}$ and the initial state $x(0)$ according to the formulas

$$x(j) = \mu_{n,j}^{-1} \cdot \left(A^j x(0) + \sum_{\ell=0}^{j-1} A^{j-\ell-1} B_\ell u(\ell) \right), \tag{1.3.8}$$

$$y(j) = \mu_{n,j}^{-1} \cdot \left(CA^j x(0) + \sum_{\ell=0}^{j-1} CA^{j-\ell-1} B_\ell u(\ell) + D_j u(j) \right). \tag{1.3.9}$$

Formula (1.3.8) is established by simple induction arguments, while (1.3.9) is obtained by substituting (1.3.8) into the second equation in (1.3.7).

To write the Z-transformed version of the system-trajectory formula (1.3.8), we multiply both sides of (1.3.8) by λ^j and sum over $j \geq 0$ to get, on account of (1.3.6),

$$\begin{aligned} \widehat{x}(\lambda) &= \sum_{j=0}^{\infty} x(j) \lambda^j \\ &= \left(\sum_{j=0}^{\infty} \mu_{n,j}^{-1} A^j \lambda^j \right) x(0) + \sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} \mu_{n,j}^{-1} A^{j-k} \lambda^j \right) B_{k-1} u(k-1) \end{aligned}$$

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$$= (I - \lambda A)^{-n} x(0) + \sum_{k=1}^{\infty} \lambda^k \left(\sum_{j=0}^{\infty} \mu_{n,j+k}^{-1} A^j \lambda^j \right) B_{k-1} u(k-1)$$

$$= (I - \lambda A)^{-n} x(0) + \sum_{k=0}^{\infty} \lambda^{k+1} R_{n,k+1}(\lambda A) B_k u(k).$$

The same procedure applied to (1.3.9) gives

$$\widehat{y}(\lambda) = C(I - \lambda A)^{-n} x(0) + \sum_{k=0}^{\infty} \lambda^k \left(\mu_{n,k}^{-1} D_k + \lambda C R_{n,k+1}(\lambda A) B_k \right) u(k)$$

$$= \mathcal{O}_{n,C,A} x(0) + \sum_{k=0}^{\infty} \lambda^k \Theta_{n,k}(\lambda) u(k), \tag{1.3.10}$$

where

$$\mathcal{O}_{n,C,A} \colon \; x \mapsto \sum_{j=0}^{\infty} \left(\mu_{n,j}^{-1} C A^j x \right) \lambda^j = C(I - \lambda A)^{-n} x \tag{1.3.11}$$

is the n-observability operator and where

$$\Theta_{n,k}(\lambda) = \mu_{n,k}^{-1} D_k + \lambda C R_{n,k+1}(\lambda A) B_k \qquad (k = 0, 1, \ldots)$$

is the family of transfer functions.

Note that observability of the output pair (C, A) in the classical sense (1.2.5) is also equivalent to the injectivity of the n-observability operator $\mathcal{O}_{n,C,A}$ (1.3.11). Following [23], we say that the output pair (C, A) is *n-output stable* if $\mathcal{O}_{n,C,A}$ is bounded as an operator from $\mathcal{X}$ into $\mathcal{A}_{n,\mathcal{Y}}$.

We note next that the transfer function $\Theta_{n,k}(z)$ encodes the result of a pulse input-vector u being applied at time $j = k$:

$$\widehat{y}(\lambda) = \Theta_{n,k}(\lambda) \cdot \lambda^k u \quad \text{if} \quad x(0) = 0 \quad \text{and} \quad u(j) = \delta_{j,k} u$$

(where δ_{jk} stands for the Kronecker symbol). In fact the functions $\Theta_{n,k}(\lambda)$ could have been derived in this way and then one could arrive at input–output relation (1.3.10) via superposition of all these time-k impulse responses. There is a notion of *conservative* for a system of the form (1.3.7) involving the system matrix $\left[\begin{smallmatrix} A & B_j \\ C & D_j \end{smallmatrix} \right]$ being unitary with respect to an appropriate choice of weights (see formulas (6.7) and (6.21) in [23]). When these metric constraints are satisfied, the associated transfer-function family $\{\Theta_{n,k}\}$ serves as a representer of a shift-invariant subspace in the weighted Bergman space while the image space of an observability operator $\mathcal{O}_{n,C,A}$ is the model for a backward-shift-invariant subspace in $\mathcal{A}_{n,\mathcal{Y}}$ (see [23, 24]).

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1.4 The Hardy–Fock Space Setting

The classical results on the system (1.2.1) admit nice extensions to a number of multivariable settings, both commutative and noncommutative. In this section, we recall the case where the Hardy space $H^2_{\mathcal{Y}}$ is replaced by what we shall call the Hardy–Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$.

To define the Hardy–Fock space, we let $\mathbb{F}^+_d$ denote the unital free semigroup (i.e., monoid) generated by the set of d letters $\{1, \dots, d\}$. Elements of $\mathbb{F}^+_d$ are words of the form $i_N \dots i_1$ where $i_\ell \in \{1, \dots, d\}$ for each $\ell \in \{1, \dots, N\}$ with multiplication given by concatenation. The unit element of $\mathbb{F}^+_d$ is the empty word denoted by $\emptyset$. For $\alpha = i_N \dots i_1 \in \mathcal{F}_d$, we let $|\alpha|$ denote the number N of letters in α and we let $\alpha^\top := i_1 \dots i_N$ denote the *transpose* of α. We let $z = (z_1, \dots, z_d)$ to be a collection of d formal noncommuting variables and let $\mathcal{Y}\langle\langle z\rangle\rangle$ denote the set of noncommutative formal power series $\sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha$ where $f_\alpha \in \mathcal{Y}$ and where

$$z^\alpha = z_{i_N} z_{i_{N-1}} \dots z_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \dots i_1. \tag{1.4.1}$$

The Hardy–Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ is then defined as

$$H^2_{\mathcal{Y}}(\mathbb{F}^+_d) = \left\{ \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z\rangle\rangle \colon \sum_{\alpha \in \mathbb{F}^+_d} \|f_\alpha\|^2_{\mathcal{Y}} < \infty \right\}. \tag{1.4.2}$$

Note that when $d = 1$, we are back to the vectorial Hardy spaces discussed in Section 1.1. The Hardy–Fock-space counterpart of (1.2.1) is the system

$$\Sigma_{\mathbf{U}}\colon \left\{ \begin{array}{rcl} x(1\alpha) & = & A_1 x(\alpha) + B_1 u(\alpha) \\ \vdots & \vdots & \vdots \\ x(d\alpha) & = & A_d x(\alpha) + B_d u(\alpha) \\ y(\alpha) & = & Cx(\alpha) + Du(\alpha) \end{array} \right. \tag{1.4.3}$$

which evolves along the free semigroup $\mathbb{F}^+_d$, and, for each $\alpha \in \mathbb{F}^+_d$, the state vector $x(\alpha)$, input signal $u(\alpha)$, and output signal $y(\alpha)$ take values in the *state space* $\mathcal{X}$, *input space* $\mathcal{U}$, and *output space* $\mathcal{Y}$. The *system matrix* $\mathbf{U}$ has the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \vdots \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}. \tag{1.4.4}$$

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Such systems were introduced in [46] and with further elaboration in [35] and [36]; following [35] we call this type of system a *noncommutative Fornasini–Marchesini linear system.*

We extend the noncommutative functional calculus (1.4.1) from noncommuting indeterminates $z = (z_1, \ldots, z_d)$ to a d-tuple of operators $\mathbf{A} = (A_1, \ldots, A_d)$ by letting

$$\mathbf{A}^\alpha := A_{i_N} A_{i_{N-1}} \ldots A_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \ldots i_1 \in \mathbb{F}_d^+, \tag{1.4.5}$$

where the multiplication is now operator composition. Letting

$$Z(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} \otimes I_{\mathcal{X}}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}, \tag{1.4.6}$$

we next observe that

$$(Z(z)A)^j = \left(\sum_{i=1}^d z_i A_i \right)^j = \sum_{\alpha \in \mathbb{F}_d^+ \colon |\alpha| = j} \mathbf{A}^\alpha z^\alpha \quad \text{for all} \quad j \geq 0 \tag{1.4.7}$$

and therefore,

$$(I - Z(z)A)^{-1} = \sum_{j=0}^\infty (Z(z)A)^j = \sum_{j=0}^\infty \sum_{\alpha \in \mathbb{F}_d^+ \colon |\alpha| = j} \mathbf{A}^\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} \mathbf{A}^\alpha z^\alpha.$$

Application of the formal noncommutative Z-transform

$$\{x(\alpha)\}_{\alpha \in \mathbb{F}_d^+} \mapsto \widehat{x}(z) = \sum_{\alpha \in \mathbb{F}_d^+} x(\alpha) z^\alpha \tag{1.4.8}$$

to the system (1.4.3) then gives

$$\begin{aligned} \widehat{x}(z) &= (I - Z(z)A)^{-1} x(\emptyset) + (I - Z(z)A)^{-1} Z(z) B \widehat{u}(z), \\ \widehat{y}(z) &= \mathcal{O}_{C,\mathbf{A}} x(\emptyset) + \Theta_{\mathbf{U}}(z) \widehat{u}(z), \end{aligned}$$

where

$$\mathcal{O}_{C,\mathbf{A}} \colon \; x \mapsto C(I - Z(z)A)^{-1} x = \sum_{\alpha \in \mathbb{F}_d^+} (C \mathbf{A}^\alpha x) z^\alpha \tag{1.4.9}$$

is the observability operator of the pair $(C, \mathbf{A})$ and where $\Theta_{\mathbf{U}}(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ is given by

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$$\begin{aligned}\Theta_{\mathbf{U}}(z) &= D + C(I - Z(z)A)^{-1}Z(z)B \\ &= D + \sum_{\alpha\in\mathbb{F}_d^+}\sum_{j=1}^{d} C\mathbf{A}^{\alpha}B_j z^{\alpha} z_j.\end{aligned} \tag{1.4.10}$$

Thus, the initial state $x = x_\emptyset$ is uniquely determined by the output signal $\widehat{y}(z)$ when the input signal $\widehat{u}(z)$ is taken to be zero exactly when $\mathcal{O}_{C,\mathbf{A}}$ is injective; when this is the case, we say that the output pair $(C, \mathbf{A})$ is *observable*. The pair $(C, \mathbf{A})$ is called *output stable* if $\mathcal{O}_{C,\mathbf{A}}$ is bounded as an operator from $\mathcal{X}$ into $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$, and *exactly observable* if $\mathcal{O}_{C,\mathbf{A}}$ is bounded and bounded below. As in the single-variable case, the system matrix (1.4.4) being unitary corresponds to a notion of energy conservation; for details on this we refer to [46]. The d-tuple $\mathbf{A} = (A_1, \ldots, A_d)$ is called *strongly stable* if

$$\lim_{N\to\infty} \sum_{\alpha\in\mathbb{F}_d^+ :\, |\alpha|=N} \|\mathbf{A}^{\alpha}x\|^2 \to 0 \quad \text{for all } x \in \mathcal{X}. \tag{1.4.11}$$

Again, as in the single-variable case, the condition (1.4.11) in conjunction with $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ being isometric guarantees that $\mathcal{O}_{C,\mathbf{A}}$ is a partial isometry and that the operator $M_{\Theta_{\mathbf{U}}} : f(z) \mapsto \Theta_{\mathbf{U}}(z)f(z)$ of multiplication by the transfer function (1.4.10) acts isometrically as an operator from $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ into $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ (see [30, 33, 46] for details). The representation $\mathcal{M} = \Theta_{\mathbf{U}} \cdot H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ amounts to the Beurling–Lax representation for right-shift-invariant subspaces (i.e., $f(z) \in \mathcal{M} \Rightarrow f(z) \cdot z_j \in \mathcal{M}$ for $j = 1, \ldots, d$) while backward-shift-invariant subspaces arise as the range of an observability operator $\mathcal{O}_{C,\mathbf{A}}$ (see [30, 33, 46, 142]).

1.5 Weighted Bergman–Fock Spaces

We introduce a family of weighted Bergman–Fock spaces as a multivariable noncommutative counterpart of standard weighted Bergman spaces; the system-theoretic point of view presented here combines the single-variable setting handled in [23, 24] with the unweighted multivariable setting from Section 1.4. Given an integer $n \geq 1$, the free semigroup $\mathbb{F}_d^+$, and the coefficient Hilbert space $\mathcal{Y}$, we let

$$\mathcal{A}_{n,\mathcal{Y}}(\mathbb{F}_d^+) = \left\{ \sum_{\alpha\in\mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z\rangle\rangle : \sum_{\alpha\in\mathbb{F}_d^+} \mu_{n,|\alpha|} \cdot \|f_\alpha\|^2_{\mathcal{Y}} < \infty \right\} \tag{1.5.1}$$

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where, according to (1.3.1), $\mu_{n,|\alpha|} = \frac{|\alpha|!\,(n-1)!}{(n+|\alpha|-1)!}$. We propose to consider the following multidimensional system with evolution along the free semigroup $\mathbb{F}_d^+$:

$$\Sigma_{\{\mathbf{U}_\alpha\},n}: \begin{cases} x(1\alpha) &= \frac{n+|\alpha|}{|\alpha|+1} A_1 x(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{1,\alpha} u(\alpha) \\ \quad\vdots & \quad\vdots \qquad\qquad\qquad \vdots \\ x(d\alpha) &= \frac{n+|\alpha|}{|\alpha|+1} A_d x(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{d,\alpha} u(\alpha) \\ y(\alpha) &= \quad C x(\alpha) + \binom{n+|\alpha|-1}{|\alpha|} D_\alpha u(\alpha) \end{cases} \tag{1.5.2}$$

with the d-tuple of state-space operators $\mathbf{A} = (A_1, \ldots, A_d)$ and the state-output operator $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Here, in addition, we have a family of system matrices and the family of input spaces indexed by $\alpha \in \mathbb{F}_d^+$:

$$\mathbf{U}_\alpha = \begin{bmatrix} A & \widehat{B}_\alpha \\ C & D_\alpha \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U}_\alpha \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \; \widehat{B}_\alpha = \begin{bmatrix} B_{1,\alpha} \\ \vdots \\ B_{d,\alpha} \end{bmatrix} \tag{1.5.3}$$

together with additional α-dependent weights in the system equations indexed by the natural number n. Upon running the system (1.5.2) with a fixed initial condition $x(\emptyset) = x \in \mathcal{X}$, we get recursively

$$x(\alpha) = \mu_{n,|\alpha|}^{-1} \cdot \Big(\mathbf{A}^\alpha x + \sum_{\alpha'' j \alpha' = \alpha} \mathbf{A}^{\alpha''} B_{j,\alpha'} u(\alpha') \Big), \tag{1.5.4}$$

$$y(\alpha) = \mu_{n,|\alpha|}^{-1} \cdot \Big(C\mathbf{A}^\alpha x + \sum_{\alpha'' j \alpha' = \alpha} C\mathbf{A}^{\alpha''} B_{j,\alpha'} u(\alpha') + D_\alpha u(\alpha) \Big). \tag{1.5.5}$$

Making use of notation (1.4.6) and equality (1.4.7), we observe that

$$(I - Z(z)A)^{-n} = \sum_{j=0}^\infty \mu_{n,j}^{-1} \cdot \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = j} \mathbf{A}^\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|}^{-1} \mathbf{A}^\alpha z^\alpha, \tag{1.5.6}$$

and then define $R_{n,k}(Z(z)A)$ via formal power series

$$R_{n,k}(Z(z)A) = \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|+k}^{-1} \mathbf{A}^\alpha z^\alpha. \tag{1.5.7}$$

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We next apply the noncommutative Z-transform (1.4.8) to (1.5.4) and then invoke (1.5.6) and (1.5.7) to get

$$\begin{aligned}\widehat{x}(z) &= \sum_{\alpha\in\mathbb{F}_d^+} \mu_{n,|\alpha|}^{-1}\Big(\mathbf{A}^\alpha x + \sum_{\alpha'' j\alpha'=\alpha} \mathbf{A}^{\alpha''} B_{j,\alpha'} u(\alpha')\Big) z^\alpha \\ &= \sum_{\alpha\in\mathbb{F}_d^+} \big(\mu_{n,|\alpha|}^{-1}\mathbf{A}^\alpha x\big) z^\alpha \\ &\quad + \sum_{\alpha'\in\mathbb{F}_d^+}\Big(\sum_{\alpha''\in\mathbb{F}_d^+} \mu_{n,|\alpha''|+|\alpha'|+1}^{-1}\mathbf{A}^{\alpha''} z^{\alpha''}\Big)\Big(\sum_{j=1}^d z_j B_{j,\alpha'}\Big) z^{\alpha'} u(\alpha') \\ &= (I - Z(z)A)^{-n} x + \sum_{\alpha\in\mathbb{F}_d^+} R_{n,|\alpha|+1}(Z(z)A) Z(z)\widehat{B}_\alpha z^\alpha u(\alpha).\end{aligned}$$

The same procedure applied to (1.5.5) now gives

$$\begin{aligned}\widehat{y}(z) &= C(I - Z(z)A)^{-n} x \\ &\quad + \sum_{\alpha\in\mathbb{F}_d^+}\Big(C R_{n,|\alpha|+1}(Z(z)A) Z(z)\widehat{B}_\alpha + \mu_{n,|\alpha|}^{-1} D_\alpha\Big) z^\alpha u(\alpha) \\ &= \mathcal{O}_{n,C,\mathbf{A}}(z) x + \sum_{\alpha\in\mathbb{F}_d^+} \Theta_{n,\mathbf{U}_\alpha}(z) z^\alpha u(\alpha), \end{aligned} \tag{1.5.8}$$

where the first term on the right presents the n-observability operator

$$\mathcal{O}_{n,C,\mathbf{A}}(z)x = C(I - Z(z)A)^{-n} x = \sum_{\alpha\in\mathbb{F}_d^+} \mu_{n,|\alpha|}^{-1}(C\mathbf{A}^\alpha x) z^\alpha \tag{1.5.9}$$

associated with the state-space d-tuple $\mathbf{A}$ and the state-output operator C and where

$$\Theta_{n,\mathbf{U}_\alpha}(z) = \mu_{n,|\alpha|}^{-1} D_\alpha + C R_{n,|\alpha|+1}(Z(z)A) Z(z)\widehat{B}_\alpha \tag{1.5.10}$$

is the family of transfer functions indexed by $\alpha \in \mathbb{F}_d^+$. One can see that the notion of the n-observability operator (1.5.9) generalizes the single-variable notion (1.3.11) as well as the unweighted multivariable one in (1.4.9). We say that the output pair $(C, \mathbf{A})$ is *n-observable* if $\mathcal{O}_{n,C,\mathbf{A}}$ is injective; from (1.5.9) we see that this is equivalent to $(C, \mathbf{A})$ being observable when viewed as an output pair for an unweighted system as in (1.4.9), i.e., observability is equivalent to

$$\bigcap_{\alpha\in\mathbb{F}_d^+} \operatorname{Ker} C\mathbf{A}^\alpha = \{0\}. \tag{1.5.11}$$

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We say that the output pair $(C, \mathbf{A})$ is *n-output stable* if $\mathcal{O}_{n,C,\mathbf{A}}$ is bounded as an operator from $\mathcal{X}$ into $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{F}_d^+)$ and *exactly n-observable* if also $\mathcal{O}_{n,C,\mathbf{A}}$ is bounded below.

In parallel with the discussion at the end of Section 1.3, we use the formula (1.5.8) to view the transfer function $\Theta_{n,\mathbf{U}_\alpha}(z)$ as encoding the result of a pulse input vector u being applied at position $\alpha \in \mathbb{F}_d^+$ with zero initial state,

$$\widehat{y}(z) = \Theta_{n,\mathbf{U}_\alpha}(z) \cdot z^\alpha u \quad \text{if} \quad x_\emptyset = 0 \quad \text{and} \quad u(\beta) = \delta_{\alpha,\beta} u.$$

A preliminary notion of *noncommutative n-Bergman conservative system* for systems of the form (1.5.2) will be developed in Section 7.1. The associated *n-Bergman inner family* is the main ingredient for one version of a Beurling–Lax representation for a forward-shift-invariant subspace for this setting (see Section 7.2). We shall see that backward-shift-invariant subspaces in this setting arise as the range of an n-observability operator form $\mathcal{O}_{n,C,\mathbf{A}}$ for an appropriate choice of a state-space d-tuple $\mathbf{A}$ and an state-output operator C.

1.6 Overview

The main goal of this book is to carry out the program outlined above in themes #1 to #4 in a unified free noncommutative multivariable setting. We conclude this introduction with an outline of the chapters to follow.

0. Upgrade of preliminaries: The vectorial Hardy space $H^2_\mathcal{Y}$ will be replaced by either a Hardy–Fock space $H^2_\mathcal{Y}(\mathbb{F}_d^+)$ (1.4.2), the more general weighted Bergman–Fock space $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{F}_d^+)$ (1.5.1), or the still more general weighted Hardy–Fock space $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}_d^+)$ for an admissible weight $\boldsymbol{\omega}$ (2.2.1) which is introduced in Section 2.2. Reproducing kernels and their associated reproducing kernel Hilbert spaces are replaced by noncommutative formal kernels and their associated noncommutative formal reproducing kernel Hilbert spaces (NFRKHSs) introduced in Chapter 2. The shift operator $S_\mathcal{Y}$ and backward shift $S_\mathcal{Y}^*$ are replaced by the shift-operator d-tuple $\mathbf{S}_{\boldsymbol{\omega},R} = (S_{\boldsymbol{\omega},R,1}, \ldots, S_{\boldsymbol{\omega},R,d})$ (2.2.3) and its adjoint right-shift-operator tuple $\mathbf{S}^*_{\boldsymbol{\omega},R} = (S^*_{\boldsymbol{\omega},R,1}, \ldots, S^*_{\boldsymbol{\omega},R,d})$ (2.2.4) on the space $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}_d^+)$ introduced in Chapter 2:

$$S_{\boldsymbol{\omega},R,j} \colon \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha z_j \quad \text{for} \quad 1 \leq j \leq d.$$

1. Backward-shift-invariant subspaces and ranges of observability operators: We consider an output pair coming for a system with weights

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as in (2.2.11) and characterize when the associated observability operator $\mathcal{O}_{\boldsymbol{\omega}, C, \mathbf{A}}$ as in (2.2.13) maps the state-space $\mathcal{X}$ boundedly into the weighted Hardy–Fock space $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$. The solution is again in terms of the existence of a positive semidefinite solution of a certain linear matrix inequality (see Theorem 4.2.2). To develop the ideas further, it suffices to normalize the solution H to be $H = I_{\mathcal{X}}$. Then, there is a notion of $\boldsymbol{\omega}$-contractive and $\boldsymbol{\omega}$-isometric output pair $(C, \mathbf{A})$ and a notion of $\boldsymbol{\omega}$-stability for the state-operator tuple $\mathbf{A} = (A_1, \dots, A_d)$. Contractively included backward-shift-invariant subspaces of $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$ are characterized as ranges of normalized contractive $\boldsymbol{\omega}$-observability operators with lifted norm. The inclusion of the backward-shift-invariant subspace in $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$ is isometric exactly when $\mathbf{A}$ is $\boldsymbol{\omega}$-strongly stable.

2. **Forward-shift-invariant subspaces and contractive multipliers and their realizations:** There are three types of scenarios:

   (a) The result most like the classical case described in Ssection 1.2.1 is for the case of a contractive multiplier Θ from a Hardy–Fock space $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ to another Hardy–Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$. Such Θ are characterized by being the transfer function of a Hardy–Fock system $\Sigma_{\mathbf{U}}$ (1.4.3) with system matrix $\mathbf{U}$ unitary, and with Θ strictly inner exactly when $\mathbf{A} = (A_1, \dots, A_d)$ is strongly stable in the sense of (1.4.11). Multipliers of this type serve as the McCullough–Trent-type Beurling–Lax representer for a shift-invariant subspace $\mathcal{M}$ isometrically contained in $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$.

   (b) The next case is that of a contractive multiplier Θ from a Hardy–Fock space $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ into a general weighted Hardy–Fock space $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$. It turns out that any such contractive multiplier Θ can be factored as $\Theta(z) = \boldsymbol{\Psi}_{\boldsymbol{\gamma}}(z) \cdot \Theta_0(z)$ where $\boldsymbol{\Psi}_{\boldsymbol{\gamma}}(z)$ is a fixed contractive multiplier from the Hardy–Fock space $H^2_{\ell^2_{\mathcal{Y}}(\mathbb{F}^+_d)}(\mathbb{F}^+_d)$ into $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$ first identified in the authors' paper [25], and where S_0 is a contractive multiplier from the Hardy–Fock space $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ to the Hardy–Fock space $H^2_{\ell^2_{\mathcal{Y}}(\mathbb{F}^+_d)}(\mathbb{F}^+_d)$. Of course the drawback (and the feature that makes everything work) of this is that the coefficient space $\ell^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ for the intermediate Hardy–Fock space $H^2_{\ell^2_{\mathcal{Y}}(\mathbb{F}^+_d)}(\mathbb{F}^+_d)$ is decidedly infinite-dimensional. Multipliers of this type serve as the Beurling–Lax representer for shift-invariant subspaces of $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$ which are contractively contained in $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$.

   (c) The case where Θ is a contractive multiplier from a Hardy–Fock space $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ into a general weighted Hardy–Fock space $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}^+_d)$ which

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is *Bergman inner* (see Definition 3.5.1), a notion of *inner* arising from the quest for a Beurling theorem and notion of *wandering subspace* in the context of Bergman spaces going to work of Aleman, Duren, Hedenmalm, Khavinson, Korenblum, Richter, Shapiro, and Sundberg in the 1990s (see [14, 77, 78, 95, 96, 98]). In this case, we realize Θ as the initial transfer function $\Theta_{\boldsymbol{\omega}, \mathbf{U}_\emptyset}(z)$ in the family of transfer functions $\{\Theta_{\boldsymbol{\omega}, \mathbf{U}_\alpha}(z)\}$ (2.2.14) arising from the time-varying noncommutative system of the form (2.2.11) (the upgrade of (1.5.4)–(1.5.5) required to handle the more general weight $\boldsymbol{\omega}$ in place of the standard Bergman weight $\boldsymbol{\mu}_n$), where the system matrices (1.5.3) are required to satisfy an additional nested set of metric constraints. A Bergman-inner family $\{\Theta_{\boldsymbol{\omega},\alpha}\}$ can be viewed as a kind of "time-varying" multiplier $\bigoplus z^\alpha u_\alpha \mapsto \bigoplus_{\alpha \in \mathbb{F}_d^+} \Theta_{\boldsymbol{\omega}, \mathbf{U}_\alpha}(z) z^\alpha u_\alpha$. This "time-varying" multiplication operator is *strictly inner* in the sense that it is an isometry from the "time-varying" Hardy–Fock space $\bigoplus_{\alpha \in \mathbb{F}_d^+} z^\alpha \mathcal{U}_\alpha$ into $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}_d^+)$, but with the cost that one must compute a power series Θ_α anew at each α (by e.g., solving a Cholesky factorization problem specified by computations already done to compute Θ'_α for α' with $|\alpha'| < |\alpha|$); see Chapter 7. The single-variable case is worked out by the authors in [23] and [24].

3. **Weighted Hardy–Fock space decompositions into backward- and forward-shift-invariant subspaces:** If $\mathcal{N} = \operatorname{Ran} \mathcal{O}_{\boldsymbol{\omega}, C, \mathbf{A}}$ is a backward-shift-invariant subspace contractively included in $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}_d^+)$, then there is a more involved version of the factorization procedure (1.2.6) leading to a realization-type formula, involving the universal function $\boldsymbol{\Psi}_{\boldsymbol{\gamma}}$ mentioned in #2(b) above as a factor, for a contractive multiplier Θ which is a Beurling–Lax representer for the Brangesian complement $\mathcal{M} = \mathcal{N}^{[\perp]}$ of $\mathcal{N}$. If $\mathcal{N}$ is a backward-shift-invariant, then $\mathcal{N}^{[\perp]}$ is forward-shift-invariant. We then have the Brangesian minimal decomposition

$$H^2_{\mathcal{Y}} = \mathcal{N} + \mathcal{M} = \operatorname{Ran} \mathcal{O}_{\boldsymbol{\omega}, C, \mathbf{A}} + \Theta \cdot H^2_{\boldsymbol{\omega}, \mathcal{U}}(\mathbb{F}_d^+).$$

If $\mathcal{N}$ is contained in $H^2_{\boldsymbol{\omega}, \mathcal{Y}}(\mathbb{F}_d^+)$ isometrically, then this decomposition is orthogonal. However, we have not identified $\mathcal{O}_{\boldsymbol{\omega}, C, \mathbf{A}}$ and Θ as the observability operator and the transfer function of the same system, i.e., while $\mathcal{O}_{\boldsymbol{\omega}, C, \mathbf{A}}$ is of the form (2.2.15), $\Theta(z)$ is not of the form (2.2.16); it is not clear what this should mean anyhow as there are many choices of α to choose from in (2.2.16). In this analysis, an ad hoc algebra of kernels takes over and we lose the connection with system theory.

Moreover, if we start with a forward-shift-invariant subspace $\mathcal{M} \subset H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}^+_d)$ contractively included in $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}^+_d)$, it is not always the case that $\mathcal{N} = \mathcal{M}^{[\perp]}$ is a backward-shift-invariant. For an example, see the end of the report [25]. If it happens to be the case that $\mathcal{M}^{[\perp]}$ is a backward-shift-invariant, then we can realize $\mathcal{N} := \mathcal{M}^{\perp]}$ as being of the form $\mathcal{O}_{\boldsymbol{\omega},C,\mathbf{A}}$ for a normalized contractive $\boldsymbol{\omega}$-observability operator $\mathcal{O}_{C,\mathbf{A}}$, and we are back to the setting of the previous paragraph.

However, if $\mathcal{M}$ is a forward-shift-invariant subspace isometrically contained in $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}^+_d)$, then by the Beurling–Lax theorem discussed in #2(c), we can model $\mathcal{M}$ as of the form $\mathcal{M} = \bigoplus_{\alpha \in \mathbb{F}^+_d} \Theta_\alpha(z) z^\alpha \mathcal{U}_\alpha$ for a Bergman-inner family $\{\Theta_\alpha\}$ which is the transfer function of the "time-varying" linear system of the form (2.2.11). Then, $\mathcal{N} = \mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{\boldsymbol{\omega},C,\mathbf{A}}$ where $\mathcal{O}_{\boldsymbol{\omega},c,\mathbf{A}}$ is the observability operator associated with the same system (2.2.11). In this more elaborate setting, we recover the conservative (so $\mathcal{W} = \{0\}$) version of the orthogonal decomposition (1.2.10) perfectly.

4. **Model theory for c.n.c. $*$-$\boldsymbol{\omega}$-hypercontractive operator tuples:** There is a class of operator tuples $\mathbf{T} = (T_1, \dots, T_d)$ naturally associated with an admissible weight $\boldsymbol{\omega}$, called $*$-$\boldsymbol{\omega}$-hypercontractive operator tuples, namely, operator tuples $\mathbf{T} = (T_1, \dots, T_d)$ with adjoint $\mathbf{A} = \mathbf{T}^* = (T_1^*, \dots, T_d^*)$ which are $\boldsymbol{\omega}$-hypercontractive in the sense of Definition 4.2.7 later. An illustrative special case is the case where $\boldsymbol{\omega} = \boldsymbol{\mu}_n$. Then we say that $\mathbf{T} = (T_1, \dots, T_d)$ is $*$-$\boldsymbol{\omega}$-hypercontractive if it is the case that

$$(I - B_{\mathbf{T}^*})^m[I] = \sum_{\alpha \in \mathbb{F}^+_d \colon |\alpha| \le m} (-1)^{|\alpha|} \left(\left[\begin{smallmatrix} m \\ |\alpha| \end{smallmatrix} \right] \right) T^{\alpha^\top} T^{*\alpha} \succeq 0 \qquad (1.6.1)$$

for $1 \le m \le n$, where in general $B_{\mathbf{T}^*}[X] = \sum_{j=1}^d T_j X T_j^*$. Our goal is to push the approach sketched in Section 1.2.3 to get a corresponding model theory for this general class of freely noncommutative operator tuples. Much of this theory can be developed without introducing an analog of the Sz.-Nagy–Foias characteristic function. As in the work of Popescu [153], there is a notion of characteristic function for general c.n.c. $*$-$\boldsymbol{\omega}$-hypercontractions, but there is no clear understanding of for which (possibly strictly proper) subclass of c.n.c. $*$-hypercontractive tuples the characteristic function exists. When it does exist it is a complete unitary invariant in the sense that an equivalence class of unitarily equivalent hypercontractive operator tuples is in one-to-one correspondence with equivalence classes of characteristic functions, but the notion of equivalence for characteristic functions may be somewhat weaker than the notion of *coincidence* appearing in the Sz.-Nagy–Foias theory.

We also discuss a generalization of the setting $\boldsymbol{\omega} = \boldsymbol{\mu}_n$ whereby one fixes a free noncommutative function p given by a formal power series in freely noncommuting arguments $z = (z_1, \ldots, z_d)$

$$p(z) = \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha z^\alpha.$$

We assume that $p(z)$ is a *regular*, meaning that

$$p_\emptyset = 0, \quad p_\alpha > 0 \text{ if } |\alpha| = 1, \quad p_\alpha \geq 0 \text{ for all } \alpha \in \mathbb{F}_d^+,$$

and that the series $p(z)$ has positive radius of convergence $\rho > 0$: *the series*

$$p(\mathbf{A}) = \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha \mathbf{A}^\alpha$$

converges absolutely whenever $\| \begin{bmatrix} A_1 & \cdots & A_d \end{bmatrix} \| < \rho$. We then consider the class of operator tuples $\mathbf{T} = (T_1, \ldots, T_d)$ which are $*$-(p,n)-*hypercontractive* in the sense that condition (1.6.1) is replaced by

$$(1 - p)^m (B_{\mathbf{T}^*})[I] \succeq 0 \text{ for } m = 1, \ldots, n. \tag{1.6.2}$$

In Chapter 9, we sketch how to carry out the whole program #1 to #4 sketched in Section 1.2.4 for this setting. This generalization originates in the work of Pott [155] for the univariate case and is handled by Popescu in [150] for the case $\boldsymbol{\omega} = \boldsymbol{\mu}_1$ and in [153] for the case $\boldsymbol{\omega} = \boldsymbol{\mu}_n$. The setting in [153] has an additional layer of flexibility: there is a constraint on the freeness of the lack of commutativity in the components $T_1, \ldots, T_d$ of the tuple $\mathbf{T}$; in particular the set of constraints $T_i T_j = T_j T_i$ implies that one can pick up the commutative version of the setting as a special case. Let us mention that the related paper [152] has still another layer of flexibility which allows one to pick up the commutative polydisk and the freely noncommutative ball as special cases.

The outline of the book is as follows:

Chapter 2 introduces the upgrades of all the preliminaries mentioned as part of theme #0 mentioned above.

Chapter 3 collects all the results on realization of contractive multipliers in various settings mentioned as part of theme #2. Here we use both linear system-theory and reproducing kernel methods. The analysis in this chapter builds on and clarifies the material from [30, 33, 46] treating the unweighted Hardy–Fock space case.

Chapter 4 is primarily concerned with theme #1, i.e., observability operators and observability operator ranges as models for backward-shift-invariant subspaces. Here we take a careful look at the characterization of when an observability operator $\mathcal{O}_{\boldsymbol{\omega},C,\mathbf{A}}$ maps the state space $\mathcal{X}$ boundedly into $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}^+_d)$ in terms of existence of a positive semidefinite solution of an associated Stein equation, as well as related questions as to when $\mathcal{O}_{\boldsymbol{\omega},C,\mathbf{A}}$ is isometric and when $\mathbf{A}$ is strongly stable in various senses. The special more concrete case $\boldsymbol{\omega} = \boldsymbol{\mu}_n$ for an $n \in \mathbb{N}$ gets special treatment, as here some special features occur. Here we also develop the basic properties of the model shift operator-tuple $\mathbf{S}_{\boldsymbol{\omega},R}$ which will be needed in later chapters for model-theory applications. We also present results on shifted observability operators and shifted observability gramians which will be needed in Chapter 7 when explaining how to characterize and construct Bergman-inner families (theme #2(c)). As a bonus we also present a converse characterization of which $\boldsymbol{\omega}$-isometric operator tuples are unitarily equivalent to such a shift-operator tuple, as the shift part in the Wold's decomposition for a more general class of operator tuples that we call $\mathcal{C}(\boldsymbol{\omega})$, thereby generalizing results of Olofsson, Giselsson, and Wennman [89, 137] and Eschmeier–Langendörfer [80].

Chapter 5 provides the details concerning Beurling–Lax representation theorems mentioned in themes #2(a) and #2(b) above.

Chapter 6 deals with yet another flavor of Beurling–Lax representation theorem for the weighted Hardy–Fock space setting not mentioned as part of theme #2 above, namely, Beurling–Lax representations based on the idea of quasi-wandering subspace introduced in the work of Izuchi et al. and Chen [63, 105, 106]. Here we show how the work of these authors can be adapted to our free noncommutative multivariable weighted Hardy–Fock space setting.

Chapter 7 is concerned exclusively with Beurling–Lax representation results mentioned as part of theme #2(c) above, exclusively for shift-invariant subspaces $\mathcal{M}$ isometrically included in $H^2_{\boldsymbol{\omega},\mathcal{Y}}(\mathbb{F}^+_d)$, by making use of Bergman-inner families $\{\Theta_{\boldsymbol{\omega},\alpha}\}_{\alpha\in\mathbb{F}^+_d}$ rather than a single contractive multiplier Θ. In this way, we arrive at a more orthogonal Beurling–Lax representation, closer to but more complicated than the classical case. These multipliers can be realized as transfer functions of the more complicated time-varying noncommutative systems (1.5.2) or more generally (2.2.11).

As a bonus topic not mentioned in our list of themes, we also obtain analogs of the expansive multiplier property and some results on characterizations of Bergman-inner multipliers as extremal solutions of interpolation problems. These results are analogous to results of Duren, Hedenmalm, Khavinson, Shapiro, Sundberg, and Vukotić [77, 95, 175] for the univariate case.

Chapter 8 fleshes out the applications of the content in the preceding chapters to the context of operator-model theory, i.e., theme #4 in the list of themes discussed above.

Chapter 9, as already mentioned, deals with the (p,n) setting where the weighted Bergman-Fock space is adjusted to handle the model theory for $*$-(p,n)-hypercontractive operator tuples $\mathbf{T}$ as in (1.6.2). What is required is a (p,n)-adaptation of the results from Chapter 8. In order to make this self-contained, it turns out to be essential to sketch how many of the results from the earlier chapters adapt to the (p,n)-setting. When $p(z) = z_1 + \cdots + z_d$, we recover all the previous results for the special case $\boldsymbol{\omega} = \boldsymbol{\mu}_n$.

1.7 Notes

While linear input/state/output systems of the form (1.2.1) are standard in systems theory, those of the form (1.4.3) are not so much. In the 1960s they came up in various guises in the theory of finite automata and formal languages as well as realization theory for nonlinear systems (see Fliess [81], Frazho [84], Schützenberger, [165], and Berstel–Reutenauer [49]) and in the 1990s in the theory and applications of stochastic multiscale systems [48, 64]. More recently, there has been a spurt of activity in the theory of free noncommutative functions [9, 109] and their still-evolving applications to noncommutative-operator functional calculus [8, 172], free-noncommutative real analytic geometry [101], and free probability [123]. In much of this work, the system (1.4.3) is hidden in the background; what is of interest is that a given function of freely noncommuting matrix or operator arguments can be realized in the form $\Theta_{\mathbf{U}}(z)$ (1.4.10) (where one plugs in operator or matrix arguments $Z_1, \ldots, Z_d$ for the free indeterminates $z_1, \ldots, z_d$ (see [102])). linear systems, and model theory for

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2
Formal Reproducing Kernel Hilbert Spaces

As was mentioned in Section 1.3, the standard weighted Bergman space $\mathcal{A}_{n,\mathcal{Y}}$ can be viewed as a reproducing kernel Hilbert space with reproducing kernel given by (1.3.2). It is useful to have a similar point of view for the weighted Bergman–Fock spaces discussed in Section 1.5.

2.1 Basic Definitions

In this section, we review the notion of *formal reproducing kernel Hilbert space* developed in [45, Section 3].

Given a collection of freely noncommuting indeterminates $z = (z_1, \dots, z_d)$, we suppose that we are given a Hilbert space $\mathcal{H}$ whose elements are formal power series

$$f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z \rangle\rangle, \qquad f_\alpha \in \mathcal{Y}, \tag{2.1.1}$$

with coefficients from a coefficient Hilbert space $\mathcal{Y}$. We say that $\mathcal{H}$ is a NFRKHS (*noncommutative formal reproducing kernel Hilbert space*) if, for each $\beta \in \mathbb{F}_d^+$, the linear operator $\Phi_\beta \colon \mathcal{H} \mapsto \mathcal{Y}$ defined by

$$\Phi_\beta \colon f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_\beta, \tag{2.1.2}$$

is continuous. As any such power series is completely determined by the list of its coefficients $\alpha \mapsto f_\alpha$ for $\alpha \in \mathbb{F}_d^+$, equivalently we can view the elements $f(z)$ as the functions $\alpha \mapsto f_\alpha$ on $\mathbb{F}_d^+$. Hence, by the noncommutative Aronszajn theory of reproducing kernel Hilbert spaces (see e.g. [45, Theorem 1.1]), there is a positive kernel $K \colon \mathbb{F}_d^+ \times \mathbb{F}_d^+ \to \mathcal{L}(\mathcal{Y})$ so that $\mathcal{H}$ is the

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reproducing kernel Hilbert space associated with K. To spell this out, in the present context we denote the value of K at (α, β) by $K_{\alpha,\beta} \in \mathcal{L}(\mathcal{Y})$ rather than $K(\alpha, \beta)$. Since we view an element $f \in \mathcal{H}$ as a formal power series (2.1.1) rather than as a function $\alpha \mapsto f_\alpha$ on $\mathbb{F}_d^+$, we write, for a given $\beta \in \mathbb{F}_d^+$ and $y \in \mathcal{Y}$, the element $\Phi_\beta^* y \in \mathcal{H}$ as $\Phi_\beta^* y = K_\beta(\cdot)y$, where

$$K_\beta(z)y = \sum_{\alpha \in \mathbb{F}_d^+} K_{\alpha,\beta}\, y\, z^\alpha. \tag{2.1.3}$$

Then, the reproducing kernel property can be written as

$$\langle f,\, K_\beta(\cdot)y\rangle_{\mathcal{H}} = \langle f,\, \Phi_\beta^* y\rangle_{\mathcal{H}} = \langle \Phi_\beta f,\, y\rangle_{\mathcal{Y}} = \langle f_\beta,\, y\rangle_{\mathcal{Y}}. \tag{2.1.4}$$

We can make the notation more suggestive of the classical case as follows. Let $\overline{\zeta} = (\overline{\zeta}_1, \ldots, \overline{\zeta}_d)$ be a second d-tuple of noncommuting indeterminates. Given a coefficient Hilbert space $\mathcal{C}$, we can use the $\mathcal{C}$-inner product to define pairings

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} \mapsto \mathbb{C}\langle\langle \zeta \rangle\rangle \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}} \mapsto \mathbb{C}\langle\langle \zeta \rangle\rangle$$

(where $\mathbb{C}\langle\langle \zeta \rangle\rangle$ is the space of formal power series in the set of formal conjugate indeterminates $\zeta = (\zeta_1, \ldots, \zeta_d)$ with coefficients in $\mathbb{C}$) by

$$\begin{aligned}
\Big\langle c, \sum f_\alpha \overline{\zeta}^\alpha \Big\rangle_{\mathcal{C} \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} &= \sum \langle c, f_\alpha \rangle_{\mathcal{C}}\, \zeta^{\alpha^\top}, \\
\Big\langle \sum f_\alpha \zeta^\alpha, c \Big\rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}} &= \sum \langle f_\alpha, c \rangle_{\mathcal{C}}\, \zeta^\alpha.
\end{aligned}$$

These pairings can be seen as special cases of the more general pairing

$$\left\langle \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha \zeta^\alpha,\, \sum_{\beta \in \mathbb{F}_d^+} g_\beta \overline{\zeta}^\beta \right\rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} = \sum_{\alpha \in \mathbb{F}_d^+} \left[\sum_{\beta, \gamma \colon \alpha = \gamma^\top \beta} \langle f_\beta, g_\gamma \rangle_{\mathcal{C}} \right] \zeta^\alpha, \tag{2.1.5}$$

which can be written more suggestively as

$$\begin{aligned}
\langle f(\zeta), g(\overline{\zeta}) \rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} &= \left\langle \sum f_\alpha \zeta^\alpha,\, \sum g_\beta \overline{\zeta}^\beta \right\rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} \\
&= g(\overline{\zeta})^* f(\zeta)
\end{aligned} \tag{2.1.6}$$

if we set

$$g(\overline{\zeta})^* = \left(\sum g_\beta \overline{\zeta}^\beta \right)^* = \sum g_\beta^* \zeta^{\beta^\top},$$

where we view $g_\beta^* \in \mathcal{L}(\mathcal{C}, \mathbb{C})$ as a linear functional on $\mathcal{C}$ so that

$$g_\beta^* f_\alpha = \langle f_\alpha, g_\beta \rangle_{\mathcal{C}} \quad \text{for any} \quad f_\alpha \in \mathcal{C}.$$